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DUAL L_p JOHN ELLIPSOIDS

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Abstract In this paper, the dual L_p John ellipsoids, which include the classical Löwner ellipsoid and the Legendre ellipsoid, are studied. The dual L_p versions of John's inclusion and Ball's volume-ratio inequality are shown. This insight allows for a unified view of some basic results in convex geometry and reveals further the amazing duality between Brunn–Minkowski theory and dual Brunn–Minkowski theory.

Keywords: Löwner ellipsoid; Legendre ellipsoid; L_p John ellipsoid; dual L_p John ellipsoids

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1. Introduction

The excellent paper by Lutwak *et al*. [**28**] shows that the classical John ellipsoid JK, the Petty ellipsoid [**10, 30**] and a recently discovered 'dual' of the Legendre ellipsoid [**24**] are all special cases ($p = \infty, 1, 2$) of a family of L_p ellipsoids, E_pK, which can be associated with a fixed convex body K. This insight allows for a unified view of, alternate approaches to and extensions of some basic results in convex geometry. Motivated by their research, we have studied the dual L_p John ellipsoids and show that the classical Löwner ellipsoid and the Legendre ellipsoid are special cases ($p = \infty, 2$) of this family of ellipsoids. Bastero and Romance [**3**] had shown this in a different way. Based on our characterization of dual L_p John ellipsoids, we present an L_p version of John's inclusion and show that the dual of Ball's volume-ratio inequality holds not only for the John ellipsoid, but also for all the dual L_p John ellipsoids.

An often used fact in both convex and Banach space geometry is that associated with each convex body K is a unique ellipsoid of minimal volume ellipsoid containing K. The ellipsoid is called the *Löwner ellipsoid* (or Löwner–John ellipsoid) of K. Here we denote the Löwner ellipsoid of K by $\tilde{J}K$, since it can be regarded as the dual of the John ellipsoid JK (the maximal volume ellipsoid contained in K). The Löwner–John ellipsoid is extremely useful (see, for example, [**1, 6**] for applications).

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Two important results concerning the Löwner ellipsoid are the dual form of John's inclusion and the dual form of Ball's volume-ratio inequality [**1**]. The dual form of John's inclusion states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\frac{1}{\sqrt{n}}\tilde{J}K \subseteq K \subseteq \tilde{J}K. \tag{1.1}$$

A consequence of Barthe's reverse Brascamp–Lieb inequality [**2**] is the outer volume-ratio inequality which can be regarded as the dual form of Ball's volume-ratio inequality: if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\frac{V(K)}{V(\tilde{J}K)} \geqslant \frac{2^n}{n!\,\omega_n}, \tag{1.2}$$

with equality if and only if K is a cross-polytope. Here ω_n denotes the volume of the unit ball, B, in $\mathbb{R}^n$.

A positive-definite $n \times n$ real symmetric matrix A generates an ellipsoid, $\varepsilon(A)$, in $\mathbb{R}^n$, defined by

$$\varepsilon(A) = \{x \in \mathbb{R}^n : x \cdot Ax \leqslant 1\},$$

where $x \cdot Ax$ denotes the standard inner product of x and Ax in $\mathbb{R}^n$.

Associated with a convex body $K \subset \mathbb{R}^n$ is its Legendre ellipsoid, $\Gamma_2 K$, which is the inertial ellipsoid of classical mechanics and can be generated by the matrix $[m_{ij}(K)]^{-1}$, where

$$m_{ij}(K) = \frac{n+2}{V(K)} \int_K (e_i \cdot x)(e_j \cdot x)\,\mathrm{d}x,$$

with $e_1, \dots, e_n$ denoting the standard basis for $\mathbb{R}^n$ and $V(K)$ denoting the n-dimensional volume of K.

The Legendre ellipsoid is an important ellipsoid that is closely related to the isotropic position and the well-known slicing problem (for more information and its important applications, see [**16**, **17**, **29**]). Recently, Lutwak *et al*. [**24**] defined a new ellipsoid $\Gamma_{-2}K$ which is a natural dual of the Legendre ellipsoid $\Gamma_2 K$. They proved that $\Gamma_{-2}K \subset \Gamma_2 K$ and noted that this is a geometrical analogue of the Cramer–Rao inequality [**26**]. The recent work of Ludwig [**18**] clearly demonstrates the importance of these two ellipsoids.

2. Dual L_p mixed volume

Lutwak introduced dual mixed volumes in [**21**] (see [**22**] for a summary of their properties), which is the beginning of dual Brunn–Minkowski theory. For general reference, the reader may wish to consult [**5**, **35**]. More recent work in dual Brunn–Minkowski theory can be found in [**7**, **8**, **14**, **15**, **20**, **38**].

In recent years, L_p-Brunn–Minkowski theory has received considerable attention and a lot of work has been done to develop this theory [**4**, **13**, **19**, **24**–**26**, **28**, **33**, **36**]. For quick reference we recall some basic results from the theory here.

A convex body in Euclidean n-dimensional space, $\mathbb{R}^n$, is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. For a convex body Q let $h_Q : \mathbb{R}^n \to \mathbb{R}$ denote its *support* *function*; i.e. for $x \in \mathbb{R}^n$, we have $h_Q(x) = \max\{x \cdot y : y \in Q\}$, where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. If Q contains the origin in its interior, then we will use Q^* to denote the polar of Q; i.e.

$$Q^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } y \in Q\}.$$

Obviously, for $\phi \in \mathrm{GL}(n)$,

$$(\phi Q)^* = \phi^{-\mathrm{T}} Q^*, \tag{2.1}$$

where $\phi^{-\mathrm{T}}$ denotes the inverse of the transpose of ϕ.

The *radial function* $\rho(Q, \cdot) = \rho_Q(\cdot) : \mathbb{R}^n \to \mathbb{R}$ associated with a set $Q \subset \mathbb{R}^n$ that is compact and star-shaped (with respect to the origin) is defined for $x \neq 0$ by $\rho_Q(x) = \max\{\lambda \geqslant 0 : \lambda x \in Q\}$. If ρ_Q is positive and continuous, Q is called a *star body*. Obviously, for $x \neq 0$ and $\phi \in \mathrm{GL}(n)$,

$$\rho_{\phi Q}(x) = \rho_Q(\phi^{-1} x). \tag{2.2}$$

Two star bodies K and L are said to be dilates if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

It is easy to verify that if A is a positive-definite $n \times n$ real symmetric matrix, then the support function of the ellipsoid $\varepsilon(A) = \{x \in \mathbb{R}^n : x \cdot Ax \leqslant 1\}$ is given by

$$h^2_{\varepsilon(A)}(u) = u \cdot A^{-1} u,$$

for $u \in S^{n-1}$. Thus, for a star body K,

$$h_{\Gamma_2 K}(u)^2 = \frac{n+2}{V(K)} \int_K |u \cdot x|^2 \, \mathrm{d}x = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^2 \rho_K(v)^{n+2} \, \mathrm{d}S(v), \tag{2.3}$$

for $u \in S^{n-1}$.

The normalized L_p polar projection body of K, $\Gamma_{-p} K$, for $p > 0$ is defined as the body whose radial function, for $u \in S^{n-1}$, is given by

$$\rho^{-p}_{\Gamma_{-p} K}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^p \, \mathrm{d}S_p(K, v).$$

For more details on the $\Gamma_{-p} K$ see [**28**].

Given $p > 0$, for star bodies K, L, and $\varepsilon > 0$, the L_p-harmonic radial combination $K \tilde{+}_{-p} \varepsilon \cdot L$ is the star body defined by

$$\rho(K \tilde{+}_{-p} \varepsilon \cdot L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}.$$

The dual L_p mixed volume $\tilde{V}_{-p}(K, L)$ [**25**] of the star bodies K, L, can be defined by

$$\frac{n}{-p} \tilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \tilde{+}_{-p} \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.4}$$

The definition (2.4) and the polar coordinate formula for volume give the following integral representation of the dual L_p mixed volume $\tilde{V}_{-p}(K, L)$ of the star bodies K, L [**25**]:

$$\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+p} \rho_L(u)^{-p} \, \mathrm{d}S(u). \tag{2.5}$$

From the integral representation (2.5), it follows immediately that, for each star body K,

$$\tilde{V}_{-p}(K,K) = V(K). \tag{2.6}$$

From (2.2) and the definition of L_p-harmonic radial combination it follows immediately that, for an L_p-harmonic radial combination of star bodies K and L,

$$\phi(K \tilde{+}_{-p} \varepsilon \cdot L) = \phi K \tilde{+}_{-p} \varepsilon \cdot \phi L.$$

This observation, together with the definition of the dual L_p mixed volume $\tilde{V}_{-p}$, shows that for $\phi \in \mathrm{SL}(n)$ and star bodies K, L we have $\tilde{V}_{-p}(\phi K, \phi L) = \tilde{V}_{-p}(K, L)$ or, equivalently,

$$\tilde{V}_{-p}(\phi K, L) = \tilde{V}_{-p}(K, \phi^{-1} L). \tag{2.7}$$

We will require a basic inequality regarding the dual L_p mixed volume $\tilde{V}_{-p}$. The dual L_p mixed volume inequality for $\tilde{V}_{-p}$ is that for star bodies K, L,

$$\tilde{V}_{-p}(K, L) \geqslant V(K)^{(n+p)/n} V(L)^{-p/n}, \tag{2.8}$$

with equality if and only if K and L are dilates. This inequality is an immediate consequence of the Hölder inequality [**12**] and integral representation (2.5).

It will be helpful to introduce a volume-normalized version of dual L_p mixed volumes. If K and L are star bodies that contain the origin in their interiors, then for each real $p > 0$ define

$$\bar{V}_{-p}(K, L) = \left(\frac{\tilde{V}_{-p}(K, L)}{V(K)}\right)^{1/p} = \left[\frac{1}{nV(K)} \int_{S^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right)^p \rho_K(u)^n \,\mathrm{d}S(u)\right]^{1/p}, \tag{2.9}$$

and for $p = \infty$ define

$$\bar{V}_{-\infty}(K, L) = \max\left\{\frac{\rho_K(u)}{\rho_L(u)} : u \in S^{n-1}\right\}. \tag{2.10}$$

Note that

$$\frac{1}{n} \rho_K(\cdot)^n \frac{\mathrm{d}S(\cdot)}{V(K)}$$

is a probability measure on S^{n-1}. Unless ρ_K/ρ_L is constant on S^{n-1}, it follows from (2.9), (2.10) and Jensen's inequality [**12**] that

$$\bar{V}_{-p}(K, L) < \bar{V}_{-q}(K, L), \tag{2.11}$$

for $0 < p < q \leqslant \infty$, and

$$\lim_{p \to \infty} \bar{V}_{-p}(K, L) = \bar{V}_{-\infty}(K, L).$$

From (2.2), (2.5) and (2.9) it follows immediately that, for $\lambda > 0$ and $p \in (0, \infty]$,

$$\bar{V}_{-p}(\lambda K, L) = \lambda \bar{V}_{-p}(K, L) \quad \text{and} \quad \bar{V}_{-p}(K, \lambda L) = \lambda^{-1} \bar{V}_{-p}(K, L). \tag{2.12}$$

From (2.7), (2.9) and (2.12) we find that, for $\phi \in \mathrm{GL}(n)$ and $p \in (0, \infty]$,

$$\bar{V}_{-p}(\phi K, \phi L) = \bar{V}_{-p}(K, L). \tag{2.13}$$

Finally, we will require the fact that

$$\bar{V}_{-\infty}(K, L) \leqslant 1 \quad \text{if and only if } K \subseteq L. \tag{2.14}$$

This is a direct consequence of definition (2.10).

3. Dual L_p John ellipsoids

Throughout, we assume that $p \in (0, \infty]$ and that K is a convex body that contains the origin in its interior. E will always denote an origin-centred ellipsoid.

3.1. Optimization problems

Given a convex body K in $\mathbb{R}^n$ that contains the origin in its interior, find an ellipsoid, amongst all origin-centred ellipsoids, which solves the following constrained maximization problem:

$$\max \left(\frac{\omega_n}{V(E)}\right)^{1/n} \quad \text{subject to } \bar{V}_{-p}(K, E) \leqslant 1. \tag{$\tilde{S}_p$}$$

A maximal ellipsoid will be called an $\tilde{S}_p$ solution for K. The dual problem is

$$\min \bar{V}_{-p}(K, E) \quad \text{subject to } \left(\frac{\omega_n}{V(E)}\right)^{1/n} \geqslant 1. \tag{$\bar{S}_p$}$$

A minimal ellipsoid will be called an $\bar{S}_p$ solution for K.

The solutions to $(\tilde{S}_p)$ and $(\bar{S}_p)$ differ by only a scale factor.

Lemma 3.1. *Suppose that $0 < p \leqslant \infty$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. If E is an ellipsoid centred at the origin that is an $\tilde{S}_p$ solution for K, then*

$$\bar{V}_{-p}(K, E)E \tag{3.1 a}$$

is an $\bar{S}_p$ solution for K. If E' is an ellipsoid centred at the origin that is an $\bar{S}_p$ solution for K, then

$$\left(\frac{\omega_n}{V(E')}\right)^{1/n} E' \tag{3.1 b}$$

is an $\tilde{S}_p$ solution for K.

The existence of a solution for $(\bar{S}_p)$ is guaranteed by the Blaschke selection theorem and the following proposition, which is given by Bastero and Romance [**3**].

Proposition 3.2 (Bastero and Romance [3]). *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies with the origin in their interior. Then*

$$\lim_{\phi \in \mathrm{SL}(n),\, \|\phi\| \to \infty} \tilde{V}_{-p}(\phi K, L) = +\infty, \quad 0 < p \leqslant \infty.$$

Lemma 3.1 now guarantees a solution to ($\tilde{S}_p$) as well.

Theorem 3.3. *Suppose that $p > 0$ and that K is a convex body in $\mathbb{R}^n$ which contains the origin in its interior. Then ($\tilde{S}_p$) and ($\bar{S}_p$) have unique solutions. Moreover, an ellipsoid E solves ($\bar{S}_p$) if and only if it satisfies*

$$\tilde{V}_{-p}(K, E)\rho_{E^*}(x)^{-2} = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p} \rho_E(v)^{2-p} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n, \tag{3.2 a}$$

and an ellipsoid E solves ($\tilde{S}_p$) if and only if it satisfies

$$V(K)\rho_{E^*}(x)^{-2} = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p} \rho_E(v)^{2-p} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n. \tag{3.2 b}$$

By Lemma 3.1, only the assertions about an $\bar{S}_p$ solution require a proof. The existence of a solution has already been established, and only the uniqueness and the characterization statements require proof.

In order to establish Theorem 3.3, we first prove a lemma that shows that, without loss of generality, we may assume that the ellipsoid E is the unit ball, B, in $\mathbb{R}^n$.

Lemma 3.4. *Suppose that $p > 0$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. If $\phi \in \mathrm{GL}(n)$, then*

$$\tilde{V}_{-p}(\phi^{-1}K, B)|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \rho_{\phi^{-1}K}(v)^{n+p} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n, \tag{3.3 a}$$

if and only if

$$\tilde{V}_{-p}(K, \phi B)\rho_{(\phi B)^*}(x)^{-2} = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p} \rho_{\phi B}(v)^{2-p} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n. \tag{3.3 b}$$

Proof. From (2.5), it is clear that, for $\lambda > 0$,

$$\tilde{V}_{-p}(\lambda K, L) = \lambda^{n+p}\tilde{V}_{-p}(K, L) \quad \text{and} \quad \tilde{V}_{-p}(K, \lambda L) = \lambda^{-p}\tilde{V}_{-p}(K, L).$$

Therefore, it suffices to prove the lemma for $\phi \in \mathrm{SL}(n)$. First note that

$$\tilde{V}_{-p}(K, \phi B)\rho_{(\phi B)^*}(x)^{-2} = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p} \rho_{\phi B}(v)^{2-p} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n$$

is equivalent to

$$\tilde{V}_{-p}(\phi^{-1}K, B)|\phi^{\mathrm{T}} x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p} |\phi^{-1}v|^{p-2} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n.$$

Let

$$\frac{\phi^{-1}v}{|\phi^{-1}v|} = v'.$$

Then

$$\tilde{V}_{-p}(\phi^{-1}K, B)|\phi^{\mathrm{T}}x|^2 = \int_{S^{n-1}} |x \cdot \phi v'|^2 \rho_K(\phi v')^{n+p}\, \mathrm{d}S(\phi v') \qquad \text{for all } x \in \mathbb{R}^n.$$

That is

$$\tilde{V}_{-p}(\phi^{-1}K, B)|\phi^{\mathrm{T}}x|^2 = \int_{S^{n-1}} |\phi^{\mathrm{T}}x \cdot v'|^2 \rho_{\phi^{-1}K}(v')^{n+p}\, \mathrm{d}S(v') \quad \text{for all } x \in \mathbb{R}^n.$$

Since x is arbitrary, we get

$$\tilde{V}_{-p}(\phi^{-1}K, B)|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \rho_{\phi^{-1}K}(v)^{n+p}\, \mathrm{d}S(v) \qquad \text{for all } x \in \mathbb{R}^n.$$

□

Proof of Theorem 3.3. The proof of this theorem is similar to that of [**28**, Theorem 2.2]. We first show that if E is an $\tilde{S}_p$ solution for K, then

$$\tilde{V}_{-p}(K, E)\rho_{E^*}(x)^{-2} = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p} \rho_E(v)^{2-p}\, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n.$$

Lemma 3.4 shows that we may assume that $E = B$.

Suppose that $T \in \mathrm{SL}(n)$ and choose $\varepsilon_0 > 0$ sufficiently small that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the matrix $I + \varepsilon T$ is invertible. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, define $T_\varepsilon \in \mathrm{SL}(n)$ by

$$T_\varepsilon = \frac{I + \varepsilon T}{\det(I + \varepsilon T)^{1/n}}.$$

Since $\det(T_\varepsilon) = 1$, the ellipsoid $E_\varepsilon = T_\varepsilon^{\mathrm{T}} B$ has volume ω_n. The fact that B is an $\tilde{S}_p$ solution implies that $\tilde{V}_{-p}(K, B) \leqslant \tilde{V}_{-p}(K, E_\varepsilon)$ for all ε, and hence we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} \rho_K(v)^{n+p} \rho_{E_\varepsilon}(v)^{-p}\, \mathrm{d}S(v) = 0,$$

or equivalently,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} \rho_K(v)^{n+p} \det(I + \varepsilon T)^{p/n} |(I + \varepsilon T)^{-1} v|^p\, \mathrm{d}S(v) \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} \rho_K(v)^{n+p} \det(I + \varepsilon T)^{p/n} |v - \varepsilon T v + O(\varepsilon^2)|^p\, \mathrm{d}S(v) \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} \rho_K(v)^{n+p} \det(I + \varepsilon T)^{p/n} |v \cdot v - 2\varepsilon v \cdot T v + O(\varepsilon^2)|^{p/2}\, \mathrm{d}S(v). \end{aligned}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \det(I + \varepsilon T) = \mathrm{tr}(T)$$

and the integrand depends smoothly on ε (for small ε), we have

$$\tilde{V}_{-p}(K,B)\,\mathrm{tr}(T) = \int_{S^{n-1}} \rho_K(v)^{n+p}(v \cdot Tv)\,\mathrm{d}S(v).$$

Choosing an appropriate T for each $i, j \in \{1, \dots, n\}$ gives

$$\tilde{V}_{-p}(K,B)\delta_{ij} = \int_{S^{n-1}} \rho_K(v)^{n+p}(v \cdot e_i)(v \cdot e_j)\,\mathrm{d}S(v),$$

which in turn gives

$$\tilde{V}_{-p}(K,B)|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p}\,\mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n,$$

as desired.

Conversely, we suppose that

$$\tilde{V}_{-p}(K,B)\rho_{B^*}(x)^{-2} = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p}\rho_B(v)^{2-p}\,\mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n, \tag{3.4}$$

and shall prove that if $|E| = \omega_n$, then

$$\tilde{V}_{-p}(K,E) \geqslant \tilde{V}_{-p}(K,B),$$

with equality if and only if $E = B$. Equivalently, we shall prove that if P is a positive-definite symmetric matrix with $\det(P) = 1$, then

$$\left[\frac{1}{n\tilde{V}_{-p}(K,B)} \int_{S^{n-1}} \rho_K(v)^{n+p}\rho_{PB}(v)^{-p}\,\mathrm{d}S(v)\right]^{1/p} \geqslant 1, \tag{3.5}$$

i.e.

$$\left[\frac{1}{n\tilde{V}_{-p}(K,B)} \int_{S^{n-1}} \rho_K(v)^{n+p}|P^{-1}v|^p\,\mathrm{d}S(v)\right]^{1/p} \geqslant 1, \tag{3.6}$$

with equality if and only if $|P^{-1}v| = 1$ for all $v \in S^{n-1}$. In order to establish (3.6) we shall prove that

$$\begin{aligned}\left[\frac{1}{n\tilde{V}_{-p}(K,B)} \int_{S^{n-1}} \rho_K(v)^{n+p}|P^{-1}v|^p\,\mathrm{d}S(v)\right]^{1/p} &\\ \geqslant \exp\left[\frac{1}{n\tilde{V}_{-p}(K,B)} \int_{S^{n-1}} \rho_K(v)^{n+p}\log|P^{-1}v|\,\mathrm{d}S(v)\right]&\\ \geqslant 1.&\end{aligned} \tag{3.7}$$

The first inequality is a direct consequence of Jensen's inequality, with equality if and only if there exists a $c > 0$ such that $|P^{-1}v| = c$ for all $v \in S^{n-1}$.

Write P^{-1} as $O^{\mathrm{T}}DO$, where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and O is an orthogonal matrix. To establish our inequality we need to show that

$$\int_{S^{n-1}} \rho_K(v)^{n+p} \log |P^{-1}v| \, \mathrm{d}S(v) \geqslant 0. \tag{3.8}$$

First note that

$$\tilde{V}_{-p}(OK, B)|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \rho_{OK}(v)^{n+p} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n.$$

From the fact that O is orthogonal and D is diagonal, and from the concavity of the log function, and the above inequality, we have

$$\begin{aligned} \int_{S^{n-1}} \rho_K(v)^{n+p} \log |P^{-1}v| \, \mathrm{d}S(v) &= \int_{S^{n-1}} \rho_K(v)^{n+p} \log |O^{\mathrm{T}}DOv| \, \mathrm{d}S(v) \\ &= \int_{S^{n-1}} \rho_K(O^{\mathrm{T}}u)^{n+p} \log |O^{\mathrm{T}}Du| \, \mathrm{d}S(O^{\mathrm{T}}u) \\ &= \int_{S^{n-1}} \rho_{OK}(u)^{n+p} \log |Du| \, \mathrm{d}S(u) \\ &\geqslant \tfrac{1}{2} \int_{S^{n-1}} \rho_{OK}(u)^{n+p} (u_1^2 \log \lambda_1^2 + \cdots + u_n^2 \log \lambda_n^2) \, \mathrm{d}S(u) \\ &= \tilde{V}_{-p}(OK, B) \sum_{i=1}^{n} \log \lambda_i = 0. \end{aligned}$$

Here $u_i = u \cdot e_i$.

From the strict concavity of the log function it follows that the equality in the above inequality is possible only if $u_{i1} \cdots u_{iN} \neq 0$ implies that $\lambda_{i1} \cdots \lambda_{iN} \neq 0$ for $u \in S^{n-1}$. Thus, $|Du| = \lambda_i$ when $u_i \neq 0$ for $u \in S^{n-1}$. Now the equality in (3.6) would also force $|P^{-1}v| = c$ for all $v \in S^{n-1}$, or equivalently $|Du| = c$ for all $u \in S^{n-1}$, so we have $\lambda_i = c$ for all i. This, together with the fact that $\lambda_1 \cdots \lambda_n = 1$, shows that equality in (3.7) would imply that $D = I$ and hence $P = I$. □

Theorem 3.3 shows that problem $(\tilde{S}_p)$ has a unique solution when $0 < p < \infty$. Now consider the case $p = \infty$ of $(\tilde{S}_p)$. With the aid of (2.14), we can rephrase $(\tilde{S}_\infty)$ as follows. Among all origin-centred ellipsoids, find an ellipsoid which solves the following constrained maximization problem:

$$\max \left(\frac{\omega_n}{V(E)} \right)^{1/n} \quad \text{subject to } K \subseteq E. \tag{$\tilde{S}_\infty$}$$

From the duality, it is easily shown that a minimizing ellipsoid in $(\tilde{S}_\infty)$ is unique [**9**]. In fact, if K is origin-symmetric, then $\tilde{E}_\infty K$ is the classical Löwner ellipsoid $\tilde{J}K$ of K.

Definition 3.5. Suppose that $0 < p \leqslant \infty$ and that K is a convex body in $\mathbb{R}^n$ which contains the origin in its interior. Among all origin-centred ellipsoids, the unique ellipsoid that solves the constrained maximization problem

$$\max_E \left(\frac{1}{V(E)}\right) \quad \text{subject to } \bar{V}_{-p}(K,E) \leqslant 1$$

will be called the dual L_p John ellipsoid of K and will be denoted by $\tilde{E}_p K$. Among all origin-centred ellipsoids, the unique ellipsoid that solves the constrained minimization problem

$$\min_E \bar{V}_{-p}(K,E) \quad \text{subject to } V(E) = \omega_n$$

will be called the normalized dual L_p John ellipsoid of K and will be denoted by $\bar{\tilde{E}}_p K$.

From (2.12) and (2.14) we immediately obtain the following lemma.

Lemma 3.6. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and if $0 < p \leqslant \infty$, then, for $\phi \in \mathrm{GL}(n)$,*

$$\tilde{E}_p \phi K = \phi \tilde{E}_p K.$$

Obviously, $\tilde{E}_p B = B$, and from Lemma 3.6 we see that if E is an ellipsoid that is centred at the origin, then $\tilde{E}_p E = E$.

From (2.3) and Theorem 3.3, we immediately obtain the following lemma.

Lemma 3.7. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\tilde{E}_2 K = \Gamma_2 K.$$

4. Generalizations of John's inclusion

The dual form of John's inclusion (1.1) states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\frac{1}{\sqrt{n}} \tilde{J} K \subseteq K \subseteq \tilde{J} K.$$

In this section, we shall prove a dual L_p version of this inclusion.

If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and $p \geqslant 1$, the L_p-centroid body $\Gamma_p K$ [**24**] is defined by

$$h_{\Gamma_p K}(u) = \left(\frac{n+p}{V(K)} \int_K |u \cdot x|^p \, \mathrm{d}x\right)^{1/p}, \tag{4.1}$$

for $u \in S^{n-1}$. Define $\Gamma_\infty K = \lim_{p\to\infty} \Gamma_p K$. From the definition of $\Gamma_p K$, it is easily shown that, when K is origin-symmetric, $\Gamma_\infty K = K$.

The L_p-centroid body, which is closely connected with the L_p-projection body, is important in L_p-Brunn–Minkowski theory. Lutwak *et al*. [**23, 25**] found many L_p-analogue inequalities of classical inequalities which include L_p versions of the Busemann–Petty centroid inequality and Petty projection inequality. Moreover, they proved sharp affine

L_p Sobolev inequalities using the L_p-Petty projection inequality [**27**]. Recent work by Yaskin and Yaskina [**37**] also shows the importance of the L_p-centroid body.

From the definition of $\Gamma_p K$, it is easily shown that if $\lambda > 0$, then $\Gamma_p \lambda K = \lambda \Gamma_p K$. Moreover, for $\phi \in \mathrm{GL}(n)$,

$$\Gamma_p \phi K = \phi \Gamma_p K. \tag{4.2}$$

Lemma 4.1. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\tilde{E}_p K \begin{cases} \subseteq \Gamma_p K & 1 \leqslant p < 2; \\ \supseteq \Gamma_p K & 2 < p \leqslant \infty. \end{cases}$$

Proof. Lemma 3.6 and (4.2) show that it suffices to prove the inclusions when $\tilde{E}_p K = B$. For $1 \leqslant p < 2$,

$$\begin{aligned} h_{\Gamma_p K}(u) &= \left(\frac{n+p}{V(K)} \int_K |u \cdot x|^p \,\mathrm{d}x \right)^{1/p} \\ &= \left(\frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p} \,\mathrm{d}S(v) \right)^{1/p} \\ &\geqslant \left(\frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^2 \rho_K(v)^{n+p} \,\mathrm{d}S(v) \right)^{1/p} \\ &= 1. \end{aligned}$$

This gives $\tilde{E}_p K = B \subseteq \Gamma_p K$ when $1 \leqslant p < 2$.

When $2 < p < \infty$, the inequality is reversed. Thus, $\tilde{E}_p K = B \supseteq \Gamma_p K$ for $p > 2$. The case $p = \infty$ follows from the definition of $\tilde{E}_\infty K$ and the fact that $\Gamma_\infty K = K$. □

Of course, the case of $p = 2$ of Lemma 4.1 is known as $\tilde{E}_2 K = \Gamma_2 K$.

Theorem 4.2. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\Gamma_q K \begin{cases} \subseteq n^{1/q-1/2} \tilde{E}_p K & \text{when } 1 \leqslant q \leqslant p \leqslant 2, \\ \supseteq n^{1/q-1/2} \tilde{E}_p K & \text{when } 2 \leqslant p \leqslant q \leqslant \infty. \end{cases}$$

Proof. Lemma 3.6 and (4.2) show that it suffices to prove the inclusions when $\tilde{E}_p K = B$. So, definition (3.5) gives $\tilde{V}_{-p}(K, B) = V(K)$. Suppose that $1 \leqslant q \leqslant p \leqslant 2$. Then

$$\begin{aligned} h_{\Gamma_q K}(u) &= \left(\frac{n+q}{V(K)} \int_K |u \cdot x|^q \,\mathrm{d}x \right)^{1/q} \\ &= \left(\frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^q \rho_K(v)^{n+q} \,\mathrm{d}S(v) \right)^{1/q} \\ &= n^{1/q} \left(\frac{1}{nV(K)} \int_{S^{n-1}} [|u \cdot v| \rho_K(v)]^q \rho_K(v)^n \,\mathrm{d}S(v) \right)^{1/q} \\ &\leqslant n^{1/q} \left(\frac{1}{nV(K)} \int_{S^{n-1}} [|u \cdot v| \rho_K(v)]^p \rho_K(v)^n \,\mathrm{d}S(v) \right)^{1/p} \end{aligned}$$

$$= n^{1/q}\left(\frac{1}{n\tilde{V}_{-p}(K,B)}\int_{S^{n-1}}|u\cdot v|^p\rho_K(v)^{n+p}\,\mathrm{d}S(v)\right)^{1/p}$$
$$\leqslant n^{1/q}\left(\frac{1}{n\tilde{V}_{-p}(K,B)}\int_{S^{n-1}}|u\cdot v|^2\rho_K(v)^{n+p}\,\mathrm{d}S(v)\right)^{1/2}$$
$$= n^{1/q}\left(\frac{1}{nV(K)}\int_{S^{n-1}}|u\cdot v|^2\rho_K(v)^{n+p}\,\mathrm{d}S(v)\right)^{1/2}$$
$$= n^{1/q-1/2}.$$

Thus, $\Gamma_q K \subseteq n^{1/q-1/2}\tilde{E}_p K$.

When $2 \leqslant p \leqslant q < \infty$, the inequality above is reversed. Thus, $\Gamma_q K \supseteq n^{1/q-1/2}\tilde{E}_p K$ when $2 \leqslant p \leqslant q < \infty$. The case $q = \infty$ follows from the definition of $(\tilde{S}_\infty)$ and the fact that $\Gamma_\infty K = K$. □

Choosing $q = \infty$ gives the following corollary.

Corollary 4.3. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then, for $2 \leqslant p \leqslant \infty$,*

$$\frac{1}{\sqrt{n}}\tilde{E}_p K \subseteq K.$$

Lutwak *et al*. [**28**] presented the following L_p version of John's inclusion.

Corollary 4.4 (Lutwak *et al*. [28]). *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$E_p K \begin{cases} \supseteq \Gamma_{-p} K \supseteq n^{1/2-1/p} E_p K & \text{when } 0 \leqslant p \leqslant 2; \\ \subseteq \Gamma_{-p} K \subseteq n^{1/2-1/p} E_p K & \text{when } 2 \leqslant p \leqslant \infty. \end{cases}$$

By taking $p = q$ in Theorem 4.2 and combining the inclusions with those of Lemma 4.1, we obtain the dual L_p version of John's inclusion, as follows.

Corollary 4.5. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$\tilde{E}_p K \begin{cases} \subseteq \Gamma_p K \subseteq n^{1/p-1/2}\tilde{E}_p K & \text{when } 1 \leqslant p \leqslant 2, \\ \supseteq \Gamma_p K \supseteq n^{1/p-1/2}\tilde{E}_p K & \text{when } 2 \leqslant p \leqslant \infty. \end{cases}$$

5. Volume-ratio inequalities

In the following sections, we will give some important properties about dual L_p John ellipsoids, which are dual forms of corresponding properties about L_p John ellipsoids given by Lutwak *et al*. [**28**].

Theorem 5.1 (Lutwak *et al*. [28]). *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and $0 < p \leqslant q \leqslant \infty$, then*

$$V(E_q K) \leqslant V(E_p K).$$

We present a dual form of the above theorem.

Theorem 5.2. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and $0 < p \leqslant q \leqslant \infty$, then*

$$V(\tilde{E}_p K) \leqslant V(\tilde{E}_q K).$$

Proof. From definition (2.5), (2.10) together with Jensen's inequality, it follows that, for $0 < p \leqslant q \leqslant \infty$,

$$\begin{aligned}\left(\frac{\tilde{V}_{-p}(K,L)}{V(K)}\right)^{1/p} &= \left[\frac{1}{nV(K)}\int_{S^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^p \rho_K(u)^n \, \mathrm{d}S(u)\right]^{1/p} \\ &\leqslant \left[\frac{1}{nV(K)}\int_{S^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^q \rho_K(u)^n \, \mathrm{d}S(u)\right]^{1/q} \\ &= \left(\frac{\tilde{V}_{-q}(K,L)}{V(K)}\right)^{1/q}.\end{aligned}$$

The above inequality, together with Definition 3.5, immediately gives the desired results. □

In general, the L_p John ellipsoid $E_p K$ is not contained in K (except when $p = \infty$). However, when $p \geqslant 1$, the volume of $E_p K$ is always dominated by the volume of K.

Theorem 5.3 (Lutwak *et al.* [28]). *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and $1 < p \leqslant \infty$, then*

$$V(E_p K) \leqslant V(K),$$

with equality for $p > 1$ if and only if K is an ellipsoid centred at the origin, and equality for $p = 1$ if and only if K is an ellipsoid.

Similarly, the dual L_p John ellipsoid $\tilde{E}_p K$ is not contain K (except when $p = \infty$). However, the volume of K is always dominated by the volume of $\tilde{E}_p K$.

Theorem 5.4. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior and $0 < p \leqslant \infty$, then*

$$V(\tilde{E}_p K) \geqslant V(K),$$

with equality if and only if K is an ellipsoid.

Proof. It is sufficient to prove the case of $p < \infty$. From Definition 3.5 and the dual L_p-Minkowski inequality (2.8), we obtain

$$V(K) = \tilde{V}_{-p}(K, \tilde{E}_p K) \geqslant V(K)^{(n+p)/n} V(\tilde{E}_p K)^{-p/n},$$

with equality if and only if K and $\tilde{E}_p K$ are translates. □

Lutwak *et al.* have shown that Ball's volume-ratio inequality holds not only for the John ellipsoid, but also for the L_p John ellipsoids.

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Theorem 5.5 (Lutwak *et al.* [28]). *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then, for $0 < p \leqslant \infty$,*

$$V(K) \leqslant \frac{2^n}{\omega_n} V(E_p K),$$

with equality if and only if K is a parallelotope.

Theorem 5.2 and the dual form of the Ball volume inequality (1.2) immediately give the dual L_p version of the Ball volume-ratio inequality as follows.

Theorem 5.6. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then, for $0 < p \leqslant \infty$,*

$$V(K) \geqslant \frac{2^n}{n!\ \omega_n} V(\tilde{E}_p K).$$

6. Intersections of convex bodies

If $p \in (0, \infty]$ and if K is an origin-symmetric convex body in $\mathbb{R}^n$, then K is said to be *dual L_p isotropic* if there exists a $c > 0$ such that

$$c|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \rho_K(v)^{n+p} \, \mathrm{d}S(v) \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 3.3 shows that K is dual L_p isotropic if and only if there exists a $\lambda > 0$ such that

$$\tilde{E}_p K = \lambda B.$$

The case for L_2 turns out to be the classical notation for isotropy.

Theorem 6.1. *If K is an origin-symmetric convex body in $\mathbb{R}^n$ that is dual L_p isotropic, then, for $1 \leqslant p \leqslant 2$,*

$$\mathrm{vol}_{n-1}(K \cap u^{\perp}) \geqslant \left[\frac{n+p}{n(p+1)}\right]^{1/p} \frac{\sqrt{n}}{(n!)^{1/n}} V(K)^{(n-1)/n}.$$

In order to prove Theorem 6.1, we first introduce a proposition given by Milman and Pajor.

Proposition 6.2 (Milman and Pajor [29]). *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then, for $p \geqslant 1$ and $u \in S^{n-1}$,*

$$\left(\frac{1}{V(K)} \int_K |u \cdot x|^p \, \mathrm{d}x\right)^{1/p} \geqslant \frac{V(K)}{2(p+1)^{1/p} \,\mathrm{vol}_{n-1}(K \cap u^{\perp})}. \tag{6.1}$$

Proof of Theorem 6.1. If inequality (6.1) holds for a body K, then it obviously holds for all dilates of the body. Thus, we may assume that $\tilde{E}_p K = B$ and

$$h_{\Gamma_p K}(u) = (n+p)^{1/p} \left(\frac{1}{V(K)} \int_K |u \cdot x|^p \, \mathrm{d}x\right)^{1/p} \geqslant \left(\frac{n+p}{p+1}\right)^{1/p} \frac{V(K)}{2 \,\mathrm{vol}_{n-1}(K \cap u^{\perp})}.$$

On the other hand,

$$\begin{aligned} h_{\Gamma_p K}(u) &= \left(\frac{n+p}{V(K)}\int_K |u\cdot x|^p\,\mathrm{d}x\right)^{1/p} \\ &= n^{1/p}\left(\frac{1}{n\tilde{V}_{-p}(K,B)}\int_{S^{n-1}} |u\cdot v|^p \rho_K(v)^{n+p}\,\mathrm{d}S(v)\right)^{1/p} \\ &\leqslant n^{1/p}\left(\frac{1}{n\tilde{V}_{-p}(K,B)}\int_{S^{n-1}} |u\cdot v|^2 \rho_K(v)^{n+p}\,\mathrm{d}S(v)\right)^{1/2} \\ &= n^{1/p}\left(\frac{1}{nV(K)}\int_{S^{n-1}} |u\cdot v|^2 \rho_K(v)^{n+p}\,\mathrm{d}S(v)\right)^{1/2} \\ &= n^{1/p-1/2}. \end{aligned}$$

Combining the two inequalities above with those in Proposition 6.2, we have

$$\mathrm{vol}_{n-1}(K\cap u^{\perp}) \geqslant \left[\frac{n+p}{n(p+1)}\right]^{1/p}\frac{\sqrt{n}}{2}V(K). \tag{6.2}$$

By Theorem 5.6, $\tilde{E}_p K = B$ implies that

$$V(K)^{1/n} \geqslant \frac{2}{(n!)^{1/n}}. \tag{6.3}$$

Combining (6.2) and (6.3) yields the desired inequality. □

If K is an origin-symmetric convex body in $\mathbb{R}^n$, the Blaschke–Santaló inequality [**34**] is the right-hand side of

$$\begin{aligned} \frac{4^n}{n!} &\leqslant V(K)V(K^*) \\ &\leqslant \omega_n^2. \end{aligned}$$

There is equality in the second line if and only if K is an ellipsoid. The first inequality is a central conjecture, known as the Mahler conjecture: among origin-symmetric convex bodies the *volume-product* is minimized by cubes and cross-polytopes. The first inequality has been verified for the class of zonoids (and their polars) by Reisner [**31**, **32**] (see also [**11**]).

For the volumes of the L_p John ellipsoids of polar reciprocal convex bodies we have the following result.

Theorem 6.3 (Lutwak *et al.* [28]). *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then, for $0 < p \leqslant \infty$,*

$$\begin{aligned} n^{-n/2}\omega_n^2 &\leqslant V(E_pK)V(E_pK^*) \\ &\leqslant \omega_n^2. \end{aligned}$$

with equality in the second line if and only if K is an ellipsoid and equality in the first line if K is a cube or the octahedron.

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We also have the following similar result.

Theorem 6.4. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then, for $0 < p \leqslant \infty$,*

$$n^{-n/2}\omega_n^2 \leqslant V(\tilde{E}_p K)V(\tilde{E}_p K^*) \leqslant n^{n/2}\omega_n^2.$$

Proof. From

$$\frac{1}{\sqrt{n}}\tilde{E}_\infty K \subseteq K \subseteq \tilde{E}_\infty K \quad \text{and} \quad V(K) \leqslant V(\tilde{E}_p K) \leqslant V(\tilde{E}_\infty K),$$

we obtain

$$n^{-n/2}V(\tilde{E}_\infty K) \leqslant V(K) \leqslant V(\tilde{E}_p K) \leqslant V(\tilde{E}_\infty K). \tag{6.4}$$

From $\sqrt{n}\tilde{E}_\infty^* K \supseteq K^* \supseteq \tilde{E}_\infty^* K$ and the definition of $\tilde{E}_\infty K$,

$$V(\tilde{E}_\infty^* K) \leqslant V(K^*) \leqslant V(\tilde{E}_p K^*) \leqslant V(\tilde{E}_\infty K^*) \leqslant n^{n/2}V(\tilde{E}_\infty^* K). \tag{6.5}$$

By combining (6.4), (6.5) and the fact that $V(\tilde{E}_\infty K)V(\tilde{E}_\infty^* K) = \omega_n^2$, we obtain the desired result. □

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References

1. K. Ball, Volume ratios and a reverse isoperimetric inequality, *J. Lond. Math. Soc.* **44** (1991), 351–359.
2. F. Barthe, On a reverse form of the Brascamp–Lieb inequality, *Invent. Math.* **134** (1998), 335–361.
3. J. Bastero and M. Romance, Positions of convex bodies associated to extremal problems and isotropic measures, *Adv. Math.* **184** (2004), 64–88.
4. W. Chen, L_p Minkowski problem with not necessarily positive data, *Adv. Math.* **201** (2006), 77–89.
5. R. J. Gardner, *Geometric tomography* (Cambridge University Press, 1995).
6. R. J. Gardner, The Brunn–Minkowski inequality, *Bull. Am. Math. Soc.* **39** (2002), 355–405.
7. R. J. Gardner and S. Vassallo, The Brunn–Minkowski inequality, Minkowski first inequality, and their duals, *J. Math. Analysis Applic.* **245** (2000), 502–512.
8. R. J. Gardner, E. B. Vedel Jensen and A. Volčič, Geometric tomography and local stereology, *Adv. Appl. Math.* **30** (2003), 397–423.
9. A. Giannopoulos and V. D. Milman, Extremal problems and isotropic positions of convex bodies, *Israel J. Math.* **117** (2000), 29–60.
10. A. Giannopoulos and M. Papadimitrakis, Isotropic surface area measures, *Mathematika* **46** (1999), 1–13.
11. Y. Godon, M. Meyer and S. Reisner, Zonoids with minimal volume-product: a new proof, *Proc. Am. Math. Soc.* **104** (1988), 273–276.
12. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge University Press, 1934).

13. C. Hu, X. Ma and C. Shen, On the Christoffel–Minkowski problem of Firey's p-sum, *Calc. Var. PDEs* **21** (2004), 137–155.
14. D. A. Klain, Star valuations and dual mixed volumes, *Adv. Math.* **121** (1996), 80–101.
15. D. A. Klain, Invariant valuations on star-shaped sets, *Adv. Math.* **125** (1997), 95–113.
16. K. Leichtweiss, *Affine geometry of convex bodies* (Barth, Heidelberg, 1998).
17. J. Lindenstrauss and V. D. Milman, The local theory of normed spaces and its applications to convexity, *Handbook of convex geometry*, Volumes A and B, pp. 1149–1220 (North-Holland, Amsterdam, 1993).
18. M. Ludwig, Ellipsoids and matrix-valued valuations, *Duke Math. J.* **119** (2003), 159–188.
19. M. Ludwig, Minkowski valuations, *Trans. Am. Math. Soc.* **357** (2005), 4191–4213.
20. M. Ludwig, Intersection bodies and valuations, *Am. J. Math.* **28** (2006), 1409–1428.
21. E. Lutwak, Dual mixed volumes, *Pac. J. Math.* **58** (1975), 531–538.
22. E. Lutwak, Intersection bodies and dual mixed volumes, *Adv. Math.* **71** (1988), 232–261.
23. E. Lutwak and G. Y. Zhang, Blaschke–Santaló inequalities, *J. Diff. Geom.* **47** (1997), 1–16.
24. E. Lutwak, D. Yang and G. Y. Zhang, A new ellipsoid associated with conves bodies, *Duke Math. J.* **104** (2000), 375–390.
25. E. Lutwak, D. Yang and G. Y. Zhang, L_p affine isoperimetric inequalities, *J. Diff. Geom.* **56** (2000), 111–132.
26. E. Lutwak, D. Yang and G. Y. Zhang, The Cramer–Rao inequality for star bodies, *Duke Math. J.* **112** (2002), 59–81.
27. E. Lutwak, D. Yang and G. Y. Zhang, Sharp affine L_p Sobolev inequalities, *J. Diff. Geom.* **62** (2002), 17–38.
28. E. Lutwak, D. Yang and G. Y. Zhang, L_p John ellipsoids, *Proc. Lond. Math. Soc.* **90** (2005), 497–520.
29. V. D. Milman and A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, in *Geometric aspects of functional analysis*, Springer Lecture Notes in Mathematics, Volume 1376, pp. 64–104 (Springer, 1989).
30. C. M. Petty, Surface area of a convex body under affine transformations, *Proc. Am. Math. Soc.* **12** (1961), 824–828.
31. S. Reisner, Random polytopes and the volume-product of symmetric convex bodies, *Math. Scand.* **57** (1985), 386–392.
32. S. Reisner, Zonoids with minimal volume-product, *Math. Z.* **192** (1986), 339–346.
33. D. Ryabogin and A. Zvavitch, The Fourier transform and Firey pfojections of convex bodies, *Indiana Univ. Math. J.* **53** (2004), 667–682.
34. L. A. Santaló, Un invariante afin para los cuerpos convexos del espacio de n dimensiones, *Portugaliae Math.* **8** (1949), 155–161.
35. R. Schneider, *Convex bodies: the Brunn–Minkowski theory* (Cambridge University Press, 1993).
36. C. Schütt and E. Werner, Surface bodies and p-affine surface area, *Adv. Math.* **187** (2004), 98–145.
37. V. Yaskin and M. Yaskina, Centroid bodies and comparison of volumes, *Indiana Univ. Math. J.* **55** (2006), 1175–1194.
38. C. Zhao and G. Leng, Inequalities for dual quermassintegrals of mixed intersection bodies, *Proc. Indian Acad. Sci. Math. Sci.* **115** (2005), 79–91.

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