A CENSUS OF FINITE AUTOMATA

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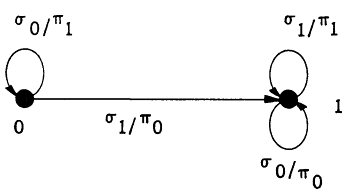
1. Introduction. A *finite automaton* may be thought of as a possible abstraction of a digital computer. Imagine a tape or sequence of letters from some alphabet being fed into a device with a finite number of internal states. When the device is in a particular state and receives an input letter, the system passes to another internal state and prints a letter on an output tape. This operation is continued until the entire input sequence has been processed.

Both Harary **(6)** and Ginsburg **(4)** have focused attention on the previously unsolved problem of counting the number of equivalence classes of finite automata. In the present paper, this problem is solved completely by proving a variety of theorems about the enumeration of functions.

Let $\Sigma_k = \{\sigma_0, \ldots, \sigma_{k-1}\}$ be the *input alphabet* and $\Pi_p = \{\pi_0, \ldots, \pi_{p-1}\}$ be the *output alphabet.*

Definition 1.1. A *finite automaton* is a system $S = \langle S, f, g \rangle$ where $S = \{s_0, \ldots, s_{n-1}\}$ is a non-void set of *internal states*, f is the *direct transition function* which maps $S \times \Sigma_k \rightarrow S$, while g is the *output function* which maps $S \times \Sigma_k \rightarrow \Pi_p$.

It is convenient to represent a finite automaton by a directed graph whose nodes represent internal states and whose labeled branches denote transitions. An example is indicated in Figure 1 along with the tabular representation of f and g which is an alternative definition of the automaton.



f	σ_0	σ_1
s_0	s_0	s_1
s_1	s_1	s_1

g	σ_0	σ_1
s_0	π_1	π_0
s_1	π_0	π_1

FIGURE 1. An example of a finite automaton.

Definition 1.2. Two automata $S = \langle S, f_1, g_1 \rangle$ and $T = \langle T, f_2, g_2 \rangle$ are *isomorphic* if there exists a one-to-one mapping α from S *onto* T such that

Received August 12, 1963. The research reported herein has been sponsored by the U.S. Air Force Office of Scientific Research, the Department of the Army, and the Department of the Navy (Grant No. AF-AFOSR-139-63).

$$\alpha(f_1(s, \sigma)) = f_2(\alpha(s), \sigma)$$

and

$$g_1(s, \sigma) = g_2(\alpha(s), \sigma).$$

Clearly, isomorphism of automata is an equivalence relation and hence decomposes the family of all machines into equivalence classes.

There are further refinements which can be made on the concept of equivalence. For instance, the machines in Figure 2 are not isomorphic, but differ

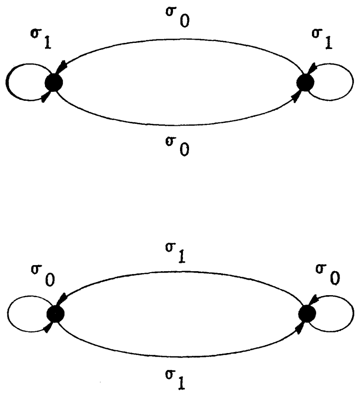


FIGURE 2. Two non-isomorphic automata which are equivalent under an input permutation.

only by a permutation of the inputs. This suggests widening our definition by allowing arbitrary permutations of the letters of Σ_k. For similar reasons, one might allow any arbitrary permutation of the elements of Π_p.

More formally, we have the following definitions.

Definition 1.3. Two automata $S = \langle S, f_1, g_1\rangle$ and $T = \langle T, f_2, g_2\rangle$ are *equivalent with respect to an input permutation* if there exists $\alpha \in \mathfrak{S}_n$, $\beta \in \mathfrak{S}_k$ such that

$$\begin{aligned} \alpha f_1(s, \sigma) &= f_2(\alpha(s), \beta(\sigma)), \\ g_1(s, \sigma) &= g_2(\alpha(s), \beta(\sigma)). \end{aligned}$$

(As usual, denote the symmetric group of degree n by $\mathfrak{S}_n$.)

Definition 1.4. Two automata $S = \langle S, f_1, g_1\rangle$ and $T = \langle T, f_2, g_2\rangle$ are said to be *equivalent with respect to input and output permutations* if there exist $\alpha \in \mathfrak{S}_n$, $\beta \in \mathfrak{S}_k$, and $\gamma \in \mathfrak{S}_p$ such that

$$\begin{aligned} \alpha f_1(s, \sigma) &= f_2(\alpha(s), \beta(\sigma)), \\ \gamma g_1(s, \sigma) &= g_2(\alpha(s), \beta(\sigma)). \end{aligned}$$

Before proceeding to the enumeration, the background material from the literature is assembled.

2. Combinatorial background. In this section, the pertinent combinatorial background will be presented. Let $\mathfrak{G}$ be a permutation group defined on a set S. Let the order of $\mathfrak{G}$ be g and the degree of $\mathfrak{G}$ be n. Two elements s_i and s_j in S are called *equivalent* if there exists a permutation α in $\mathfrak{G}$ such that $\alpha(s_i) = s_j$. The number of transitivity classes is given by a theorem due to Froebenius (**1**, p. 191).

THEOREM 2.1. *Let $\mathfrak{G}$ be a permutation group of order g acting on a set S. The number of equivalence classes induced by $\mathfrak{G}$ is*

$$(1/g)\textstyle\sum_c n_c\, I(c),$$

where the sum is over all conjugate classes c, n_c is the cardinality of c, and $I(c)$ is the number of fixed points of S under any permutation in c.

For the purposes of the present paper, the only group to be considered will be the symmetric group, but the generalization to arbitrary groups is immediate.

Let $\alpha \in \mathfrak{S}_n$ have cycle structure $(j_1, \ldots, j_n)$; that is, j_i cycles of length i for $i = 1, \ldots, n$. Clearly

$$\sum_{i=1}^{n} ij_i = n. \tag{1}$$

Conjugate elements of $\mathfrak{S}_n$ have the same cycle structure and conversely.

COROLLARY 2.2. *Let $\mathfrak{S}_n$ act on a set S. The number of equivalence classes induced by $\mathfrak{S}_n$ is*

$$\frac{1}{n!}\sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}}\, I(j), \tag{2}$$

where the sum is over all non-negative integral solutions of equation (1) *and $I(j)$ denotes the number of elements of S fixed by any permutation of $\mathfrak{S}_n$ with cycle structure $(j_1, \ldots, j_n)$.*

The coefficient in (2) which is the number of permutations having cycle structure $(j_1, \ldots, j_n)$ will henceforth be denoted by $c_{(j)}$; that is,

$$c_{(j)} = \frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}}$$

There will be an occasion to use a new and powerful theorem of de Bruijn (**3**). For this theorem it is convenient to develop the concept of the cycle index polynomial.

Let $\mathfrak{G}$ be a permutation group of order g acting on a set D of cardinality s while $\mathfrak{H}$ is a permutation group of order h acting on a set of R of cardinality r. Consider the class of functions from D into R and call two functions f_1 and f_2 equivalent if there exists an $\alpha \in \mathfrak{G}$ and a permutation $\beta \in \mathfrak{H}$ such that for every $d \in D, f_1(\alpha(d)) = \beta f_2(d)$. The number of equivalence classes of functions is desired. In order to state the theorem of de Bruijn which solves this problem, the following definition must be introduced.

Definition 2.3. If $\mathfrak{G}$ is a permutation group of order g acting on a set S of cardinality s and if $f_1, \ldots, f_s$ are s indeterminates, then the *cycle index polynomial* of $\mathfrak{G}$ is defined as

$$Z_{\mathfrak{G}}(f_1, \ldots, f_s) = \frac{1}{g} \sum_{(j)} g_{(j)} f_1^{j_1} \ldots f_s^{j_s},$$

where the equation is summed over all partitions of s and $g_{(j)}$ is the number of elements having cycle structure $(j_1, \ldots, j_s)$.

THEOREM 2.4 (de Bruijn). *The number of classes of functions $f: D \to R$ with a permutation group $\mathfrak{G}$ of degree s and order g acting on D and a group $\mathfrak{H}$ of degree r and order h acting on R is given by*

$$Z_{\mathfrak{G}}\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_s}\right) Z_{\mathfrak{H}}(h_1, \ldots, h_r)$$

evaluated at $z_1 = \ldots = z_s = 0$, where

$$h_i = \exp\left(\sum_{k=1}^{s/i} i z_{ki}\right) \qquad \textit{for } i = 1, \ldots, r.$$

Theorem 4 is generally applied in actual problems through the use of the following lemma.

LEMMA 2.5. *A term $h_1^{j_1} \ldots h_r^{j_r}$ in $Z_{\mathfrak{H}}$ gives rise to*

$$Z_{\mathfrak{G}}\left(\sum_{t|1} t j_t, \ldots, \sum_{t|s} t j_t\right).$$

There is a convenient product of permutation groups which was defined and used by Harary **(5)** in his study of bicoloured graphs. Let $\mathfrak{A}$ and $\mathfrak{B}$ be permutation groups of order m and n operating on disjoint object sets X and Y of cardinality a and b respectively. The Cartesian product $\mathfrak{A} \times \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ is defined on $X \times Y$ as

$$(\alpha, \beta)(x, y) = (\alpha(x), \beta(y)).$$

It is important to be able to compute the cycle index of $\mathfrak{A} \times \mathfrak{B}$ from the cycle indices of $\mathfrak{A}$ and of $\mathfrak{B}$. This is accomplished by defining a cross operation on cycle index polynomials. The pertinent result of Harary is the following.

THEOREM 2.6 (Harary). *If*

$$Z_{\mathfrak{A}} = \frac{1}{m} \sum_{(i)} a_{(i)} \prod_{r=1}^{a} g_r^{i_r} \qquad \textit{and} \quad Z_{\mathfrak{B}} = \frac{1}{h} \sum_{(j)} b_{(j)} \prod_{s=1}^{b} h_s^{j_s},$$

then

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$$Z_{\mathfrak{A}\times\mathfrak{B}} = Z_{\mathfrak{A}} \times Z_{\mathfrak{B}} = \frac{1}{m}\frac{1}{n}\sum_{(i)}\sum_{(j)} a_{(i)}b_{(j)} \prod_{r=1}^{a}\prod_{s=1}^{b} f_{\langle r,s\rangle}^{i_r j_s (r,s)}$$

where $\langle r, s\rangle$ is the least common multiple of r and s, while (r, s) is the greatest common divisor of r and s.

It is often convenient to use these theorems for terms of the cycle index polynomials rather than the entire polynomial. Such a procedure often results in elegant symbolic proofs and is easily verified to be a valid method of proof **(8)**.

3. Principle for counting automata. The following restatement of Theorem 2.1 enables the enumeration of automata to be carried out by enumerating classes of functions.

THEOREM 3.1. *The number of classes of non-isomorphic finite automata with n internal states defined over input alphabet Σ_k and output alphabet Π_p is*

$$\frac{1}{n!}\sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}} F(j)G(j),$$

where the sum is over all partitions of n and $F(j)$ denotes the number of functions $f: S \times \Sigma_k \to S$ left invariant by a permutation α having cycle structure $(j_1, \ldots, j_n)$ applied to both domain and range. Similarly, $G(j)$ is the number of functions from $S \times \Sigma_k$ into Π_p invariant under α.

The result of Theorem 3.1 becomes slightly more complicated when one allows arbitrary permutations in $\mathfrak{S}_k$ over Σ_k or in $\mathfrak{S}_p$ over Π_p.

THEOREM 3.2. *The number of equivalence classes of finite automata with n internal states and allowing arbitrary permutations of $\mathfrak{S}_k$ is*

$$\frac{1}{n!}\frac{1}{k!}\sum_{(j)}\sum_{(l)} \frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}} \frac{k!}{\prod_{i=1}^{k} l_i!\, i^{l_i}} F(j, l)\, G(j, l),$$

where $F(j, l)$ is the number of transition functions which are fixed by a permutation of the states with cycle structure $(j) = (j_1, \ldots, j_n)$ and a permutation of the inputs with cycle structure $(l) = (l_1, \ldots, l_k)$.

THEOREM 3.3. *The number of equivalence classes of finite automata with n internal states allowing $\mathfrak{S}_k$ on Σ_k while $\mathfrak{S}_p$ permutes Π_p is given by*

$$\frac{1}{n!}\frac{1}{k!}\frac{1}{p!}\sum_{(j)}\sum_{(l)}\sum_{(m)} \frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}} \frac{k!}{\prod_{i=1}^{k} l_i!\, i^{l_i}} \frac{p!}{\prod_{i=1}^{p} m_i!\, i^{m_i}} F(j, l)\, G(j, l, m),$$

where $G(j, l, m)$ is the number of output functions invariant under permutations $\alpha \in \mathfrak{S}_n$, $\beta \in \mathfrak{S}_k$, $\gamma \in \mathfrak{S}_p$ such that α has cycle structure $(j_1, \ldots, j_n)$, β has cycle structure $(l_1, \ldots, l_k)$, and γ has cycle structure $(m_1, \ldots, m_p)$.

Proofs. The above theorems are restatements of Theorem 2.1. The only additional observation that must be made is that the number of automata left invariant by α is the product of the number of transition functions and the number of output functions left invariant by α.

4. The number of output functions. First, we determine $G(j) = G(j_1, \ldots, j_n)$ in the case of no permutations on Σ_k or on Π_p.

THEOREM 4.1. *The number of functions $g: S \times \Sigma_k \to \Pi_p$ left invariant by a permutation of $\mathfrak{S}_n$ with cycle structure $(j_1, \ldots, j_n)$ is*

$$G(j) = p^J, \qquad \textit{where } J = k \sum_{i=1}^{n} j_i.$$

Proof. There are p choices for each element of the range, and there are kj_1 choices for elements of S in cycles of length $1, \ldots, kj_n$ choices for elements of S in cycles of length n. Thus there are

$$\prod_{i=1}^{n} p^{kj_i} = p^J$$

such functions.

The immediate generalization to arbitrary permutations of Σ_k is now indicated.

THEOREM 4.2. *The number of output functions $g: S \times \Sigma_k \to \Pi_p$ which are fixed by a permutation $\alpha \in \mathfrak{S}_n$ having cycle structure $(j_1, \ldots, j_n)$ and $\beta \in \mathfrak{S}_k$ with cycle structure $(l_1, \ldots, l_k)$ is*

$$G(j, l) = p^{J'}, \qquad \textit{when } J' = \sum_{r=1}^{n} \sum_{s=1}^{k} j_r\, l_s (r, s),$$

where (r, s) denotes the greatest common divisor of r and s.

Proof. The cycle structure of $\alpha \in \mathfrak{S}_n$ is denoted by

$$\prod_{r=1}^{n} g_r^{\,j_r}$$

and the cycle structure of $\beta \in \mathfrak{S}_k$ by

$$\prod_{s=1}^{k} h_s^{\,l_s}.$$

Therefore the cycle structure induced on $S \times \Sigma_k$ is given as:

$$\left(\prod_{r=1}^{n} g_r^{\,j_r}\right) \times \left(\prod_{s=1}^{k} h_s^{\,l_s}\right) = \prod_{r=1}^{n} \prod_{s=1}^{k} f_{\langle r, s\rangle}^{\,j_s\, l_s (r, s)}.$$

There are p choices in the range for each and every one of the domain elements, so the total number of invariant functions is

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$$\prod_{r=1}^{n}\prod_{s=1}^{k} p^{j_r l_s (r,s)} = p^{J'}$$

The last generalization to be allowed is the case when arbitrary permutations of Σ_k and of Π_p are allowed.

THEOREM 4.3. *The number of output functions $g:S \times \Sigma_k \to \Pi_p$ left invariant by $\alpha \in \mathfrak{S}_n$, $\beta \in \mathfrak{S}_k$,* and *$\gamma \in \mathfrak{S}_p$ having cycle structure $(j_1, \ldots, j_n)$, $(l_1, \ldots, l_k)$, and $(m_1, \ldots, m_p)$ respectively is*

$$G(j, l, m) = \prod_{r=1}^{n}\prod_{s=1}^{k}\left(\sum_{t\mid\langle r,s\rangle} tm_t\right)^{j_r l_s(r,s)},$$

where $\langle r, s\rangle$ and (r, s) denote the least common multiple and greatest common divisor of r and s respectively.

Proof. The desired number is given by the de Bruijn result specialized to the case where

$$Z_{\mathfrak{G}} = \prod_{r=1}^{n}\prod_{s=1}^{k} f_{\langle r,s\rangle}^{j_r l_s(r,s)}$$

and

$$Z_{\mathfrak{H}} = \prod_{t=1}^{p} g_t^{m_t}.$$

One application of Lemma 2.5 gives the result.

5. The number of transition functions. The enumeration of finite automata will be completed by enumerating the number of non-isomorphic functions $f:S \times \Sigma_k \to S$. The present problem of determining the $F(j)$ is more complicated than the corresponding problem for output functions. The reason for the added complication is that the permutation α with cycle structure $(j_1, \ldots, j_n)$ acts on the range of the transition function as well as on part of the domain of the function.

THEOREM 5.1. *The number of transition functions $f:S \times \Sigma_k \to S$ fixed under a permutation of S with cycle structure $(j_1, \ldots, j_n)$ is*

$$F(j) = \prod_{i=1}^{n}\left(\sum_{d\mid i} dj_d\right)^{kj_i}.$$

Proof. When $k = 1$, this theorem reduces to a result of Davis (**2**, Theorem 6). To prove the result, let α be a permutation with cycle structure $f_1^{j_1} \ldots f_n^{j_n}$. Since permutations of Σ_k are not allowed, the cycle index of the group on Σ_k is f_1^k. The entire group on the domain is

$$f_1^{j_1} \ldots f_n^{j_n} \times f_1^k = f_1^{kj_1} f_2^{kj_2} \ldots f_n^{kj_n}.$$

Using de Bruijn's theorem with

$$Z_{\mathfrak{G}} = f_1^{kj_1} \ldots f_n^{kj_n} \quad \text{and} \quad Z_{\mathfrak{H}} = f_1^{j_1} \ldots f_n^{j_n},$$

we obtain

$$\prod_{i=1}^{n} \Big(\sum_{d \mid i} dj_d \Big)^{kj_i}.$$

It is also easy to give a direct proof.

Now the generalization to arbitrary permutations in $\mathfrak{S}_k$ is given.

THEOREM 5.2. *The number of transition functions $f{:}S \times \Sigma_k \rightarrow S$ fixed under $\alpha \in \mathfrak{S}_n$ and $\beta \in \mathfrak{S}_k$ where α has cycle structure $(j_1, \ldots, j_n)$ and β has cycle structure $(l_1, \ldots, l_k)$ is*

$$F(j, l) = \prod_{r=1}^{n} \prod_{s=1}^{k} \Big(\sum_{t \mid \langle r,s \rangle} tj_t \Big)^{j_r l_s (r,s)}.$$

Proof. We use the de Bruijn theorem with

$$Z_{\mathfrak{G}} = \prod_{r=1}^{n} g_r^{j_r} \times \prod_{s=1}^{k} h_s^{l_s} = \prod_{r=1}^{n} \prod_{s=1}^{k} f_{\langle r,s \rangle}^{j_r l_s (r,s)}$$

and

$$Z_{\mathfrak{H}} = \prod_{i=1}^{n} f_i^{j_i}.$$

An application of Lemma 2.5 yields

$$F(j, l) = \prod_{r=1}^{n} \prod_{s=1}^{k} \Big(\sum_{d \mid \langle r,s \rangle} dj_d \Big)^{j_r l_s (r,s)}.$$

6. The main theorems. Collecting the results of the previous sections gives the formulas for the number of finite automata.

THEOREM 6.1. *The number of classes of non-isomorphic finite automata $S = \langle S, f, g \rangle$ with $S = \{s_0, \ldots, s_{n-1}\}$ over input alphabet Σ_k and output alphabet Π_p is*

$$\frac{1}{n!} \sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i! \, i^{j_i}} \prod_{i=1}^{n} \Big(p \sum_{d \mid i} dj_d \Big)^{kj_i}.$$

Proof. Cf. Theorems 3.1, 4.1, and 5.1.

THEOREM 6.2. *The number of classes of finite automata $S = \langle s, f, g \rangle$ with $S = \{s_0, \ldots, s_{n-1}\}$ over input alphabet Σ_k with $\mathfrak{S}_k$ acting on Σ_k and with output alphabet Π_p is given by*

$$\frac{1}{n!} \frac{1}{k!} \sum_{(j)} \sum_{(l)} \frac{n!}{\prod_{i=1}^{n} j_i! \, i^{j_i}} \frac{k!}{\prod_{i=1}^{k} l_i! \, i^{l_i}} \prod_{r=1}^{n} \prod_{s=1}^{k} \Big(p \sum_{t \mid \langle r,s \rangle} tj_t \Big)^{j_r l_s (r,s)}.$$

Proof. Cf. Theorems 3.2, 4.2, and 5.2.

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THEOREM 6.3. *The number of classes of finite automata* $S = \langle S, f, g\rangle$ *with* $S = \{s_0, \ldots, s_{n-1}\}$ and $\mathfrak{S}_k$ *operating on* Σ_k *with* $\mathfrak{S}_p$ *operating on* Π_p *is given by*

$$\frac{1}{n!}\frac{1}{k!}\frac{1}{p!}\sum_{(j)}\sum_{(l)}\sum_{(m)}\frac{n!}{\prod_{i=1}^{n} j_i!\, i^{j_i}}\,\frac{k!}{\prod_{i=1}^{k} l_i!\, i^{l_i}}\,\frac{p!}{\prod_{i=1}^{p} m_i!\, i^{m_i}}$$
$$\times \prod_{r=1}^{n}\prod_{s=1}^{k}\left\{\left(\sum_{t\mid\langle r,s\rangle} tj_t\right)\left(\sum_{d\mid\langle r,s\rangle} dm_d\right)\right\}^{j_r l_s(r,s)}.$$

Proof. Cf. Theorems 3.3, 4.3, and 5.2.

One can obtain a lower bound on the number of automata from the theorems of this section.

COROLLARY 6.4. *The number of classes of non-isomorphic automata with* n *internal states over input alphabet* Σ_k *and output alphabet* Π_p *is not smaller than*

$$\frac{1}{n!}(pn)^{kn}.$$

COROLLARY 6.5. *The number of equivalence classes of* n*-state automata with* $\mathfrak{S}_k$ *on* Σ_k *is not smaller than*

$$\frac{1}{n!}\frac{1}{k!}(pn)^{kn}.$$

COROLLARY 6.6. *The number of equivalence classes of* n*-state automata with* $\mathfrak{S}_k$ *on* Σ_k *and* $\mathfrak{S}_p$ *on* Π_p *is not less than*

$$\frac{1}{n!}\frac{1}{k!}\frac{1}{p!}(pn)^{kn}.$$

Proofs. In every case, replace the sum by the first term.

It will now be shown that as the number of states increases without limit, the number of equivalence classes of automata approaches the lower bounds of the previous three theorems.

THEOREM 6.7. *The number of classes of non-isomorphic automata is asymptotic to*

$$(np)^{kn}/n!$$

as n *increases without bound.*

THEOREM 6.8. *The number of classes of finite automata with* $\mathfrak{S}_k$ *acting on* Σ_k *is asymptotic to*

$$(np)^{kn}/(n!\,k!)$$

as n *increases without limit.*

THEOREM 6.9. *The number of classes of finite automata with* $\mathfrak{S}_k$ *acting on* Σ_k *and* $\mathfrak{S}_p$ *on* Π_p *is asymptotic to*

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$$(np)^{kn}/(n!\,k!\,p!)$$

as n increases without limit.

Proof. The number of classes is given by

$$\frac{1}{n!\,k!\,p!}\left((pn)^{kn} + \theta\right).$$

The method of proof is to bound θ from above and show that

$$\lim_{n\to\infty} \frac{\theta}{(pn)^{kn}} = 0.$$

An upper bound for θ comes from replacing every term of the polynomial expansion of Theorem 6.3 by the largest non-identity term. This occurs when

$$\begin{aligned} j_1 &= n-2, & j_2 &= 1,\\ l_1 &= k-2, & l_2 &= 1,\\ m_1 &= p-2, & m_2 &= 1. \end{aligned}$$

Thus

$$\theta \leqslant (n!\,k!\,p! - 1)((n-2)(p-2))^{(n-2)(k-2)}(np)^{n-2}(np)^{k-2}(np)^2,$$
$$\theta < n!\,k!\,p!\,((n-2)(p-2))^{(n-2)(k-2)}(np)^{n+k-2} < (np)^{(n-2)(k-2)+n+k-2},$$
$$\theta < n!\,k!\,p!\,(np)^{nk-k-n+2}.$$

Therefore,

$$\rho(n,k,p) = \frac{\theta}{(np)^{kn}} < \frac{n!\,k!\,p!}{(np)^{k+n-2}} \sim \frac{\sqrt{2\pi}\,k!\,p!\,e^{-n}}{p^{k+n-2}n^{k-5/2}}.$$

Finally

$$\lim_{n\to\infty} \rho(n,k,p) = 0.$$

7. Connected machines. For many purposes, it is convenient to consider machines which are connected. We enumerate the connected machines in this section.

Definition 7.1. An automaton is *connected* if and only if its (undirected) graph is connected.

Let $a_{n,k,p}$ be the number of equivalence classes of automata with n internal states defined over Σ_k and Π_p. Let $A(x,y,z)$ be the generating function for $a_{n,k,p}$, that is

$$A(x,y,z) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty} a_{n,k,p}\,x^n y^k z^p.$$

Let $C(x,y,z)$ be the corresponding series for the number of connected automata.

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THEOREM 7.2. *The generating function $C(x, y, z)$ for the number of classes of connected automata is obtained from the generating function $A(x, y, z)$ of the total number of classes of automata by the relation*

$$\log(1 + A(x, y, z)) = \sum_{i=1}^{\infty} \frac{1}{i} C(x^i, y^i, z^i).$$

Proof. This theorem is a restatement of Pólya's result on the number of connected graphs (**7**, p. 457).

Using Theorem 7.2 and Theorems 6.7–6.9, it is a simple matter to show that the number of classes of connected automata is asymptotic to the limits given in Theorems 6.7–6.9. The following result follows from this discussion.

THEOREM 7.3. *Almost all automata are connected.*

8. Numerical results. Numerical calculations are presented for modest values of n, k, and p. Since it is easy to prove that binary inputs are completely general, the enumeration is restricted to the case $k = 2$.

In Table I, the number of non-isomorphic binary machines without outputs is given, along with the number of connected non-isomorphic machines. In Table II, input permutations are permitted.

TABLE I

THE NUMBER OF NON-ISOMORPHIC BINARY AUTOMATA

n	$a_{n,2,1}$	$c_{n.2,1}$
1	1	1
2	10	9
3	129	119

TABLE II

THE NUMBER OF NON-ISOMORPHIC BINARY AUTOMATA UNDER INPUT PERMUTATIONS

n	$a_{n,2,1}$	$c_{n,2,1}$
1	1	1
2	7	6
3	74	67

In Figure 3, the 10 classes of automata are shown. It is clear that there is only one disconnected machine. Furthermore, input permutation causes i and ii, vi and vii, viii and ix, to become equivalent.

https://doi.org/10.4153/CJM-1965-010-9 Published online by Cambridge University Press

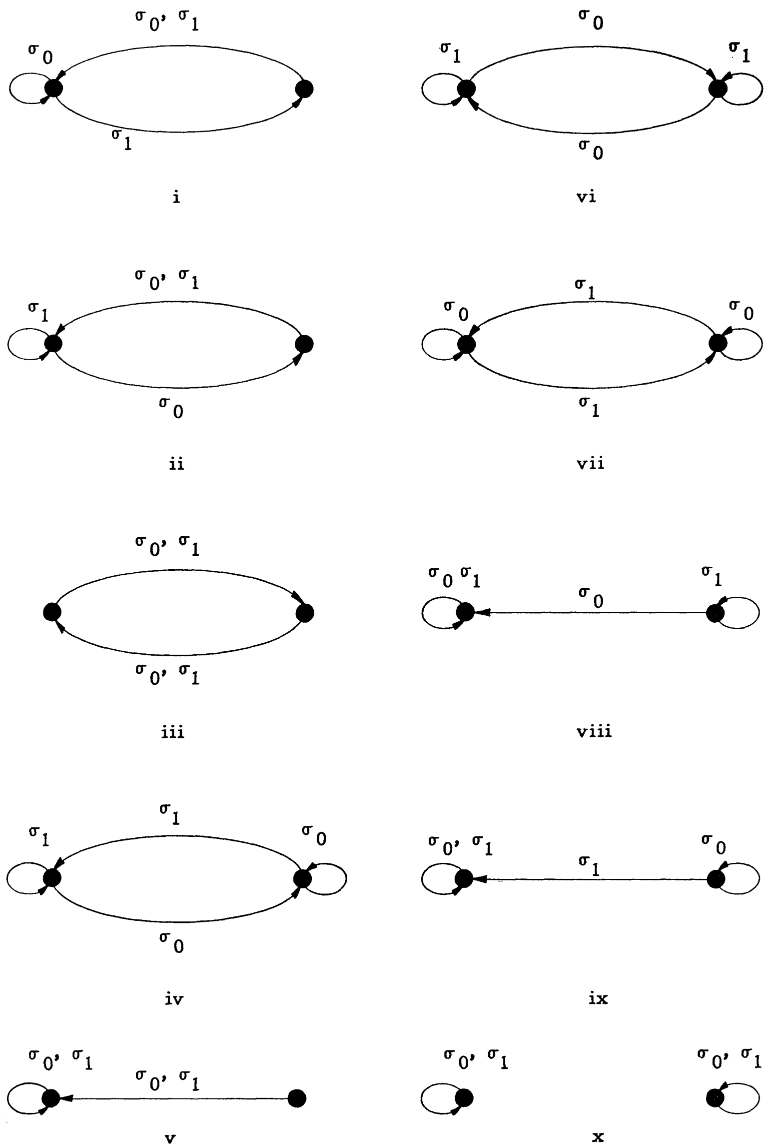

FIGURE 3. The non-isomorphic binary automata without output for $n = 2$.

The remaining calculations are presented in Tables III through V.

TABLE III
THE NUMBER OF NON-ISOMORPHIC AUTOMATA WITH BINARY INPUT AND OUTPUT

n	$a_{n,2,2}$	$c_{n,2,2}$
1	4	4
2	136	126
3	7860	7336

TABLE IV
THE NUMBER OF EQUIVALENCE CLASSES OF AUTOMATA WITH $\mathfrak{S}_2$ ON Σ_2 AND BINARY OUTPUT

n	$a_{n,2,2}$	$c_{n,2,2}$
1	3	3
2	76	70
3	4003	3783

TABLE V
THE NUMBER OF EQUIVALENCE CLASSES OF AUTOMATA WITH $\mathfrak{S}_2$ ON Σ_2 AND $\mathfrak{S}_2$ ON Π_2

n	$a_{n,2,2}$	$c_{n,2,2}$
1	2	2
2	44	41
3	2038	1952

9. Comments and unsolved problems. Loosely speaking, an automaton $S = \langle S, f, g \rangle$ is said to be *strongly connected* if for every pair of states $s, t \in S$ there exists a sequence of inputs which causes S to leave state s and go into state t. Thus, strongly connected machines correspond to strongly connected digraphs. The general enumeration problem for strongly connected digraphs (automata) is still unsolved.

In the present context, the automata did not have a designated initial state. The enumeration problem in this case has been studied by Vyssotsky in unpublished work. The rooted case follows from our results by employing the same device used in enumerating rooted graphs, i.e. the number of classes of rooted n-state machines is given by the expression for the $(n-1)$-state machines with j_1 replaced by $j_1 + 1$. The details are in **(7)**.

Acknowledgment. I wish to thank P. G. Neumann for confirming my numerical results by reference to his tabulation of classes of machines and O. Ibarra for checking my calculations.

Added in Proof: C. Radke has recently enumerated the strongly connected digraphs, thus disposing of a problem mentioned in Section 9. Another unsolved problem is the determination of the number of minimal automata with n states.

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