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# CONTINUOUS ERGODIC EXTENSIONS AND FIBRE BUNDLES

ROBERT J. ZIMMER

**1. Introduction.** If a locally compact group $G$ acts as a measure preserving transformation group on a Lebesgue space $X$, then there is a naturally induced unitary representation of $G$ on $L^2(X)$, and one can study the action on $X$ by means of this representation. The situation in which the representation has discrete spectrum (i.e., is the direct sum of finite dimensional representations) and the action is ergodic was examined by von Neumann and Halmos when $G$ is the integers or the real line [**7**], and by Mackey for general non-abelian $G$ [**10**]. This theory was generalized to the case of extensions of ergodic actions by the author in [**11**] and [**12**]. There we consider a pair of ergodic $G$-spaces $X$ and $Y$ with an equivariant measure preserving map between them. Then $L^2(X)$ becomes a $G$-Hilbert bundle over $Y$, and we say that $X$ has *relatively discrete spectrum over* $Y$ if $L^2(X)$ is a direct sum of finite dimensional $G$-invariant subbundles. Theorems 4.3, 6.2, 6.4 of [**11**] are the generalization to extensions of the main von Neumann-Halmos-Mackey theorems, and imply the latter when $Y$ is taken to be a point.

In all these considerations, one is dealing solely with measure spaces and Borel isomorphisms. If, in addition, the spaces involved have a topological structure and the finite dimensional $G$-invariant subspaces (or subbundles) consist of continuous functions, it is natural to inquire as to what extent the measure-theoretic theory will actually hold in a topological category. For a single $G$-space $X$, this situation is essentially understood, and it is the aim of this paper to examine this question for extensions. Our main result will be a topological version of the structure theorem [**11**, Theorem 4.3] of the measure theoretic theory. The latter maintains that any extension with relatively discrete spectrum is (Borel) isomorphic to a type of skew product action in a product space. In the topological category, one would like to conclude that an extension should be (topologically) isomorphic to a certain type of action in a fibre bundle, which we call a homogeneous extension. (See Section 2 for detailed definitions.) This however is not true, but we do show, at least in the case in which $Y$ is minimal under $G$, that the extensions in question are inverse limits of homogeneous extensions. Furthermore, we identify a certain class of extensions, which we call finitely generated, that we do show to be homogeneous.

Continuous extensions of the type to be considered in this paper have arisen in the study of minimal distal transformation groups and extensions on compact

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spaces. Based on the work of Ellis and Furstenberg, Knapp [**9**] showed that the isometric extensions introduced by Furstenberg [**4**] actually have the function theoretic properties that we are considering. As a consequence, we are able to conclude a strengthened version of Furstenberg's structure theorem for minimal distal actions on compact metric space [**4**], showing that every such action is built up from a point by taking homogeneous extensions and inverse limits. The main theorems of this paper were in fact announced in this context of distal actions in [**13**]. It has been pointed out to the author, however, that a short proof in this context is available through the machinery that has been built up around minimal actions on compact spaces. (See [**3**], for example.) We remark that the techniques and results of this paper are dependent on neither compactness nor minimality of the action on $X$.

The organization of this paper is as follows. Section 2 presents the definitions of the various types of extensions we will be considering-homogeneous, finitely generated, and those possessing "topological" relatively discrete spectrum. It also includes some preliminary results and elementary properties of these extensions. Section 3, which is the main part of the paper, is devoted to a proof that finitely generated extensions with a minimal base are homogeneous. Section 4 concludes with applications and examples.

**2. Preliminaries.** Suppose $G$ is a locally compact group and $X$ and $Y$ locally compact metrizable spaces. (We shall throughout take local compactness to include the condition of second countability.) We suppose that there is a jointly continuous right $G$-action on both $X$ and $Y$, and that $p : X \to Y$ is continuous, surjective, and equivariant. We shall further suppose that $G$ preserves a probability measure $\mu(\nu)$ on $X(Y)$ whose support is all of $X(Y)$, and that $p$ is measure preserving, i.e. $p_*(\mu) = \nu$. This is just the situation studied in [**11**], now with the additional assumptions of topological structure. We can decompose $\mu$ with respect to $\nu$ over the fibers of $p$ and write $\mu = \int^{\oplus} \mu_y d\nu(y)$. This is a measure-theoretic construction and in the topological framework we will need the stronger condition that $y \to \mu_y$ is continuous, in the sense that for any $f \in C_0(X)$ (the continuous functions on $X$ that vanish at $\infty$), $y \to \int f d\mu_y$ is continuous, and that the support of $\mu_y$ is $p^{-1}(y)$. We shall then call $\mu_y$ a *continuous decomposition of* $\mu$. Such a decomposition does not always exist. It will exist, for example, if $X$ is a fibre bundle over $Y$ whose structure group consists of measure preserving homeomorphisms of the fibre. If $X$ is compact, minimal, and an isometric extension of $Y$, such a decomposition will also exist. In the compact case, continuous decompositions have been investigated in some detail by Glasner [**5**].

*Definition* 2.1. Under the above assumptions, we will call $X$ a *continuous ergodic extension of* $Y$.

If $f \in C_0(X)$, and $y \in Y$, by $f^y$ we shall mean $f|p^{-1}(y)$. By virtue of the fact that $y \to \mu_y$ is continuous, we have the following readily verified facts, which we shall use below without further explicit mention.

PROPOSITION 2.2. (i) *If* $f, g \in C_0(X)$, *then* $y \to \langle f^y | g^y \rangle_y$ *is continuous. Here* $\langle \, | \, \rangle_y$ *is the inner product on* $L^2(p^{-1}(y), \mu_y)$.

ii) *If* $f_1, \ldots, f_n \in C_0(X)$ *and* $\{f_i^y\}$ *are linearly independent, then* $\{f_i^z\}$ *are linearly independent for all* $z$ *in some neighborhood of* $y$. (This follows since the latter will be linearly independent if and only if the determinant of the matrix $(\langle f_i^z | f_j^z \rangle_z)$ is not 0.)

iii) *If* $f_1^y, \ldots, f_n^y$ *are linearly independent, then there are* $g_1, \ldots, g_n \in C_0(X)$ *such that* (a) $\{g_i^z\}$ *are orthonormal in some neighborhood of* $y$. (b) span $[g_1^z, \ldots, g_n^z] =$ span $[f_1^z, \ldots, f_n^z]$ *in some neighborhood of* $y$.

(c) span $[g_1^z, \ldots, g_n^z] \subset$ span $[f_1^z, \ldots, f_n^z]$ *for all* $z \in Y$. (This can be achieved by restricting to a suitable neighborhood of $y$, applying the Gram-Schmidt process fibre-by-fibre, and then extending by multiplying by a suitable function in $C_0(Y)$. See [**9**, Lemma 6.2] for details of the technique.)

iv) *If* $\{g_i^z\}_{i=1,\ldots,n}$ *are orthonormal for* $z \in U \subset Y$, *and* $f$ *is a bounded function on* $p^{-}(U)$ *such that* $f^z \in$ span $\{g_i^z\}$, *then* $f$ *is continuous on* $p^{-1}(U)$ *if and only if* $y \to \langle f^y | g_i^y \rangle_y$ *is continuous on* $U$ *for each* $i$.

Suppose now that $X$ has relatively discrete spectrum over $Y$ [**11**, Definition 5.1]. This means that $L^2(X) = \sum^{\oplus} V_i$ where $V_i = \int^{\oplus} V_i^y d\nu$ are $G$-invariant and $\dim V_i^y < \infty$. This again is a purely measure-theoretic notion, and our main object of study will be the topological version of this notion, which we now describe. We note first that $C_0(p^{-1}\Omega)$ is a $C_0(\Omega)$-module for $\Omega \subset Y$. If $V \subset C_0(X)$ is a subspace, we will call $V$ *regular on* $\Omega$ if

(i) $V$ is closed under multiplication by $C_0(\Omega)$, and

(ii) there are $f_1, \ldots, f_n \in V$ such that $\{f_i^y\}$ is a basis of $V^y = \{f^y | f \in V\}$ for each $y \in \Omega$.

We will then call $\{f_i\}$ a *local basis for* $V$ *on* $\Omega$. If $f_i^y$ are orthonormal we will call $\{f_i\}$ a *local orthonormal basis*. We will call $V$ *regular* if it is regular on each set of an open covering of $Y$. We remark that by the technique described in Proposition 2.2 (iii), if $V$ is regular we can assume $\{f_i\}$ is a local orthonormal basis.

LEMMA 2.3. *If* $V$ *is regular and* $f \in C_0(X)$, *then* $P_1 f$ *and* $P_2 f$ *are in* $C(X)$, *where* $P_1$ *and* $P_2$ *are orthogonal projections of* $L^2(X)$ *onto* $\bar{V}$ *and* $V^{\perp}$, *respectively*.

*Proof.* $P_1 = \int^{\oplus} P_1^y$ where $P_1^y : L^2(p^{-1}(y), \mu_y) \to V^y$, and on a suitable open set, $P_1 f = \sum \langle f | f_i^y \rangle f_i^y$ where $\{f_i\}$ is a local orthonormal basis. We then also have $P_2 f = f - P_1 f$.

*Definition* 2.4. If $p : X \to Y$ is a continuous ergodic extension, then $X$ *has proper relatively discrete spectrum over* $Y$ if there are regular $G$-invariant subspaces $V_i \subset C_0(X)$ such that $\sum V_i$ is uniformly dense in $C_0(X)$.

We remark that if $X$ has proper relatively discrete spectrum over $Y$, then $X$ has relatively discrete spectrum over $Y$. We also note that in light of Lemma 2.3, we can choose the $V_i$ to be mutually orthogonal. Our aim is to describe the structure of extensions with proper relatively discrete spectrum in terms of locally trivial extensions.

*Definition* 2.5. If $X$ is a continuous ergodic extension of $Y$, then $X$ is a *homogeneous extension of* $Y$ if

(i) $X$ is a fibre bundle over $Y$, with fibre $H/H_0$, and structure group $H$, where $H$ is a compact metric group and $H_0 \subset H$ is a closed subgroup;

(ii) the projection $p : X \to Y$ intertwines the $G$-actions;

(iii) for each $y_0 \in Y$, there is an open set $\Omega \subset Y$, with $y_0 \in \Omega$ and an admissible homeomorphism of $p^{-1}(\Omega)$ with $\Omega \times H/H_0$ such that $y, z \in \Omega$ with $y \cdot g = z$ implies: there is $\alpha(y, g) \in H$ such that the action of $g$ taking $p^{-1}(y)$ to $p^{-1}(z)$ corresponds to the map $\{y\} \times H/H_0 \to \{z\} \times H/H_0$ defined by translation by $\alpha(y, g)$.

We show below that every extension with proper relatively discrete spectrum is the inverse limit of homogeneous extensions (if $y$ is minimal).

We now derive some first consequences of definition 2.4 and the results of [**11**].

LEMMA 2.6. *Suppose $V_i$ are in Definition* 2.4 *and are mutually orthogonal. Let $R(y)$ be the algebraic direct sum $\sum V_i^y$. Then if $f \in C_0(X)$ is contained in a $G$-invariant regular subspace $W$ of $C_0(X)$, then $f^y \in R(y)$ for all $y$.*

*Proof*. Let $P_i = \int^{\oplus} P_i^y$ be orthogonal projection of $L^2(X)$ onto $\bar{V}_i$. From [**11**, Theorems 3.14 and 4.3] each equivalence class of $G$-Hilbert bundles contained in $L^2(X)$ appears with only finite multiplicity. It follows that $P_i f = 0$ in $L^2$ for all but finitely many $i$. Thus, for some $n$ and almost all $y$, $f^y \in \sum_{i=1}^n V_i^y$. But then the continuity of $f$ implies that this holds for all $y$, proving the lemma.

COROLLARY 2.7. *$R(y)$ is an algebra for all $y \in Y$.*

We remark that if $t \in G$ and $y, z \in Y$ with $yt = z$, then $t$ defined a homeomorphism of $p^{-1}(y)$ with $p^{-1}(z)$. Furthermore, this map takes $\mu_y$ to $\mu_z$. (This follows from the $G$-invariance of $\mu$, the essential uniqueness of decompositions of measures, and the continuity of $y \to \mu_y$.) Thus, there is an induced unitary map $t^* : C_0(p^{-1}(z)) \to C_0(p^{-1}(y))$ and this map is an algebra isomorphism of $R(z)$ with $R(y)$.

PROPOSITION 2.8. *If $Y$ is minimal, then $p$ is a proper map, i.e. $p^{-1}(C)$ is compact if $C$ is compact.*

*Proof*. We can consider $L^2(Y) \subset L^2(X)$ as a $G$-invariant sub-Hilbert bundle. Because of the ergodicity of $G$ on $X$, this bundle will be inequivalent to any other $G$-invariant subbundle of $L^2(X)$ [**11**, Theorems 3.14, 4.3], and it follows that for some $i$, $\bar{V}_i = L^2(Y)$, where $\bar{V}_i$ is the $L^2$ closure. In particular, there is an $f \in C_0(X)$, $f \neq 0$, which is constant almost everywhere on almost all fibres. Since $f$ is continuous, $f$ must be constant on almost all fibres. Now

$$x \to f(x) - \int f^{p(x)} d\mu_{p(x)}$$

is continuous, and since $f$ is constant on almost all fibres, this must be 0 almost everywhere. Hence, it is identically 0 by continuity and this implies $f$ is constant on all fibres. Since $f \neq 0$, there is a compact set $U \subset Y$ with non-empty interior such that $f(p(x)) \neq 0$ for $x \in p^{-1}(U)$. Since $f \in C_0(X)$, $p^{-1}(U)$ must be compact. Because $p$ is $G$-invariant and $G$ acts minimally on $Y$, it follows readily that $p$ is proper, completing the proof.

Since $X$ has relatively discrete spectrum over $Y$, the structure theorem [**11**, Theorem 4.3] shows that $X$ is Borel isomorphic (modulo null sets) as an extension of $Y$ to $Y \times {}_{\alpha}K/K_0$ where $K$ is a compact group, $K_0 \subset K$ is a closed subgroup, and $\alpha : Y \times G \to K$ is a minimal cocycle with $K_\alpha = K$[**11**, Definition 3.7]. Call $S = K/K_0$ and let $R(S)$ be the set of continuous functions on $S$ that are contained in finite dimensional $K$-invariant subspaces. Under the Borel isomorphism of $X$ with $Y \times S$, we will have $R(y)$ corresponding to $R(S)$ for almost all $y$, and hence, for almost all $y$, $R(y) = R(S)$ as algebras via a unitary (as a map of $L^2$ spaces) algebra isomorphism.

We now introduce the concept of a finitely generated continuous ergodic extension, which will facilitate the discussion and is of independent interest.

*Definition* 2.9. $X$ is a *finitely generated extension of* $Y$ if $R(y)$ is a finitely generated algebra for almost all $y$. Equivalently, by the remarks above, $R(S)$ is a finitely generated algebra. (We will also call $S$ a *finitely generated homogeneous space.*)

We will show in Section 3 that every finitely generated extension, with $Y$ minimal, is a homogeneous extension.

We wish now to remove the almost everywhere condition in Definition 2.9. This can be conveniently done after the introduction of some technical concepts and notation of which we will in fact make constant use throughout this paper. As a homogeneous $K$-space, $S$ has a unique $K$-invariant probability measure and we have a natural unitary representation of $K$, say $U$, on $L^2(S)$ defined by translation. For each $\pi \in \hat{K}$ (the set of equivalence classes of irreducible representations of $K$), define $H_\pi \subset L^2(S)$ by

$$H_\pi = \mathrm{sp}\{f \in L^2(S) | f \text{ is contained in a subspace } V \subset L^2(S) \text{ for which } U|V \cong \pi\}.$$

Then each $H_\pi$ is finite dimensional, and $H_\pi \subset C(S)$, the continuous complex-valued functions on $S$. We shall call $H_\pi$ the *canonical subspace of* $C(S)$ (or $L^2(S)$) *associated with* $\pi$. In $L^2(S)$, $\{H_\pi\}$ are mutually orthogonal and

$L^2(S) = \sum^{\oplus} H_\pi$. Then $R(S)$ is the algebraic direct sum $\sum H_\pi$, and is a uniformly dense *-subalgebra of $C(S)$. For $S$ finitely generated, it will be convenient to single out certain sets of generators.

*Definition* 2.10. A finite set of generators for $R(S)$, $\{f_1, \ldots, f_n\}$, will be called *proper* if

i) $f_i \perp 1$ for all $i$,

ii) each $f_i$ is in a canonical subspace, and

iii) if for $\pi \in \hat{K}$, there is some $f_i \in H_\pi$, then $H_\pi \subset \operatorname{span}\{f_i\}$.

PROPOSITION 2.11. (i) *If $\{f_i\}$ is a proper set of generators and $\{g_j\}$ is a finite set in $R(S)$ such that*

(a) *each $g_j$ is in a canonical subspace, and*

(b) $\operatorname{span}\{g_j\} = \operatorname{span}\{f_i\}$,

*then $\{g_j\}$ is a proper generating set.*

(ii) *Any finitely generated $S$ has an orthonormal proper generating set.*

Let $f_1, \ldots, f_n$ be an orthonormal proper set of generators of $R(S)$ and $L_0 \subset \hat{K}$ the finite set determined by

$$\operatorname{span}\{1, f_1, \ldots f_n\} = \sum_{\pi \in L_0}^{\oplus} H_\pi, \quad \text{and } H_\pi \neq \{0\} \text{ for } \pi \in L_0.$$

Let $H_0 = \sum_{\pi \in L_0}^{\oplus} H_\pi$. Let $C[X_1, \ldots, X_n]$ be the polynomial ring with complex coeficients. There is a unique surjective algebra homomorphism $\Phi : C[X_1, \ldots, X_n] \to R(S)$ such that $\Phi(X_i) = f_i$. Let $I = \ker \Phi$. Since $C[X_1, \ldots, X_n]$ is noetherian, $I$ is a finitely generated ideal. Choose generators $p_1, \ldots, p_k$ for $I$ and let

$$d = \max_{j=1,\ldots k} \{\text{degree}(p_j)\}.$$

Let $P_d$ be the set of polynomials in $C[X_1, \ldots, X_n]$ of degree $\leqq d$. Then $\Phi(P_d)$ consists of polynomials of degree $\leqq d$ in the $\{f_i\}$, and since $\{f_i\}$ is a proper set of generators, $\Phi(P_d)$ will be a finite dimensional $G$-invariant subspace. Hence, there is a smallest finite dimensional subspace $H_1$ of $R(S)$ such that $H_1 \supset \Phi(P_d)$ and $H_1$ is a direct sum of canonical subspaces, say $H_1 = \sum_{\pi \in L_1}^{\oplus} H_\pi$, where $L_1 \subset \hat{K}$ is a finite set. Let $H_2$ be the smallest subspace containing all products $fg$, $f$, $g \in H_1$, and which is also a direct sum of canonical subspaces. Thus, $H_2 = \sum_{\pi \in L_2}^{\oplus} H_\pi$, where $L_2 \subset \hat{K}$ is finite.

For each subspace $V_i \subset C_0(X)$, we can associate a unique element $\pi$ of $\hat{K}$. This representation has the property that as $G$-Hilbert bundles, $\bar{V}_i$ is equivalent to a subbundle of $L^2(Y; H_\pi)$, where the action on the latter is determined as restriction of the action of $G$ on $L^2(Y \times {}_\alpha S)$. Let $\bar{H}_\pi \subset C_0(X)$ be the direct sum of the (finitely many [**11**]) $V_i$ associated with $\pi$, and $H_\pi(y)$ the direct sum of the corresponding $V_i^y$. We will call $H_\pi(y)$ the *canonical subspace of $R(y)$ associated with $\pi$*. Similarly, we can form $\bar{H}_i$ and $H_i(y)$, $i = 0, 1, 2$. We remark that under the isomorphism of $R(y)$ with $R(S)$ (almost all $y$) described following the proof of Proposition 2.8, we actually have $R(y) \cong R(S)$ under a unitary algebra isomorphism that preserves canonical subspaces. In particular, for almost all $y$, $H_1(y) \cdot H_1(y) \subset H_2(y)$, and a continuity argument implies that this relation holds for all $y$. Similarly, any polynomial of degree $\leqq d$ in a basis for $H_0(y)$ will be contained in $H_1(y)$.

PROPOSITION 2.12. *If $X$ is a finitely generated extension of $Y$ with $Y$ minimal, then $R(y)$ is finitely generated for all $y$. In fact, $H_0(y)$ generates $R(y)$ for all $y$.*

*Proof.* Let $\pi \in \hat{K}$. Then there is an integer $r$ such that the space of all polynomials of degree $\leqq r$ in some basis of $H_0$ contains $H_\pi$. We remark that this is independent of the choice of basis of $H_0$, since one basis is obtained from another by linear relations. Let $Q_1, \ldots, Q_k$ be a basis of the set of polynomials over $\mathbf{C}$ in $n$ variables, of degree $\leqq r$. Choose functions $a_1, \ldots, a_n$ in $C_0(X)$ that form a local basis for $\bar{H}_0$ on some open set $\Omega \subset Y$. Let $q_i = Q_i(a_1, \ldots, a_n)$. Then $\{q_i^y\}$ will span a space containing $H_\pi(y)$ for almost all $y \in \Omega$. This follows from the fact that $R(y) \cong R(S)$ as algebras for almost all $y$ by a canonical subspace preserving map. Let $n(y)$ be the dimension of the space spanned by $\{q_i^y\}$, $m = \max\{n(y) | y \in \Omega)$ and choose $z \in \Omega$ such that $n(z) = m$. Choose a subset of $\{q_i\}$ (which by relabeling we will write $q_1, \ldots, q_m$) so that $q_1^z, \ldots, q_m^z$ are linearly independent. Then $q_1^y, \ldots, q_m^y$ will be linearly independent in a neighborhood of $z$, and hence, by the choice of $m$, span $[q_1^y, \ldots, q_m^y]$ = span $[q_1^y, \ldots, q_k^y]$ for all $y$ in an open set. By a Gram-Schmidt argument, we can find $p_1, \ldots, p_m \in C_0(X)$ such that $p_1^y, \ldots, p_m^y$ are an orthonormal basis for span $[q_1^y, \ldots, q_m^y]$ for all $y$ in an open set $\Omega'$. Now let $f \in \bar{H}_\pi$. Then, as remarked above, $f^y \in$ span $[q_1^y, \ldots, q_k^y]$ for almost all $y$ in $\Omega'$. Now for $y \in \Omega'$, $f^y \in$ span $[q_1^y, \ldots, q_k^y]$ if and only if $||f^y||^2 = \sum_j |\langle f^y | p_j^y \rangle|^2$. Since both sides of this last equation are continuous in $y$, and they are equal a.e., they must be equal on all of $\Omega'$. Thus, $f^y$ is a polynomial in $a_1^y, \ldots, a_n^y$ for all $y \in \Omega'$, which implies that $H_\pi(y)$ is in the algebra generated by $H_0(y)$ for all $y \in \Omega'$. Now for any $g \in G$ and $y \in Y$, the action of $G$ gives an algebra isomorphism of $R(yg)$ with $R(y)$ preserving canonical subspaces. By the minimality of $G$ on $Y$, it follows that $H_0(y)$ generates $H_\pi(y)$ for all $y$. Since $\pi$ is arbitrary, the proposition follows.

Before turning to the main results of this paper, we present two preliminary lemmas that we shall need below.

LEMMA 2.13. *Suppose $S$ and $T$ are compact metric spaces. Let $A(S)$, $A(T)$ be dense subalgebras of $C(S)$ and $C(T)$ respectively. Suppose $\mu$, $\nu$ are finite measures on $S$, $T$ respectively, both of which are positive on open sets. Let $W : L^2(S, \mu) \to L^2(T, \nu)$ be unitary, such that*

(i) $W(A(S)) = A(T)$

(ii) *if $f, g \in A(S)$, then $W(f \cdot g) = W(f)W(g)$.*

*Then $W|C(S)$ is a multiplicative, involutive banach algebra isomorphism between the $C^*$-algebras $C(S)$ and $C(T)$.*

*Proof.* (i) Suppose $f \in L^\infty(S)$ and $g \in A(S)$. Then there is a sequence

$f_n \in A(S)$ such that $f_n \to f$ in $L^2(S)$. Thus, $f_n \cdot g \to f \cdot g$ in $L^2(S)$ so $W(f_n g) \to W(f \cdot g)$. But $W(f_n \cdot g) = W(f_n) \cdot W(g) \to W(f)W(g)$. Hence, $W(f \cdot g) = W(f)W(g)$.

(ii) Now suppose $f \in L^\infty(S)$ and $g \in L^2(S)$. Then there is a sequence $g_n \in A(S)$ with $g_n \to g$ in $L^2(S)$. It follows that $f \cdot g_n \to f \cdot g$ in $L^2(S)$ so $W(f \cdot g_n) \to W(f \cdot g)$. By (i), $W(f \cdot g_n) = W(f)W(g_n) \to W(f)W(g)$. Thus, for each $g$, $W(f)W(g) \in L^2(T)$ and $W(f)W(g) = W(f \cdot g)$. Hence, multiplication by $W(f)$ maps $L^2(T) \to L^2(T)$ continuously, and it follows that $W(L^\infty(S)) \subset L^\infty(T)$. By applying the same technique to $W^{-1}$, we see that $W(L^\infty(S)) = L^\infty(T)$. By the argument of [**6**, p. 45], $W$ is induced by a measure preserving map $T \to S$, and in particular, $W|L^\infty(S)$ preserves $|| \ ||_\infty$. Since open sets in $S$ and $T$ have positive measure, $W|C(S)$ will preserve the sup $|| \ ||$. As $A(S)$ $(A(T))$ is uniformly dense in $C(S)$ $(C(T))$ it follows from hypothesis (i) that $W : C(S) \to C(T)$ is an isomorphism.

LEMMA 2.14. *Let $S$ be a compact metric space, $\mu$ a measure on $S$ that is positive on open sets. Suppose $A_i \subset C(S)$, $i = 1, 2, \ldots$, such that*

(i) *each $A_i$ is finite dimensional, and $\{A_i\}$ are mutually orthogonal, and*

(ii) *$A$, the algebraic direct sum of $\{A_i\}$, is a dense subalgebra of $C(S)$. Let $K$ be a closed (strong operator topology) subgroup of $U(L^2(S))$ such that for all $U \in K$, $U(A_i) = A_i$ for all $i$, and $U(f \cdot g) = U(f)U(g)$ for all $f, g \in A$.*

*Then the action of $K$ on $C(S)$ is induced by a jointly continuous action of $K$ on $S$.*

*Proof.* By Lemma 2.7, $U : C(S) \to C(S)$ is an isomorphism of $C^*$-algebras for each $U \in K$. As $S$ can be characterized as the maximal ideal space of $C(S)$, it is easy to see that it suffices to show that $K \times C(S) \to C(S)$ is jointly continuous. If $U_n \in K$, $U_n \to U$, and $f_n \in C(S)$, $f_n \to f$, then

$$||U_n f_n - Uf|| \leqq ||U_n f_n - U_n f|| + ||U_n f - Uf|| \leqq ||f_n - f|| + ||U_n f - Uf||.$$

Thus, it suffices to see that $||U_n f - Uf|| \to 0$. If $f \in A$, this follows since on a finite dimensional subspace of $C(S)$, $|| \ ||_\infty$ and $|| \ ||_2$ are equivalent. If $f \notin A$, given $\epsilon > 0$, there is $g \in A$ with $||f - g|| < \epsilon$. Then

$$\begin{aligned} ||U_n f - Uf|| &\leqq ||U_n f - U_n g|| + ||U_n g - Ug|| + ||Ug - Uf|| \\ &\leqq 2\epsilon + ||U_n g - Ug||. \end{aligned}$$

The result follows.

**3. Finitely generated extensions.** This section is devoted to a proof of the following theorem.

THEOREM 3.1. *If $X$ is a continuous ergodic extension of $Y$ with proper relatively discrete spectrum, and $Y$ is minimal, then $X$ is a homogeneous extension of $Y$.*

*Proof.* Choose a point $y_0 \in Y$. The first main step in the proof of Theorem 3.1 is the following lemma.

LEMMA 3.2. *There is a compact neighborhood $\Omega$ of $y_0$ in $Y$, and subspaces of functions, $V_n \subset C(p^{-1}(\Omega))$, $n = 0, 1, 2, \ldots$, regular on $\Omega$, such that*

(i) $\{(V_n)_y | n = 0, 1, 2, \ldots\}$ *are mutually orthogonal and finite-dimensional for each* $y \in \Omega$;

(ii) *the algebraic direct sum* $\sum (V_n)_y = R(y)$ *for each* $y \in \Omega$;

(iii) $(V_0)_y = \mathbf{C}(\subset C(p^{-1}(y)))$ *for all* $y \in \Omega$;

(iv) $(V_1)_y$ *generates* $R(y)$ *as an algebra for all* $y \in \Omega$;

(v) *if* $W_n = \sum_{i=0}^{n} V_i$, *then* $W_n \cdot W_p \subset W_{n+p}$ *for all* $n, p \geqq 0$;

(vi) *if* $y, z \in \Omega$, *and* $t \in G$ *with* $y \cdot t = z$, *then* $t^*((V_n)_z) = (V_n)y$ *for all* $n$, *where* $t^* : C(p^{-1}(z)) \to C(p^{-1}(y))$ *is the induced map.*

*Proof.* For each $y \in Y$ recall that $H_j(y) = \sum_{\pi \in L_j}^{\oplus} H_\pi(y)$, for $j = 0, 1, 2$. Then there is a compact neighborhood $\Omega'$ of $y_0$ in $Y$ such that $\bar{H}_\pi$ is regular on $\Omega'$ for all $\pi \in L_2$. For $\pi \in L_2$, choose $g_1^\pi, \ldots, g_p^\pi \in \bar{H}_\pi$ to be a local orthonormal basis for $\bar{H}_\pi$ on $\Omega'$; so that $g_1^I = 1$, where $I$ is the 1-dimensional identity representation. Let $h_1^\pi, \ldots, h_p^\pi$ be an orthonormal basis of $H_\pi$ such that

(a) for $\pi = I$, $h_1^I = 1$, and

(b) for $\pi \in L_0 - \{I\}$, $\{h_i^\pi | i = 1, \ldots, p\} = \{f_i | f_i \in H_\pi\}$.

Define, for $y \in \Omega'$, $U_\pi(y) : H(y) \to H_\pi$ to be the unitary operator with $U_\pi(y)(g_i^\pi | p^{-1}(y)) = h_i^\pi$. Then $U(y) = \sum_{\pi \in L_2}^{\oplus} U_\pi(y) : H_2(y) \to H_2$ is unitary. For notational convenience, we shall let $\{g_1, \ldots, g_m\} = \bigcup_{\pi \in L_2} \{g_i^\pi\}$ and $\{h_1, \ldots, h_m\}$ the corresponding (under $U(y)$) members of $H_2$. We can choose the ordering such that $h_i = f_i$ for $i = 1, \ldots, n$.

The proof of Lemma 3.2 now breaks up into 3 steps. First, we will show how to "continuously modify" $U(y)$ so that it preserves multiplication of elements in $H_1(y)$. Step 2 will be to show that these new operators can be extended to algebra isomorphisms. Finally, we shall use the algebra isomorphisms to construct the required regular subspaces.

*Step* 1. For each $y \in \Omega'$, we have a bilinear map $B_y : H_1(y) \times H_1(y) \to H_2(y)$, given by $B_y(f, g) = f \cdot g$. This defines a bilinear map $\bar{B}_y : H_1 \times H_1 \to H_2$ by $\bar{B}_y(f, g) = U(y)(B_y(U(y)^{-1}f, U(y)^{-1}g))$.

Now $\mathrm{Bil}(H_1; H_2)$, the space of bilinear maps from $H_1 \times H_1 \to H_2$ has a natural topology on it, being a finite dimensional vector space. We claim the map $Y \to \mathrm{Bil}(H_1; H_2)$, $y \to \bar{B}_y$, is continuous. It suffices to show that if $h_i \in H_\pi$, $h_j \in H_\sigma$, where $\pi, \sigma \in L_1$, then the map $y \to \langle \bar{B}_y(h_i, h_j) | h_k \rangle$ is continuous. (Here, $\langle \, | \, \rangle$ denotes the inner product in $H_2$.) But

$$\langle \bar{B}_y(h_i, h_j) | h_k \rangle = \langle U(y) B_y(U(y)^{-1} h_i, U(y)^{-1} h_j | h_k \rangle = \langle g_i^y g_j^y | g_k^y \rangle_y,$$

which is continuous.

Now let $A$ be the set of unitary maps $U : H_2 \to H_2$ that take every canonical subspace into itself, and is the identity on $H_I$. There is continuous right action of $A$ on $\mathrm{Bil}(H_1; H_2)$ defined by

$$(B \cdot U)(f, g) = U^{-1} B(Uf, Ug).$$

Let $B_0 \in \mathrm{Bil}(H_1; H_2)$ correspond to multiplication in $R(S)$. For almost all $y \in \Omega'$, there is an isomorphism of algebras $i(y) : R(S) \to R(y)$ such that $i_y|H_\pi : H_\pi \to H_\pi(y)$ unitarily. Thus, $U(y)i(y)|H_2$ preserves canonical subspaces, is the identity on $H_I$, and so $(U(y)i(y)|H_2) \in A$. Furthermore,

$$B_0 \cdot i(y)^{-1}U(y)^{-1} = \bar{B}_y.$$

To see this, note that

$$(B_0 \cdot i(y)^{-1}U(y)^{-1})(f, g) = U(y)i(y)B_0(i(y)^{-1}U(y)^{-1}f, i(y)^{-1}U(y)^{-1}g).$$

Since $B_0$ is multiplication, and $i(y)$ is multiplicative, this is equal to

$$\begin{aligned} &U(y)(U(y)^{-1}f \cdot U(y)^{-1}g) \\ &= U(y)B_y(U(y)^{-1}f, U(y)^{-1}g) \\ &= \bar{B}_y(f, g). \end{aligned}$$

Thus, for almost all $y \in \Omega'$, $\bar{B}_y$ and $B_0$ are in the same orbit in $\mathrm{Bil}(H_1, H_2)$ under $A$. Let $T \subset \mathrm{Bil}(H_1; H_2)$ be this $A$ orbit. Since $A$ is compact, $T$ is closed, and from the continuity in $y$ of $\bar{B}_y$, we can conclude $\bar{B}_y \in T$ for all $y$ in a neighborhood of $y_0$, which for simplicity we shall continue to denote by $\Omega'$. The map $A \to T$, $U \to \bar{B}_{y_0} \cdot U$ defines an $A$-homeomorphism from $A/A_0 \cong T$, where $A_0 \subset A$ is some closed subgroup. By its construction, $A$ is a compact Lie group, and hence $A_0$ is also. Thus, there is an open neighborhood of $[e]$ in $A/A_0$ and a section on this neighborhood of the natural projection $A \to A/A_0$. Identifying $A/A_0$ with $T$, we obtain an open neighborhood $N$ of $\bar{B}_{y_0}$ in $T$ and a continuous map $s_0 : N \to A$ such that for all $B \in N$, $B \cdot s_0(B)^{-1} = \bar{B}_{y_0}$. Choose a fixed element $U_0 \in A$ with $\bar{B}_{y_0} \cdot U_0 = B_0$, and let $s : N \to A$ be $s(B) = s_0(B)^{-1}U_0$. Then $s$ is continuous and $B \cdot s(B) = B_0$ for all $B \in N$.

Now $y \to \bar{B}_y$ is a continuous map from $\Omega'$ into $T$, and $N \subset T$ is a neighborhood of $\bar{B}_{y_0}$. Hence, there is a compact neighborhood of $y_0$, $\Omega \subset \Omega'$, such that $\bar{B}_y \in N$ for all $y \in \Omega$. Let $\lambda : \Omega \to A$ be the map $\lambda(y) = s(\bar{B}_y)$. Then $\lambda$ is continuous and $\bar{B}_y \cdot \lambda(y) = B_0$ for all $y \in \Omega$.

Now let $T(y) = \lambda(y)^{-1}U(y)$. Then for $y \in \Omega$, $T(y) : H_2(y) \to H_2$, is unitary, preserves canonical subspaces, and $T(y)(1) = 1$. We claim that if $f, g \in H_1(y)$, then

$$(*) \quad T(y)(fg) = T(y)(f) \cdot T(y)(g).$$

This follows by unravelling the definitions:

$$\begin{aligned} T(y)(fg) &= \lambda(y)^{-1}U(y)(fg) \\ &= \lambda(y)^{-1}U(y)B_y(f, g) \\ &= \lambda(y)^{-1}\bar{B}_y(U(y)f, U(y)g) \\ &= (\bar{B}_y \cdot \lambda(y))(\lambda(y)^{-1}U(y)f, \lambda(y)^{-1}U(y)g) \\ &= B_0(T(y)f, T(y)g) \\ &= T(y)f \cdot T(y)g. \end{aligned}$$

*Step* 2. We now claim that for each $y \in \Omega$, there is a unique algebra isomorphism $T_1(y) : R(S) \to R(y)$ with $T_1(y) = T(y)^{-1}$ on $H_0$. Since $H_0(y)$ and $H_0$ generate $R(y)$ and $R(S)$ respectively, as algebras, uniqueness is clear. Recall that we have a surjective algebra homomorphism $\Phi : C[X_1, \ldots, X_n] \to R(S)$ such that $\Phi(X_i) = f_i$, where $\{f_i\}$ is a proper generating set, and span $\{1, f_i\} = H_0$. For each $y$, define a homomorphism $\beta(y) : C[X_1, \ldots, X_n] \to R(y)$ by $\beta(y)(X_i) = T(y)^{-1}f_i$. We claim first that there is an algebra homomorphism $T_1(y) : R(S) \to R(y)$ such that $\beta(y)$ factors as follows:

$$\begin{tikzcd} C(X_1, \ldots, X_n) \arrow[r, "\Phi"] \arrow[dr, "\beta(y)"'] & R(S) \arrow[d, "T_1(y)"] \\ & R(y) \end{tikzcd}$$

To see this, it suffices to show that if $p \in C(X_1, \ldots, X_n)$ and $\Phi(p) = 0$, then $\beta(y)(p) = 0$. But if $p \in \ker \Phi = I$, $p = \sum_1^k q_j p_j$ where $p_j$ are, as above, the generators of $I$. So

$$\begin{aligned} \beta(y)(p) &= \sum \beta(y)(q_j)\beta(y)(p_j) \\ &= \sum \beta(y)(q_j)p_j(T(y)^{-1}f_1, \ldots, T(y)^{-1}f_n). \end{aligned}$$

Since any polynomial in $\{T(y)^{-1}f_i\}$ of degree $\leqq d$ is in $H_1(y)$, it follows from (*) that

$$p_j(T(y)^{-1}f_1, \ldots, T(y)^{-1}f_n) = T(y)^{-1}p_j(f_1, \ldots, f_n) \quad \text{for all } p_j.$$

Since $p_j \in I$, this is 0, and hence $\beta(p) = 0$. Thus $T_1(y)$ exists.

We note that by commutativity of the diagram above that $T_1(y)f_i = T(y)^{-1}f_i$, and hence $T_1(y) = T(y)^{-1}$ on $H_0$. Therefore, it remains to show that $T_1(y)$ is actually an algebra isomorphism. Since $\beta(y)$ is surjective, so is $T_1(y)$, and hence it suffices to show that $T_1(y)$ is injective. We show this first for the (conull set of) $y$ for which $R(y) \cong R(S)$ by a unitary preserving canonical subspaces. For such a $y$, choose an isomorphism $\theta : R(y) \to R(S)$ and let $\alpha = \theta \circ T_1(y)$. It suffices to show that $\alpha$ is injective. Let $a_{ij} \in \mathbf{C}$ such that $\alpha(f_i) = \sum a_{ij}f_j$, so that $(a_{ij})$ is a unitary matrix. Then the algebra homomorphism $\gamma : C[X_1, \ldots, X_n] \to C[X_1, \ldots, X_n]$ defined by $\gamma(X_i) = \sum a_{ij}X_j$ is actually an algebra automorphism of $C[X_1, \ldots, X_n]$. Furthermore, the following diagram commutes

$$\begin{tikzcd} C[X_1, \ldots, X_n] \arrow[r, "\Phi"] \arrow[d, "\gamma"'] \arrow[dr, "\omega"'] & R(S) \arrow[d, "\alpha"] \\ C[X_1, \ldots, X_n] \arrow[r, "\Phi"'] & R(S) \end{tikzcd}$$

where $\omega = \theta \circ \beta(y)$. To show that $\alpha$ is injective, it suffices to see that $\ker(\omega) \subset \ker(\Phi)$. Now $\ker(\omega) = \gamma^{-1}(\ker \Phi)$, and hence $\ker(\omega)$ is generated by $\{p_j' = \gamma^{-1}(p_j) | j = 1, \ldots, k\}$. Thus, it suffices to show that $p_j' \in \ker \Phi$. We have $\alpha\Phi(p_j') = \Phi(p_j) = 0$. But

$$\begin{aligned} \alpha\Phi(p_j') &= \alpha(p_j'(f_1, \ldots, f_n)) \\ &= p_j'(\alpha f_1, \ldots, \alpha f_n). \end{aligned}$$

On $H_0$, $\alpha = \theta \circ T(y)^{-1}$, so this equals $\theta p_j'(T(y)^{-1}f_1, \ldots, T(y)^{-1}f_n)$. But degree $p_j' \leqq d$, so from (*), this becomes $\theta T(y)^{-1}p_j'(f_1, \ldots, f_n) = \theta T(y)^{-1}\Phi(p_j')$. Since $T(y)^{-1}$ is unitary on $H_2$, and $\theta$ is an isomorphism, it follows that $\Phi(p_j') = 0$.

We now know that $T_1(y)$ is an algebra isomorphism for $y$ in a conull set in $\Omega$, and hence a dense subset of $\Omega$ if we assume, as we may, that $\Omega$ is the closure of its interior. We now claim this is true for all $y \in \Omega$. We first note a continuity property of the maps $T_1(y)$. If $f \in R(S)$, consider the function $\Delta$ defied on $p^{-1}(\Omega)$ by $\Delta(x) = (T_1(p(x))f)(x)$. We claim that $\Delta$ is continuous. Suppose first that $f = f_i$ for some $i$. Because $f_i \in H_0$,

$$T_1(p(x))f_i = T(y)^{-1}f_i = \sum c_{ij}(p(x))g_j^{p(x)},$$

where $c_{ij}$ are functions on $\Omega$ and $g_i$ are as in the beginning of the proof. Thus, to show $\Delta$ is continuous, it suffices to see that $x \to c_{ij}(p(x))$ is continuous. But

$$\begin{aligned} c_{ij}(p(x)) &= \langle U(p(x))^{-1}\lambda(p(x))f_i | g_j^{p(x)} \rangle_{p(x)} \\ &= \langle \lambda(p(x))f_i | f_j \rangle. \end{aligned}$$

Since $y \to \lambda(y)$ is continuous, so is $c_{ij}(p(x))$.

Now if $f \in R(S)$ is arbitrary, we can write $f = q(f_i, \ldots, f_n)$ where $q$ is a polynomial. Since $T_1(y)$ is an algebra homomorphism,

$$(T_1(p(x))f)(x) = q(T_1(p(x))f_1, \ldots, T_1(p(x))f_n)(x),$$

and the result for $f$ follows from the result for $f_i$.

Now suppose $g \in G$, and $y, yg \in \Omega$. Let $g^* : R(yg) \to R(y)$ be the induced algebra isomorphism. Both $\{T_1(y)f_i\}$ and $\{g^*T_1(yg)f_i\}$ are orthonormal bases for $H_0(y)$. Let $(b_{ij})$ be the unitary matrix such that $g^*T_1(yg)f_i = \sum b_{ij}T_1(y)f_j$. Let $\psi$ be the automorphism of $C[X_1, \ldots, X_n]$ defined by $\psi(X_i) = \sum b_{ij}X_j$. Then the following diagram commutes:

$$\begin{tikzcd} C[X_1, \ldots, X_n] \arrow[r, "\Phi"] \arrow[d, "\psi"'] & R(S) \arrow[r, "T_1(yg)"] & R(yg) \arrow[d, "g^*"] \\ C[X_1, \ldots, X_n] \arrow[r, "\Phi"] & R(S) \arrow[r, "T_1(y)"] & R(y) \end{tikzcd}$$

Note also that $\psi$ preserves $P_k$, the set of polynomials of degree $\leqq k$, for each $k$.

Suppose $y \in \Omega$ with $T_1(y)$ an isomorphism. Choose $q_1, \ldots, q_r$ in $C[X_1, \ldots, X_n]$ such that $\{q_i(f_1, \ldots, f_n)\}$ is a basis for $\Phi(P_k)$. Then $\{T_1(y)(q_i(f_1, \ldots, f_n))\}$ is a basis for $T_1(y)\Phi(P_k)$, and therefore, by our remarks above concerning the continuity property of $T_1(y)$, $\{T_1(z)q_i(f_1, \ldots, f_n)\}$ is linearly independent for $z$ in an open set, i.e. $\dim T_1(z)\Phi(P_k) = \dim \Phi(P_k)$. It follows from the commutative diagram above and the fact that $g^*$ is an isomorphism that $\dim T_1(yg)\Phi(P_k) = \dim T_1(y)\Phi(P_k)$. Since the action on $Y$ is minimal, this implies that $\dim T_1(y)\Phi(P_k) = \dim \Phi(P_k)$ for all $y \in \Omega$, and hence that $T_1(y)$ is injective for all $y \in \Omega$. This completes Step 2.

*Step* 3. We now construct $V_n$. Let $n_k$ be the dimension of $\Phi(P_k) \subset R(S)$. Choose polynomials $q_1, q_2, q_3 \ldots$ such that $q_1 = 1$, and for each $k$, $q_1(f_1, \ldots, f_n), \ldots, q_{n_k}(f_1, \ldots, f_n)$ are a basis for $\Phi(P_k)$ each with degree $\leqq k$. For each $y \in \Omega$, let $b_j^y \in R(y)$ be given by

$$b_j^y = T_1(y)(q_j(f_1, \ldots, f_n)), \quad \text{and}$$
$$\Delta_j(x) = b_j^{p(x)}(x), \quad \text{a function on } p^{-1}(\Omega).$$

Then, as shown above, $\Delta_j(x)$ is continuous. Let $W_k$ be the subspace of $C(p^{-1}(\Omega))$ consisting of linear combinations, with coefficients in $C(\Omega)$, of $\Delta_1(x), \ldots, \Delta_{n_k}(x)$. Then $\Delta_1, \ldots, \Delta_{n_k}$ is a local basis for $W_k$, and $(W_k)_y$ consists of polynomials of degree $\leqq k$ in $\{T_1(y)f_i\}_{i=1,\ldots,n}$. In particular, $(W_k)_y \subset R(y)$. Let $\bar{\Delta}_1(x), \bar{\Delta}_2(x), \ldots$ be obtained from $\Delta_1(x), \Delta_2(x), \ldots$ by applying the Gram-Schmidt process fibre-by-fibre so that $\bar{\Delta}_i(x)$ are continuous. Let $V_k \subset C(p^{-1}(\Omega))$, $k \geqq 0$, be the space consisting of linear combinations, with coefficients in $C(\Omega)$, of $\bar{\Delta}_{n_{k-1}+1}, \ldots, \bar{\Delta}_{n_k}$ (where $n_0 = 1$, $n_{-1} = 0$). Then $\{(V_k)_y | k = 0, 1, \ldots\}$ are mutually orthogonal for each $y \in \Omega$, and $V_k$ is a subspace of functions regular on $\Omega$. It follows from the definitions that $(V_0)_y = \mathbf{C}$, and that $(V_1)_y = H_0(y) \ominus \mathbf{C} = \operatorname{span}\{T_1(y)f_1, \ldots, T_1(y)f_n\}$. Since $\{f_1, \ldots, f_n\}$ generates $R(S)$, $\{T_1(y)f_1, \ldots, T_1(y)f_n\}$ generates $R(y)$. Condition ($v$) of Lemma 3.2 is also immediate from the definitions, and thus to prove Lemma 3.2, it remains only to verify condition (vi).

So suppose $y, z, t$ as in the statement of the lemma. We know that $t^* : R(z) \to R(y)$ is unitary, multiplicative, and preserves canonical subspaces. Thus, $t^*(H_0(z)) = H_0(y)$. Since $t^*(T_1(z)f_j) = \sum_i a_{ij}T_1(y)f_i$, we see that $t^*((W_k)_z) = (W_k)_y$. As $(W_k)_y$ is the orthogonal direct sum $(W_{k-1})_y \oplus (V_k)_y$, and a similar statement holds for $z$, the fact that $t^*$ is unitary implies via an easy induction argument that $t^*((V_k)_z) = (V_k)_y$. This completes the proof of the lemma.

We now show how Lemma 3.2 can be used to prove Theorem 3.1. The method involved is actually similar to that of the proof of Lemma 3.2 itself. We shall abandon the notation within the proof of Lemma 3.2 and start afresh.

*Proof of Theorem* 3.1. By Lemma 3.2, we can choose functions $g_1, g_2, \ldots$ on $C(p^{-1}(\Omega))$ such that $g_1 = 1$ and $\{g_{n_{k-1}+1}{}^y, \ldots, g_{n_k}{}^y\}$ is an orthonormal basis of $(V_k)_y$ for each $y \in \Omega$, where $n_k = \dim(W_k)_y$ for $k \geqq 0$, $n_{-1} = 0$. Let $R_i$ be a finite dimensional Hilbert space with $\dim R_i = n_i - n_{i-1}$, and $R$ the pre-Hilbert space obtained by taking the orthogonal algebraic direct sum. Choose $F_1, F_2, \ldots \in R$ such that $\{F_{n_{i-1}+1}, \ldots, F_{n_i}\}$ is an orthonormal basis for $R_i$. Let $\bar{R}_n = \sum_{i=0}^{n\oplus} R_i$.

Let $M$ be the space of bilinear maps $R \times R \to R$ for which $\bar{R}_n \times \bar{R}_p \to \bar{R}_{n+p}$. Give $M$ the smallest topology such that all maps $M \to \mathbf{C}$, $B \to \langle B(f, g)|h\rangle$ are continuous, where $f, g, h \in R$. This topology will also be the smallest topology for which all maps $B \to \langle B(F_i, F_j)|F_k\rangle$ are continuous. Thus, $M$ becomes a second countable Hausdorf space. Now let $K$ be the set of all unitary maps $U : R \to R$ such that $U(R_i) = R_i$ for all $i$, and $U|R_0 = I$. Then $K$ is isomorphic in a natural way to $\prod_{i=1}^{\infty} U(R_i)$, where $U(R_i)$ is the unitary group on $R_i$; thus $K$ is a compact group. We have an action of $K$ on $M$ defined by $(B \cdot U)(f, g) = U^{-1}B(Uf, Ug)$, and it is straightforward to check that this action is jointly continuous.

Let $U(y) : R(y) \to R$ be the unitary map defined by $U(y)(g_i^y) = F_i$. For each $y \in \Omega$, we obtain an element $B_y \in M$, defined by $B_y(f, g) = U_y(U(y)^{-1}f \cdot U(y)^{-1}g)$. We note that condition (v) of Lemma 3.2 ensures that $B_y \in M$. Furthermore, the map $y \to B_y$ is a continuous function from $\Omega \to M$. This can be seen exactly as in Step 1 of the proof of Lemma 3.2.

Let $y, z \in \Omega$ and $t \in G$ such that $y \cdot t = z$. Then $t^* : R(z) \to R(y)$ is unitary, and it follows from condition (vi) of Lemma 3.2 that $U(y)t^*U(z)^{-1} \in K$. Furthermore, a straightforward calculation shows that $B_y \cdot U(y)t^*U(z)^{-1} = B_z$. Thus, by the minimality of $G$ on $Y$, $\{B_y\}$ are in the same orbit in $M$ under $K$, say $M_0$, for $y$ in a dense subset of $\Omega$. Since $K$ is compact, $M_0$ is closed, and continuity of $B_y$ implies $B_y \in M_0$ for all $y \in \Omega$. Let $K_0 \subset K$ be the stability group of $B_{y_0}$. Then $K_0$ is a closed subgroup, and $U \to B_{y_0} \cdot U$ defines a $K$-homeomorphism of $K/K_0$ with $M_0$. Defining multiplication on $R$ by $f \cdot g = B_{y_0}(f, g)$, we see that $R$ becomes an algebra, and $U(y_0) : R(y_0) \to R$ is an algebra isomorphism. Since $(V_1)_{y_0}$ generates $R(y_0)$, $R_1$ generates $R$. Now $K_0 = \{U \in K | B_{y_0} \cdot U = B_{y_0}\}$, i.e.

(**) $\quad K_0 = \{U \in K | U : R \to R \text{ is an algebra isomorphism}\}$.

Because $R_1$ generates $R$, the map $K_0 \to U(R_1)$ defined by $U \to U|R_1$ is injective, and since $K_0$ is compact, this is a homeomorphism onto its image. Since $\dim R_1 < \infty$, this implies that $K_0$ is a compact lie group.

It follows from [**2**, Theorem II.5.8] that there is an open neighborhood of $[e]$ in $K/K_0$ and a continuous section on this neighborhood of the natural projection $K \to K/K_0$. Via the isomorphism of $K/K_0$ and $M_0$, we obtain an open neighborhood $N$ of $B_{y_0}$ in $M_0$, and a continuous map $s : N \to K$ such that $B = B_{y_0} \cdot s(B)$ for all $B \in N$. Since $y \to B_y$ is continuous, there is a compact neighborhood of $y_0$, $\Omega' \subset \Omega$, such that $B_y \in N$ for all $y \in \Omega'$. Let $\lambda(y) = s(B_y)^{-1}$, for $y \in \Omega'$. Then $\lambda$ is continuous, and $B_y \cdot \lambda(y) = B_{y_0}$ for $y \in \Omega'$.

Let $A_y : R(y_0) \to R(y)$ be $A_y = U(y)^{-1}\lambda(y)U(y_0)$. Then

(i) $A_y$ is unitary and $A_y((V_n)_{y_0}) = (V_n)_y$, and

(ii) $A_y$ is an algebra isomorphism. To see (ii), note that for $f, g \in R(y_0)$,

$$\begin{aligned} U(y)^{-1}\lambda(y)U(y_0)(f \cdot g) &= U(y)^{-1}\lambda(y)B_{y_0}(U(y_0)f, U(y_0)g) \\ &= U(y)^{-1}B_y(\lambda(y)U(y_0)f, \lambda(y)U(y_0)g), \end{aligned}$$

since $B_y \cdot \lambda(y) = B_{y_0}$. But from the definition of $B_y$, the last expression becomes

$$(U(y)^{-1}\lambda(y)U(y_0)f)(U(y)^{-1}\lambda(y)U(y_0)g) = (A_y f)(A_y g).$$

That $A_y(1) = 1$ follows from the fact that $\lambda(F_1) = F_1$, by the definition of $K$.

Let $Z = \Omega' \times p^{-1}(y_0)$, and $R(Z)$ be the subspace of continuous functions consisting of (finite) linear combinations of functions of the form $\alpha(y)h(z)$, where $\alpha \in C(\Omega')$ and $h \in R(y_0)$. Then $R(Z)$ is a *-subalgebra of $C(Z)$, contains the constants, and separates points (since $R(y_0)$ separates points of $p^{-1}(y_0)$).

Let

$$\bar{R} = \{f \in C(p^{-1}(\Omega'))|f^y \in R(y) \text{ for all } y \in \Omega'\}.$$

We define a map $A : R(Z) \to R$ as follows. Given $f \in R(Z)$, let

$$A(f)(x) = A_{p(x)}(f|\{p(x)\} \times p^{-1}(y_0))(x).$$

As defined, $A(f)$ is a function on $p^{-1}(\Omega')$. Its restriction to a fibre $p^{-1}(y)$ is clearly in $R(y)$, so to see that $A(f) \in \bar{R}$, it remains to show that $A(f)$ is continuous. Since $A(\sum \alpha_i h_i) = \sum \alpha_i A(h_i)$ where $\alpha_i \in C(\Omega')$ and $h_i \in R(y_0)$, it suffices to see that $A(f)$ is continuous for $f$ of the form $f(y, z) = h(z)$. Now $A(f)$ will be continuous if for each $i$, $y \to \langle A(f)^y|g_i^y\rangle_y$ is continuous, where $g_i$ are as above. But this expression is just

$$\begin{aligned}
&\langle A_y(f|\{y\} \times p^{-1}(y_0)|g_i^y\rangle_y \\
&= \langle U(y)^{-1}\lambda(y)U(y_0)h|g_i^y\rangle_y \\
&= \langle \lambda(y)U(y_0)h|F_i\rangle.
\end{aligned}$$

Since $\lambda(y)$ is continuous, this expression is continuous in $y$.

Thus, we have $A : R(Z) \to \bar{R}$, $A(1) = 1$, $A$ is linear over $C(\Omega')$, and since $A_y$ is multiplicative, so is $A$. Since each $A_y$ is injective, $A$ is also. To see that $A$ is an algebra isomorphism, it suffices to show that $g_i|p^{-1}(\Omega') \in A(R(Z))$. But a straightforward calculation shows that $A(U(y_0)^{-1}\lambda(y)^{-1}F_i) = g_i|p^{-1}(\Omega')$.

Now let $\mu'$ be the product measure $\nu \times \mu_{y_0}$ on $\Omega' \times p^{-1}(y_0)$. Then both $\mu$ and $\mu'$ are positive on open sets. Since each $A_y : R(y_0) \to R(y)$ is unitary, it follows that $A : R(Z) \to \bar{R}$ is unitary. As $R(Z)$ and $\bar{R}$ are dense in $C(Z)$ and $C(p^{-1}(\Omega'))$ respectively, by Stone-Weierstrass, $A$ extends to a unitary map $A : L^2(Z, \mu') \to L^2(p^{-1}(\Omega'), \mu)$. It follows from Lemma 2.13 that there is homeomorphism $\phi : p^{-1}(\Omega') \to Z$ that induces $A$. Since $A$ is $C(\Omega')$-linear, it readily follows that $\phi$ is a fibre preserving map. This proves that $X$ is locally trivial over $\Omega'$. We now examine how the action of $G$ on $p^{-1}(\Omega')$ carries over to an action on $Z$.

Recall the group $K_0$(**). Let $K' = U(y_0)K_0U(y_0)^{-1}$. Then for $T \in K'$, $T : R(y_0) \to R(y_0)$, $T$ is an algebra isomorphism, $T((V_n)_{y_0}) = (V_n)_{y_0}$, and $T$ is unitary. It follows from Lemma 2.14 that $K'$ acts continuously on $p^{-1}(y_0)$. If $y$, $z \in \Omega'$ and $t \in G$, $y \cdot t = z$, let $t^*$ be the induced map $t^* : C(p^{-1}(z)) \to C(p^{-1}(y))$. The map $t : p^{-1}(y) \to p^{-1}(z)$ corresponds to a map $\bar{t} : \{y\} \times p^{-1}(y_0) \to \{z\} \times p^{-1}(y_0)$ (under the homeomorphism $\phi$) such that the induced map $\bar{t}^* : C(p^{-1}(y_0)) \to C(p^{-1}(y_0))$ is given by $\bar{t}^* = A_y^{-1}t^*A_z$.

Thus, $\bar{t}^* \in K'$, and hence, under the action of $K'$ on $p^{-1}(y_0)$, $t$ corresponds to translation by some element of $K'$. It then follows from the ergodicity of $G$ on $X$ that $K'$ must actually be transitive on $p^{-1}(y_0)$ (recall $K'$ is compact), and hence $p^{-1}(y_0)$ can be identified with a homogeneous space $K'/K_0'$, where $K_0' \subset K'$ is a closed subgroup.

Finally, we verify that $X$ is a homogeneous extension of $Y$. If $t \in G$, we have a homeomorphism $\phi_t : p^{-1}(\Omega') \cdot t \to \Omega' \cdot t \times p^{-1}(y_0)$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(\Omega') \arrow[r, "t"] \arrow[d, "\phi"'] & p^{-1}(\Omega')t \arrow[d, "\phi_t"] \\ \Omega' \times p^{-1}(y_0) \arrow[r, "t \times i"] & \Omega' \cdot t \times p^{-1}(y_0), \end{tikzcd}$$

where $i : p^{-1}(y_0) \to p^{-1}(y_0)$ is the identity, i.e., $\phi_t(z) = (t \times i)\phi(zt^{-1})$. This shows that $X$ is a fibre bundle over $Y$ with fibre $p^{-1}(y_0)$, since $G$ is minimal on $X$.

We now describe the action of $G$ on $\Omega' t \times p^{-1}(y_0)$ defined by the homeomorphism $\phi_t$. Suppose $y', z' \in \Omega' t$, and $y' \cdot s = z'$ for $s \in G$. Then it is straightforward to check, in light of the previous paragraph, that the corresponding (under $\phi_t$) map

$$\{y'\} \times p^{-1}(y_0) \to \{z'\} \times p^{-1}(y_0)$$

is defined by translation by an element of $K'$. In fact, on $C(p^{-1}(y_0))$, this map induces the operator $A_{y''}{}^{-1}(tst^{-1})^* A_{z''}$, where $y'' \cdot t = y'$ and $z'' \cdot t = z'$.

Thus, to complete the proof of Theorem 3.1, it suffices to show that the structure group can be taken as $K'$. So suppose $u \in \operatorname{int}(\Omega' \cdot t) \cap \operatorname{int}(\Omega' \cdot s)$, where $s, t \in G$. Let $u = y \cdot t = z \cdot s$, with $y, z \in \operatorname{int}(\Omega')$. We then have two homeomorphisms $p^{-1}(y_0) \to p^{-1}(u)$, namely,

$$p^{-1}(u) \xrightarrow{t^{-1}} p^{-1}(y) \xrightarrow{\phi|p^{-1}(y)} p^{-1}(y_0).$$

$$p^{-1}(u) \xrightarrow{s^{-1}} p^{-1}(z) \xrightarrow{\phi|p^{-1}(z)} p^{-1}(y_0).$$

To prove that the structure group can be taken as $K'$ suffices to check that the map

$$(\phi|p^{-1}(z))s^{-1}t(\phi|(y))^{-1} : p^{-1}(y_0) \to p^{-1}(y_0)$$

lies in $K'$. But on $C(p^{-1}(y_0))$, this map induces the operator $A_y{}^{-1}t^*(s^{-1})^*A_z$. This is in $K'$, completing the proof.

*Remark.* The fact that the measure $\nu$ on $Y$ is finite and invariant was used in a significant way only in that we have applied the results of [**11**]. What we did make use of, however, is the "relative invariance" of the family $y \to \mu_y$. The results of [**11**] can presumably be reworked for $\nu$ quasi-invariant and ergodic, and thus the results of this paper may well generalize to that case (with the assumption of relative invariance.) It seems possible that this latter condition may in fact be deducible from a function space decomposition.

## 4. Applications and examples.

PROPOSITION 4.1. *If $X \to Y$ is a continuous ergodic extension with proper relatively discrete spectrum, then $X = \lim_{\leftarrow} X_i$, where $X_i$ are finitely generated extensions of $Y$. Hence, if $Y$ is minimal, $X$ is an inverse limit of homogeneous extensions.*

*Proof.* Since $X$ is separable, $\{\pi \in \hat{K} | H_\pi \neq 0\}$ is countable. Label these representations $\pi_1, \pi_2, \ldots$. For each $n$, let $B_n$ be the $C_0(Y)$-*- subalgebra generated by $\sum_1^{n\oplus} H_{\pi_i}$. Then $\bar{B}_n$ is a $G$-invariant sub-$C^*$-algebra of $C_0(X)$ containing $C_0(Y)$, so via Gelfand theory, we have an intermediate $G$ space $X_n, X \to X_n \to Y$. One readily checks that $X_n$ is a finitely generated continuous ergodic extension with proper relatively discrete spectrum, and since $\overline{\bigcup B_n} = C_0(X)$, the result follows.

Suppose now that $X$ and $Y$ are minimal, distal, compact metric $G$-spaces and $X \to Y$ is an isometric extension. Then Knapp [**9**] shows that $X$ is a continuous ergodic extension with proper relatively discrete spectrum. Hence, we have the following strengthened version of Furstenberg's structure theorem.

THEOREM 4.2. *Suppose $X$ is a minimal distal compact metric G-space. Then there is an ordinal $\xi$, metric G-spaces $X_\eta$ for each $\eta < \xi$, and continuous, surjective, non-injective G-maps $X_\eta \to X_\sigma$ for all pairs $(\eta, \sigma)$ with $\eta > \sigma$, such that, calling $X = X_\xi$*

(i) *$X_{\eta+1}$ is a homogeneous extension of $X_\eta$ for all $\eta < \xi$;*

(ii) *if $\eta$ is a limit ordinal, $\eta \leqq \xi$, then $X_\eta = \lim_{\leftarrow} \{X_\sigma | \sigma < \eta\}$*

*Proof.* This follows by a Zorn's lemma argument, exactly as in [**4**, Theorem 4.2], by making use of Proposition 4.1.

The following provides a criterion for when an extension will be finitely generated.

PROPOSITION 4.3. *Suppose $p : X \to Y$ is a continuous ergodic extension with proper relatively discrete spectrum, and normal in the sense of* [**11**, Definition 5.4]. *(We remark that normality holds in particular if $V_i(y)$ are all one dimensional where $V_i$ are as in* Definition 2.4). *If the fibres of $p$ are finite dimensional and locally connected, then $X$ is a finitely generated extension of $Y$.*

*Proof.* For almost all $y$, $L^2(p^{-1}(y)) \cong L^2(S)$ by a unitary map which when restricted to $R(y)$, is an algebra isomorphism of $R(y)$ and $R(S)$. It follows from Lemma 2.13 that $p^{-1}(y)$ is homeomorphic to $S$. Normality implies that $S$ is actually a (separable) compact group [**11**, Theorem 5.7], and since $S$ is finite dimensional and locally connected, $S$ must be a Lie group. But then $R(S)$ is finitely generated [**8**, 30.48].

Finally, we present a simple example to show that the finite generation condition is required. In fact, we exhibit compact, minimal, distal spaces $X$ and $Y$, with $X$ an isometric but not homogeneous extension of $Y$.

*Example* 4.4. Let $N$ be the nilpotent Lie group consisting of matrices of the form

$$\begin{vmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{vmatrix}$$

where $x, y, z \in \mathbf{R}$. For convenience, we denote this matrix by $[x, y, z]$. Let $D$ be the discrete subgroup of $N$ consisting of the matrices for which $x$, $y$, $z$ are integers. Then $N/D$ is a compact manifold, admitting an $N$-invariant probability measure. (For this and other statements not proven below, see [**1**].) Let $M = D[N, N]$. Then $N/M$ is torus, and we have a natural projection $N/D \to N/M$. For each positive integer $i$, let $N_i$, $D_i$, $M_i$ be copies of $N$, $D$, $M$, respectively. Let

$$X_n = \prod_{i=1}^{n} N_i/D_i, \quad Y_n = \prod_{i=1}^{n} N_i/M_i,$$

$$X = \prod_{i=1}^{\infty} N_i/D_i, \quad Y = \prod_{i=1}^{\infty} N_i/M_i.$$

$D_1 \times \ldots \times D_n$ is a discrete subgroup of the nilpotent Lie group $N_1 \times \ldots \times N_n$, and $M_1 \times \ldots \times M_n$ is the subgroup $(D_1 \times \ldots \times D_n)[N_1 \times \ldots \times N_n, N_1 \times \ldots \times N_n]$. We have

$$X_n \cong N_1 \times \ldots \times N_n/D_1 \times \ldots \times D_n,$$
$$Y_n \cong N_1 \times \ldots \times N_n/M_1 \times \ldots \times M_n,$$

and natural projection, $X_n \to Y_n$, $p : X \to Y$.

Choose a sequence of real numbers, $x_1, x_2, \ldots$, such that $\{1, x_1, x_2 \ldots\}$ is independent over the rationals. For each $i$, let $A_i$ be the matrix $[x_{2i-1}, x_{2i}, 0]$. Then $\sum_{i=1}^{\oplus} A_i$ can be considered as an element of $N_1 \times \ldots \times N_n$, and translation on the right by the one-parameter subgroup determined by this element defines an action of $\mathbf{R}$ on $X_n$ that factors to an action on $Y_n$. By the choice of $A_n$, it follows from [**1**, Corollary V.4.5] that $X_n$ is a minimal $\mathbf{R}$-space, and from [**1**, Theorem IV.3] that $\mathbf{R}$ acts distally on $X_n$. The actions on $X_n$ induce an $\mathbf{R}$-action on $X$ which factors to an action on $Y$, and it follows readily that $X$ is also minimal and distal. We now claim that $X$ is an isometric extension of $Y$. For each $i$, $N_i/D_i$ is an isometric extension of $N_i/M_i$ [**9**, Paragraph 9]. Let $\rho_i$, $K_i$ be the function and homogeneous space that exhibit this, as required in definition of isometric extension [**4**, Definition 2.2]. Let

$$\rho = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\rho_i}{1 + \rho_i}, \quad K = \prod_1^{\infty} K_i.$$

Then it is straightforward to check that $\rho$ and $K$ exhibit $X$ as an isometric extension of $Y$.

Finally, we claim that $X$ is not a homogeneous extension of $Y$. If it is, we can choose an open set $U \subset Y$ for which $p^{-1}(U) \to U$ is a trivial bundle. For an open set $U \subset Y$, there is an integer $k$ and elements $a_i \in N_i/M_i$, $i \neq k$, such that

$$\{a_1\} \times \ldots \times \{a_{k-1}\} \times N_k/M_k \times \{a_{k+1}\} \times \ldots \subset U.$$

But if $p^{-1}(U)$ is trivial over $U$, $N_k/D_k$ must be a trivial bundle over $N_k/M_k$, which is a contradiction.

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*United States Naval Academy,*
*Annapolis, Maryland*