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FORMULA FOR THE Nth PRIME NUMBER

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In this note we give a simple formula for the nth prime number. Let p_n denote the nth prime number ($p_1=2, p_2=3$, etc.). We shall show that p_n is given by the following formula.

THEOREM.

$$p_n = \sum_{i=0}^{n^2}\left(1 \dot{-} \left(\left(\sum_{j=0}^{i} r((j\dot{-}1)!^2, j)\right) \dot{-} n\right)\right).$$

Here $r(x, y)$ denotes the remainder upon division of x by y. (We take $r(x, 0)=x$). Proper subtraction, $x \dot{-} y$, is defined as follows: If $y \le x$ then $x \dot{-} y = x-y$, and if $x<y$ then $x \dot{-} y=0$.

Proof. If j is prime, Wilson's theorem asserts that $(j-1)! \equiv -1 \bmod j$. When j is composite, $(j-1)! \equiv 0 \bmod j$ with the single exception of $j=4$. If $j=4$ then $(j-1)! \equiv 2 \bmod j$. In any case we see that

$$r((j-1)!^2, j) = \begin{cases} 1 & \text{if } j \text{ is prime,} \\ 0 & \text{if } j \text{ is composite.} \end{cases} \tag{2}$$

It follows from (2) that, if $\pi(i)$ denotes the number of primes $\le i$ then

$$\pi(i) = \sum_{j=1}^{i} r((j-1)!^2, j), \qquad (i = 1, 2, 3, \ldots). \tag{3}$$

The function $C(a, n) = 1 \dot{-} ((1+a) \dot{-} n)$ is easily seen to be the characteristic function of the relation $a<n$. That is to say, $C(a, n)=1$ for $a<n$ and 0 otherwise. Now $\pi(i)<n$ if and only if $i<p_n$. Therefore $C(\pi(i), n)=1$ for $i<p_n$ and 0 otherwise. Hence we see that

$$p_n = \sum_{i=0}^{k} C(\pi(i), n), \qquad (n = 1, 2, 3, \ldots) \tag{4}$$

whenever k is large enough (i.e. $k \ge p_n - 1$).

Bertrand's postulate implies that $p_n \le 2^n$. Therefore we could take $k=2^n$ in (4). However, a much more economical bound is possible. J. Barkley Rosser and Lowell Schoenfeld [3] have proven that $p_n < n(\log n + \log\log n)$, for $5<n$. Thus we may actually take $k=n^2$ in (4).

Now for $j=0$, $r((j\dot{-}1)!^2, j)=1$. Hence (3) implies that

$$1+\pi(i) = \sum_{j=0}^{i} r((j\dot{-}1)!^2, j), \qquad (i = 0, 1, 2, \ldots). \tag{5}$$

To obtain the formula of the theorem we need only substitute (5) into (4).

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