BULL. AUSTRAL. MATH. SOC. 16A12, 16A21, 16A56
VOL. 47 (1993) [287–290]

ON PRIME ESSENTIAL RINGS

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A ring A is prime essential if A is semiprime and every prime ideal of A has a nonzero intersection with each nonzero ideal of A. We prove that any radical (other than the Baer's lower radical) whose semisimple class contains all prime essential rings is not special. This yields non-speciality of certain known radicals and answers some open questions.

Throughout this note all rings considered are associative. The terminology and basic results of radical theory can be found in [**1**, **2**].

A ring A is *prime essential* if A is semiprime and every prime ideal of A has a nonzero intersection with each nonzero ideal of A (equivalently [**3**, Proposition 1], A is semiprime and no nonzero ideal of A is a prime ring).

Prime essential rings were introduced by Rowen [**7**] and their important role in the study of special radicals was beautifully demonstrated by Gardner and Stewart in [**3**]. The present note shows yet another negative influence prime essential rings have on speciality of radicals. Namely, we prove that any radical (other than the Baer's lower radical $\mathcal{B}$) whose semisimple class contains all prime essential rings is not special.

As an application of the main result, we show that the lower radical $\mathcal{L}_2$ determined by the class of all almost nilpotent rings (that is, of rings whose every proper homomorphic image is nilpotent) is not a special radical. This gives a negative answer to a question, put in a private conversation, by Heyman. This also proves that $\mathcal{L}_2$ does not coincide with the Andrunakievic's antisimple radical $\mathcal{B}_\varphi$ and thus provides an answer to yet another question raised by van Leeuwen and Heyman in [**6**].

Finally, we use the main result to construct infinitely many supernilpotent nonspecial radicals. This leads to non-speciality of some supernilpotent radicals discussed in [**4**].

For future use we shall need the following construction of prime essential rings.

EXAMPLE 1. [**3**, Example 5]. Let A be any nonzero semiprime ring, let k be an infinite cardinal number greater than the cardinality of A and let $W(k)$ denote the set of all finite words made from a well-ordered alphabet of cardinality k, lexicographically

Received 24 March 1992

ordered. Then $W(k)$ is a semigroup with multiplication defined by $xy = \max\{x, y\}$ and the semigroup ring $A(W(k))$ is a nonzero prime essential ring whose every prime homomorphic image is isomorphic to some prime homomorphic image of the ring A.

We start with our main result which determines sufficient conditions for a radical to be nonspecial.

THEOREM 1. *Any radical $\mathcal{R}$ (other than the lower Baer's radical $\mathcal{B}$) whose semisimple class contains all prime essential rings is nonspecial.*

PROOF: Let $\mathcal{R} \neq \mathcal{B}$ be a radical whose semisimple class contains all prime essential rings. We shall prove that $\mathcal{R}$ is nonspecial.

Since every special radical is supernilpotent, we may assume that $\mathcal{R}$ is a supernilpotent radical. But a supernilpotent radical $\mathcal{R}$ is special if and only if every $\mathcal{R}$ semisimple nonzero ring has an $\mathcal{R}$ semisimple nonzero prime homomorphic image. Thus it is sufficient to indicate an $\mathcal{R}$ semisimple nonzero ring every nonzero homomorphic prime image of which is not $\mathcal{R}$ semisimple. To do so we shall adapt the ring from Example 1.

Since $\mathcal{R}$ strictly contains $\mathcal{B}$, there exists a nonzero semiprime and $\mathcal{R}$ radical ring, say A. We construct the nonzero prime essential ring $R = A(W(k))$, as described in Example 1. Now, since every prime essential ring is $\mathcal{R}$ semisimple, in particular the ring R is $\mathcal{R}$ semisimple. On the other hand, every prime homomorphic image of R, being isomorphic to some prime homomorphic image of the $\mathcal{R}$ radical ring A, is $\mathcal{R}$ radical. Thus R is a nonzero $\mathcal{R}$ semisimple ring whose every nonzero prime homomorphic image is not $\mathcal{R}$ semisimple and the result follows. ∎

REMARK. In the following example, we shall show that there exists a nonspecial radical containing a nonzero prime essential ring. Therefore the converse of Theorem 1 is not valid.

EXAMPLE 2. Let $\mathcal{R}$ be the Jenkins radical, that is the upper radical determined by the class of all prime simple rings. It is well known [5] that $\mathcal{R}$ is not hereditary and all but special. We shall now show that $\mathcal{R}$ contains a nonzero prime essential ring. To see this consider a nonzero prime ring A without proper prime homomorphic images and without minimal ideals. For example, the ring $A = \{2x/(2y+1),\ x, y \in Z,\ (2x,\ 2y+1) = 1\}$ [2, Example 10] will do. We construct the semigroup ring $A(W(k))$, as described in Example 1. Then $A(W(k))$ is a nonzero prime essential ring and every nonzero prime homomorphic image of $A(W(k))$ is isomorphic to A. Consequently, since A is far removed from being simple, it follows that $A(W(k))$ is $\mathcal{R}$ radical.

As an application of Theorem 1, now we shall answer certain open questions.

COROLLARY 1. *The lower radical $\mathcal{L}_2$ determined by the class of all almost nilpotent rings is nonspecial.*

PROOF: Since every ring without nonzero almost nilpotent ideals is $\mathcal{L}_2$ semisimple, in view of Theorem 1, it is sufficient to prove that every prime essential ring is without nonzero almost nilpotent ideals.

Suppose not and let A be a nonzero prime essential ring containing a nonzero ideal I which is almost nilpotent. Since A is semiprime, so is I. But any almost nilpotent ring is either nilpotent or prime. Thus I must be a prime ring. But this is impossible because A is prime essential. This contradiction ends the proof. $\square$

Corollary 1 together with the fact that the antisimple radical $\mathcal{B}_\varphi$ is special implies the following

COROLLARY 2. $\mathcal{L}_2 \neq \mathcal{B}_\varphi$.

For a class $\mathcal{M}$ of rings, as usual, let $\mathcal{UM}$ denote the class of all rings which cannot be homomorphically mapped onto a nonzero ring from the class $\mathcal{M}$ and let $\mathcal{SM}$ be the class of all rings without nonzero ideals from $\mathcal{M}$.

In [4] infinitely many supernilpotent radicals were constructed. The following result provides a more general construction of such radicals.

THEOREM 2. *Let $\mathcal{M}$ be any hereditary class of prime rings and let $\mathcal{C} = \{$ all nilpotent rings $\} \cup \mathcal{M}$. Then $\mathcal{USC}$ is a supernilpotent and nonspecial radical or $\mathcal{USC} = \mathcal{B}$.*

PROOF: By [4, Theorem 4], $\mathcal{USC}$ is a supernilpotent radical. Since the class of all prime essential rings is obviously contained in $\mathcal{SC}$ and $\mathcal{SC}$ is contained in $\mathcal{SUSC}$, the nonspeciality of the radical $\mathcal{USC}$ follows immediately from Theorem 1 unless $\mathcal{USC} = \mathcal{B}$. $\square$

COROLLARY 3. [4, Theorem 6]. *Let $\mathcal{P}$ be any hereditary class of prime rings containing Z_2 and $\mathcal{C} = \{$ all nilpotent rings $\} \cup \mathcal{P}$, then $\mathcal{USC}$ is a supernilpotent nonspecial radical.*

COROLLARY 4. [4, Corollary 2] *Let E be any prime ring that cannot be mapped into a field and let $\mathcal{D}$ be any hereditary class of prime rings containing Z_2 but not E nor any ideals of E. If $\mathcal{K} = \{$ all nilpotent rings $\} \cup \mathcal{D}$ then $\mathcal{USK}$ is a supernilpotent and nonspecial radical independent of the upper radical determined by all fields.*

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