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Integral Representation for $U_3 \times GL_2$

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Abstract. Gelbart and Piatetskii-Shapiro constructed various integral representations of Rankin–Selberg type for groups $G \times GL_n$, where G is of split rank n. Here we show that their method can equally well be applied to the product $U_3 \times GL_2$, where U_3 denotes the quasisplit unitary group in three variables. As an application, we describe which cuspidal automorphic representations of U_3 occur in the Siegel induced residual spectrum of the quasisplit U_4.

1 Introduction

1.1 Summary

Gelbart and Piatetskii-Shapiro [4] outlined three ways to obtain integral representations for generic cuspidal automorphic representations of groups of the type $G\times GL_n$, where G is of split rank n. Two of their methods have been worked out for unitary groups G in more detail by Watanabe [17]. We will show that a similar method works on $\mathrm{Res}_{E/F}\, GL(2) \times U_3$. Here E/F denotes a CM field extension, and U_3 is the quasisplit unitary group in three variables. Essentially the clue is to embed both groups into the quasisplit U_4, where $GL_2(E)$ can be realized as a Levi component of a maximal proper parabolic subgroup. Starting with a cuspidal automorphic representation of $\mathrm{Res}_{E/F}\, GL(2)$, one obtains an Eisenstein series on U_4. This function can be restricted to U_3 and integrated against a cuspform. Performing the standard procedure of double coset analysis and the Rankin–Selberg method, the integral decomposes into an Euler product over F of local zeta integrals. The convergence of the global integral results from the fact that U_3 is of split rank one with a center Z such that $Z(F)\backslash Z(\mathbb{A}_F)$ is compact. Therefore the rapid decay of the cuspform on the smaller group suffices for the convergence. In a completely split case, *i.e.,* on $GL_3 \times GL_4$, the analogous integral would not converge, and one has to truncate the Eisenstein series. Here this is not necessary.

In the analysis of the local zeta integrals we obtain the following results. For sufficiently large real part of s, they converge absolutely and normally in s. They can be analytically extended to a meromorphic function of $s \in \mathbb{C}$. At a finite place, the local integrals are rational functions of q^{-s}, and at unramified places they equal a degree 12 Euler factor over F associated with an explicitly given representation of the L-group of $U_3 \times \mathrm{Res}_{E/F}\, GL(2)$. We do not establish a functional equation of the local integrals, nor can we say anything more precise about the ramified local integrals. At an archimedean place, we again obtain convergence for large real part of s and

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analytic continuation. The precise determination of the archimedean zeta integrals is quite subtle, as can be seen in the work of Koseki and Oda [12]. In the present case an additional difficulty is given by the fact that these local integrals also include Whittaker functions on $GL_2(\mathbb{C})$ coming from the Levi subgroup of U_4. We expect, but do not verify here, that under suitable restrictions on the Eisenstein series these local integrals equal a product of Γ-functions times a polynomial in s, and that they coincide with the expected Langlands L-factors from the nonarchimedean places.

The present paper presents the details for an integral representation which is a special case of a more general theory which is announced and summarized in [16]. More precisely, here we investigate the properties of an integral, using Gelfand–Graev models, of the form [16, (2.7), p. 349] specialized to the case $U_3 \times GL_2$. The original motivation for this work stems from the author's thesis, in which the goal was to use this particular integral representation to obtain information about period integrals on $U(2) \times U(3)$. Namely, one can pass from $U(2)$ to $GL(2, E)$ via base change. Applying the integral representation of the present work to cuspidal automorphic representations on $GL(2, E)$ that are in the image of that lift could lead to an integral representation on $U(2) \times U(3)$, *i.e.,* on two groups, none of which is GL_n. Moreover one might be able to extract information about the nonvanishing of the central value of the Rankin–Selberg L-function in terms of $U(2)$-period integrals on $U(2) \times U(3)$. The work of Gelbart and Piatetskii-Shapiro [3] can be interpreted as giving a central value formula on $U(2)\times U(3)$ in terms of $U(2)$-period integrals for automorphic representations whose $U(2)$-part is noncuspidal. One goal of my work is to obtain such a period integral formula for cuspidal automorphic representations, which would have applications to the analogue of the Gross–Prasad conjecture (formulated in [7] for orthogonal groups) in the setting of unitary groups.

One application in that direction of the formula obtained in this paper consists of Theorem 1.3. In it we describe how the residual representations of U_4 that arise from the Siegel parabolic decompose when restricted to U_3.

1.2 Statement of the Main Results

Here are the main global and local results of this paper. The notation will be defined precisely at the beginning of Section 2.

Theorem 1.1 *Let π be an irreducible generic unitary cuspidal automorphic representation of U_3. Let τ be an irreducible unitary cuspidal automorphic representation of $GL(2, E)$. Let $\varphi \in \pi$ denote a cuspform, and let $E^*(s, g, \tau)$ denote an Eisenstein series on U_4, induced from a maximal parabolic P of type $(2, 2)$ and the representation τ of its Levi factor M. Here, U_4 denotes the quasisplit unitary group in four variables, and we identify M with $\mathrm{Res}_{E/F}\, GL(2)$. Embed U_3 into U_4 and identify it with its image. Let dh denote a fixed Haar measure on $U_3(\mathbb{A}_F)$.*

(i) *The global integral*

$$I(s, \varphi, E^*) = \int_{U_3(F)\backslash U_3(\mathbb{A}_F)} \varphi(h)E^*(s, h, \tau)\, dh \tag{1.1}$$

converges absolutely and uniformly for s in a compact subset of $\mathbb{C}$ in which the

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Eisenstein series has no poles. It thus defines a meromorphic function of $s \in \mathbb{C}$ whose poles are contained in the poles of E^*.

(ii) *The integral equals 0 unless π is generic. If π is generic, then for suitable choices of φ and E^*, and for $\Re(s)$ sufficiently large, it decomposes into a product of local integrals. More precisely, in such a situation we have the equality*

$$I(s,\varphi,E^*) = \int_{N_3(\mathbb{A}_F)\backslash U_3(\mathbb{A}_F)} W^{\varphi}(g)W^{\tau}(s,g)\,dg.$$

Here W^{φ} denotes a Whittaker function associated with φ, and $W^{\tau}(s,g)$ is a function on $U_4(\mathbb{A}_F)$ that is related to functions in a Whittaker model of τ.

We will define the space of functions $W^{\tau}(s,g)$ more precisely below. It is a representation space for U_4, and the function on M given by $m \mapsto W^{\tau}(s,m)$ is, up to a certain dependency on s, in the Whittaker model of τ. In particular, for appropriate choices of φ and the Eisenstein series, these functions decompose into a tensor product of local functions. Thus for $\Re(s)$ sufficiently large, $I(s,\varphi,E^*)$ decomposes into a product of local integrals over the places of F. We write $I(s,\varphi,E^*) = \prod_v I_v(s,W_v,W_v^{\tau})$, where the local integrals are given by

$$I_v(s,W_v,W_v^{\tau}) = \int_{N_3(F_v)\backslash U_3(F_v)} W_v(g)W_v^{\tau}(s,g)\,dg.$$

Here the functions W_v run through a Whittaker model of π_v (we suppose it exists, for otherwise the global integral is 0). The following can be said about these local integrals. Recall that we are working with explicit Eisenstein series, to be constructed below. In particular, they are suitably normalized.

Theorem 1.2 *Let v be a place of F. We denote the local component of an integral of the above type by $I_v = I_v(s,W_v,W_v^{\tau})$. Then the following assertions hold.*

(i) *The archimedean local integrals I_v converge absolutely for $\Re(s)$ sufficiently large. They have meromorphic continuation to $s \in \mathbb{C}$.*

(ii) *Let v be a finite place of F, with residue field of order q. Then the local integral I_v is a rational function in q^{-s}.*

(iii) *For a finite place v at which U_3, π and τ are unramified, and for which the data in I_v is unramified, the integral equals*

$$I_v(s,W_v,W_v^{\tau}) = L(s+\tfrac{1}{2},\pi_v \times \tau_v). \tag{1.2}$$

This is an Euler factor of degree 12 over F.

The Euler factor appearing at the unramified places can be described precisely as follows. Let ${}^LG = {}^LG^0 \rtimes \mathrm{Gal}(E/F)$ be the L-group of $G = U_3 \times \mathrm{Res}_{E/F}(GL_{2/E})$ in finite Galois form. Here ${}^LG^0 = GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$, and the nontrivial element $c \in \mathrm{Gal}(E/F)$ acts by

$$c(g,h_1,h_2)c^{-1} = \left(\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} {}^tg^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}, h_2, h_1 \right).$$

Let ρ_n denote the standard n-dimensional representation of $GL_n(\mathbb{C})$, and let *triv* denote the trivial representation of $GL_2(\mathbb{C})$. Then $\rho := \mathrm{Ind}({}^LG, {}^LG^0\ ;\ \rho_3 \otimes \rho_2 \otimes \mathrm{triv})$ is a 12-dimensional irreducible representation of ${}^LG$. The local unramified L-factor occurring in (1.2) is then given by the following Langlands type L-factor:

$$L(s+\tfrac{1}{2}, \pi_v \times \tau_v) = L(s+\tfrac{1}{2}, \pi_v \times \tau_v, \rho) = \det\left(\mathbf{1}_{12} - q_v^{-s}\rho(t(\pi_v \times \tau_v))\right)^{-1}.$$

Here $t(\pi_v \times \tau_v) \in {}^LG$ is (any element of) the semisimple conjugacy class attached to $\pi_v \times \tau_v$ by the local Langlands correspondence in the spherical case.

1.3 Application

The residual discrete spectrum of U_4 is described in [11, Theorem 1.1]. Since U_4 has, up to conjugacy, three different proper parabolic subgroups, this spectrum can be viewed as the direct sum of three subspectra, each corresponding to one class of parabolics. Here we are interested in the part coming from the Siegel parabolic, whose Levi component is isomorphic to $\mathrm{Res}_{E/F}\, GL_2$. From the results of Kon-No [11], it follows that the representations which occur in this part are induced from cuspidal representations of the Levi factor $\mathrm{Res}_{E/F}\, GL_2$ of the form $\tau \otimes |\det(\,\cdot\,)|_{\mathbb{A}_E}^{1/2}$, subject to two conditions:

(A) The central character ω_τ of τ has trivial restriction to $\mathbb{A}_F^\times$.
(B) $L(s, \tau, \mathrm{Asai})$ has a simple pole at $s = 1$.

Results by Flicker [2] then imply that τ is the image under the unstable base change from U_2 to GL_2 of a stable cuspidal L-packet on U_2.

More precisely, suppose we fix a character $\mu\colon E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$, whose restriction to $\mathbb{A}_F^\times$ equals the quadratic character associated with E/F by class field theory. We also fix an element $w_0 \in W_F - W_E$, where W_F, W_E denote the Weil group of F and E respectively. These two choices give rise to homomorphisms of L-groups:

$$\xi_1\colon {}^L(U_2) \longrightarrow {}^L(\mathrm{Res}_{E/F}\, GL_2), \tag{1.3}$$

$$\xi_2\colon {}^L(U_2 \times U_1) \longrightarrow {}^LU_3. \tag{1.4}$$

The first map is defined in [2, p. 143], where it is denoted b_κ. The second one is defined in [14, pp. 51–52], where it is denoted ξ_H. However here we insist that in the definition of ξ_2 the character μ is replaced by its inverse μ^{-1}.

Flicker [2] showed that if τ satisfies the two conditions (A) and (B) above, then there exists a stable cuspidal L-packet τ_0 on U_2 that maps to τ under the base change defined by ξ_1. Our result is the following.

Theorem 1.3 *Let τ be an irreducible unitary cuspidal automorphic representation of* $\mathrm{Res}_{E/F}\, GL_2$ *satisfying* (A) *and* (B). *Let τ_0 be the stable cuspidal L-packet on U_2 which maps to τ under the unstable base change correspondence defined by ξ_1.*

Let σ be the global Langlands quotient of $\mathrm{Ind}_{P(\mathbb{A}_F)}^{U_4(\mathbb{A}_F)}(\tau \otimes |\cdot|_{\mathbb{A}_E}^{1/2})$ that occurs in the residual spectrum of U_4. Suppose it acts on the space $V_\sigma \subset L^2_{disc}(U_4(F)\backslash U_4(\mathbb{A}_F))$.

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Via the embedding $U_3 \subset U_4$, we may view the smooth functions in V_σ as automorphic forms on U_3. Denote by $V_{\sigma,0}$ the projection of this space onto the space of cuspforms on U_3, and by σ_0 the representation of U_3 on this space.

(i) *The space $V_{\sigma,0}$ is nonempty.*

(ii) *The constituents of σ_0 are the unique generic cuspidal representations in the endoscopic L-packets $\xi_2(\tilde{\tau}_0 \times \nu)$, as ν runs through the characters of $U_1(F)\backslash U_1(\mathbb{A}_F)$.*

Here $\tilde{\tau}_0$ denotes the contragredient of τ_0. Each of the L-packets $\xi_2(\tilde{\tau}_0 \times \nu)$, or better the L-packet on U_3 associated with $\tilde{\tau}_0 \times \nu$ via the unstable base change defined by ξ_2, contains a unique generic cuspidal representation, by [5, Theorem I].

This theorem, or rather the proof that is given below, has the following corollary.

Corollary 1.4 *Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{Res}_{E/F}\, GL_2$. If there exists an irreducible unitary cuspidal automorphic representation π of U_3 and a finite set S of places of F that includes the archimedean ones such that the partial L-function $L^S(s, \pi \times \tau)$ has a simple pole at $s = 1$, then $L(s, \tau, \mathrm{Asai})$ also has a simple pole at $s = 1$.*

2 The Global Setup and Proof of Theorem 1.1

2.1 Notation

We begin by describing the algebraic groups that appear. Recall that E/F is a CM extension of number fields. Let $(V, \langle \cdot\, , \cdot \rangle)$ be a 4-dimensional hermitian space over E of Witt index 2. Fix a maximal totally isotropic subspace L inside V. Then L defines a maximal parabolic subgroup P of type $(2, 2)$ inside the unitary group of $(V, \langle \cdot\, , \cdot \rangle)$. We also fix an anisotropic line $A \subset V$ and denote its orthogonal complement by $W = A^{\perp}$. The isotropic line $L \cap W =: L_W$ inside W defines a minimal parabolic subgroup of the unitary group associated with W. More precisely, these choices give rise to the following algebraic groups over F:

$$U_4 = U(V), \text{unitary group of } (V, \langle \cdot\, , \cdot \rangle),$$
$$P = \mathrm{Stab}_{U_4}(L) = \{h \in U_4\,;\, h(L) = L\}, \text{ a maximal parabolic of type } (2, 2),$$
$$U_3 = U(W), \text{ unitary group of } (W, \langle \cdot\, , \cdot \rangle|_{W\times W}),$$
$$B_3 = \mathrm{Stab}_{U_3}(L_W), \text{ a minimal parabolic of } U_3,$$
$$B = \mathrm{Stab}_{U_4}\{(0) \subset L_W \subset L \subset (L_W)^{\perp} \subset V\}.$$

So B is a minimal parabolic subgroup of U_4 and is contained in P; U_3 is naturally embedded in U_4. Notice however that B_3 is not contained in P. Denote by N, N_3, respectively N_B, the unipotent radicals of P, B_3, respectively B. In order to fix Levi subgroups for these parabolics, we need to introduce extra structure. Choose a nonzero vector $e \in L_W$, and a second isotropic vector $e' \in W$ such that $\langle e, e' \rangle = 1$. Next we choose two nonzero vectors $a \in A, w \in W$ such that $w \perp (L_W \oplus Ee')$ and such that $\langle w, w \rangle = -\langle a, a \rangle$. Replacing w by a multiple if necessary, we may further assume that $l := w - a \in L$. Set $l' := w + a$.

With these notations, $L' = Ee' \oplus El'$ is a maximal isotropic subspace of V complementary to L. We can now pin down Levi components of our parabolic subgroups as follows. Let M_3 denote elements of U_3 which, written in matrix form with respect to the basis $\{e, w, e'\}$ of W, are diagonal. Similarly let $M_B \subset B$ denote the elements of U_4 that are diagonal with respect to the basis $\{e, l, l', e'\}$ of V. Finally let $M \subset P$ be the unique Levi factor which contains M_B. It consists of 2 by 2 block diagonal matrices with respect to this fixed basis of V.

Set $d = \langle a, a\rangle$. Then $\langle w, w\rangle = -d$ and $\langle l, l'\rangle = -2d$. Therefore with respect to the bases of W and V fixed above, the hermitian pairings are represented by the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -2d & 0 \\ 0 & -2d & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ on } V, \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & -d & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ on } W.$$

The embedding is then given explicitly by

$$U_3 \longrightarrow U_4; \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \longmapsto \begin{pmatrix} a & b & b & c \\ d/2 & (e+1)/2 & (e-1)/2 & f/2 \\ d/2 & (e-1)/2 & (e+1)/2 & f/2 \\ g & h & h & i \end{pmatrix}. \tag{2.1}$$

In what follows, we will often, by abuse of notation, identify elements of U_3 with their images in U_4. The following notation for an element in N_3 will be convenient

$$n(x, y) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & \bar{x}/d \\ 0 & 0 & 1 \end{pmatrix}, \ x, y \in E, \ d \cdot \mathrm{Tr}_{E/F}(y) = N_{E/F}(x). \tag{2.2}$$

2.2 The Global Integral

Now let π be an irreducible unitary cuspidal automorphic representation of U_3. Let V_π denote the space of cuspforms on which π acts by right translation. By multiplicity 1 [14, Theorem 13.3.1], V_π is uniquely determined by π.

Let τ be an irreducible unitary cuspidal automorphic representation of

$$\mathrm{Res}_{E/F}\, GL_2 \approx M \subset U_4.$$

We define a space of Eisenstein series on U_4, following Moeglin and Waldspurger [13]. First we establish some notations. Let $\kappa = \left(\begin{smallmatrix} 0 & 1 \\ -2d & 0 \end{smallmatrix}\right)$. For a matrix $x \in GL_2(\mathbb{A}_E)$, set $\tilde{x} = \kappa^{-1}({}^t\bar{x}^{-1})\kappa$, and $m(x) = \mathrm{diag}(x, \tilde{x}) \in M(\mathbb{A}_F)$ (a 2 by 2 block diagonal matrix in U_4). We will use the same notation when x is an element of $GL_2(E \otimes R)$ for any F-algebra R. Then the modulus character $\delta\colon P(\mathbb{A}_F) \to \mathbb{C}^\times$ is given by

$$\delta(m(x)n) = \left|N_{E/F}(\det x)\right|^2_{\mathbb{A}_F}, \quad x \in GL_2(\mathbb{A}_E), n \in N(\mathbb{A}_F).$$

Fix a compact open subgroup $K_f \subset U_4(\mathbb{A}_{F,f})$ and a maximal compact subgroup $K_\infty \subset U_4(F \otimes_{\mathbb{Q}} \mathbb{R})$. Set $K = K_\infty \times K_f$ and suppose this data chosen such that

$U_4(\mathbb{A}_F) = P(\mathbb{A}_F)K$. Then δ can be extended to a function on $U_4(\mathbb{A}_F)$, still denoted by δ, by setting $\delta(pk) := \delta(p)$, for $p \in P(\mathbb{A}_F)$ and $k \in K$. Let $\tilde{\mathfrak{I}}(\tau)$ denote the space of continuous functions $\tilde{f}$ from $U_4(\mathbb{A}_F)$ into the space of τ which are K-finite on the right and satisfy $\tilde{f}(m(x)ng) = \tau(x)\tilde{f}(g)$ for all $x \in GL_2(\mathbb{A}_E)$, $n \in N(\mathbb{A}_F)$, $g \in U_4(\mathbb{A}_F)$. Then let $\mathfrak{I}(\tau)$ denote the space of functions f on $U_4(\mathbb{A}_F)$ which are of the form $f(g) = \tilde{f}(g)(\mathbf{1_2})$ for some $\tilde{f} \in \tilde{\mathfrak{I}}(\tau)$. Given $f \in \mathfrak{I}(\tau)$, the associated Eisenstein series is defined by

$$E(s, g, f) := \sum_{\gamma \in P(F) \backslash U_4(F)} \delta(\gamma g)^{\frac{s+1}{2}} f(\gamma g). \tag{2.3}$$

It is known [13, p. 85, Proposition] that the sum defining the Eisenstein series converges absolutely and normally in s for $\Re(s)$ sufficiently large. Moreover, [13, p. 140] it has an analytic continuation to a meromorphic function of $s \in \mathbb{C}$. For a fixed value of s away from the poles, it defines an automorphic form on $U_4(\mathbb{A}_F)$; in particular, it is a function of moderate growth. Since the series converges normally in s, the growth condition is satisfied uniformly for s in any compact subset of $\mathbb{C}$ in which E has no poles. In fact, we can normalize the Eisenstein series so that the number of poles is finite. The normalizing L-factor can be determined by analyzing the action of ${}^LM$ on Lie(${}^LN$) by conjugation [6, §I.2.5]. This factor is given by the Asai L-function (as defined in [8, pp. 66–67]). Define

$$E^*(s, g, f) := L(1 + 2s, \tau, \text{Asai})\, E(s, g, f).$$

For a cuspform $\varphi \in V_\pi$ and an Eisenstein series $E^*(s, g, f)$ as above, consider the integral

$$I(s, \varphi, E^*) = \int_{U_3(F) \backslash U_3(\mathbb{A}_F)} \varphi(h) E^*(s, h, f)\, dh.$$

Lemma 2.1 *The integral converges absolutely and uniformly for s in a compact subset of $\mathbb{C}$ in which the Eisenstein series has no poles. Therefore it defines a meromorphic function of $s \in \mathbb{C}$ whose poles are contained in the poles of E^*.*

Proof Let us first show convergence. We need to define a Siegel set of U_3. Let $K_3 \subset U_3(\mathbb{A}_F)$ be a maximal compact subgroup such that the equality $U_3(\mathbb{A}_F) = B_3(\mathbb{A}_F)K_3$ holds. Recall the Levi subgroup M_3 of B_3, consisting of diagonal matrices with respect to the coordinate basis $\{e, w, e'\}$ of W. For an idele $\alpha \in \mathbb{A}_E^\times$, we denote $m_3(\alpha)$ the transformation in $U_3(\mathbb{A}_F)$ that sends e to αe, w to w, and e' to $\bar{\alpha}^{-1}e'$. Fix a compact subset $C \subset B_3(\mathbb{A}_F)$ and a positive real number c. Then we define the Siegel set

$$\Sigma = \Sigma(c, C) = \{pm_3(t)k;\ p \in C, t \in F^+, k \in K, |t| > c > 0\}.$$

Here F^+ denotes the ideles $\alpha \in \mathbb{A}_F^\times$ for which there exists a positive real number r such that $\alpha_v = r$ for every archimedean place v, and $\alpha_v = 1$ for every nonarchimedean place v. By reduction theory it is possible to choose c, C such that $U_3(\mathbb{A}_F) = U_3(F)\Sigma$.

The elements in Σ can also be written in the form $m_3(t)\cdot\omega$, for $|t| > c,\ t \in F^+$ and ω in a fixed compact subset Ω of $U_3(\mathbb{A}_F)$. Therefore to check the convergence of $I(s,\varphi,E^*)$, it suffices to show that the following integral converges uniformly for $\omega \in \Omega$.

$$\int_{\substack{t\in F^+\\ |t|>c}} \varphi(m_3(t)\omega)\, E^*(s, m_3(t)\omega, f)\, |t|^{-2}\, d^\times|t|\,. \tag{2.4}$$

The condition of slow growth says that given any compact subset D of $\mathbb{C}$ in which E^* has no poles, there exist positive constants a, b such that for any $s \in D, \omega \in \Omega, t \in F^+$ with $|t| > c$, $|E^*(s, m_3(t)\omega, f)| \le a\,|t|^b$. The condition of rapid decay says that $\varphi(m_3(t)\omega)$ satisfies the same inequality with the additional fact that b can be chosen to be any real number. (Of course the corresponding a will then depend on b.) Therefore the integral (2.4) can be majorized by a constant multiple of $\int_{t\in F^+,|t|>c} |t|^{-1}\, d^\times t$, and hence is finite. ■

Note that for the convergence of this integral it was crucial that U_3 is of split rank one. If E were globally split, *i.e.*, $E = F \oplus F$ and $U_3 \approx GL_3, U_4 \approx GL_4$, then the corresponding integral does not converge, and one needs to truncate the Eisenstein series. Suppose, for example, that we are in such a completely split situation, and that π has trivial central character. Then the global integral involves integrating over the center of GL_3, and we are essentially integrating a GL_4-Eisenstein series over it.

$$\int_{F^\times\backslash\mathbb{A}_F^\times} E\begin{pmatrix} t &&& \\ & t && \\ && t & \\ &&& 1\end{pmatrix} d^\times t.$$

By the condition of slow growth, this can only be majorized by $\max\{|t|, |t|^{-1}\}^k$ for some positive integer k, which is not enough for the integral to converge.

We can also look at a simpler example, namely the analogous integral for $GL_1 \subset GL_2$, embedded as diagonal matrices whose second entry equals 1. This is the domain of integration for the global integral representation for automorphic L-functions of GL_2. The function to be integrated is an automorphic form on GL_2, from which one subtracts the constant term of its Whittaker–Fourier expansion (see [18, (4.1), p. 199]). This is a more elementary example of the same philosophy, since on GL_2 truncating automorphic forms essentially means subtracting their constant term.

In our nonsplit case, before proving the decomposition of the global integral for certain φ and E^*, which is a standard application of the Rankin–Selberg method, we begin with a double coset analysis and some further geometric considerations.

Lemma 2.2 *The double coset space $P(F)\backslash U_4(F)/U_3(F)$ consists of only one element. In other words, $U_3(F)$ acts transitively on the set of maximal isotropic subspaces of V.*

Proof It is easier to show this result the other way around. Namely, it suffices to show that $P(F)$ acts transitively on the set of lines A' in V whose nonzero elements $a' \in A'$ satisfy $\langle a', a'\rangle \in N(E^\times)\langle a, a\rangle$. But this is well known; it follows, for example, from Witt's theorem. ■

The following lemma follows from the explicit formula (2.1).

Lemma 2.3 *Given the fixed basis of W above, consider an element h in the Borel subgroup B_3 of U_3. Write*

$$h = \begin{pmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \bar{\alpha}^{-1} \end{pmatrix}, \alpha \in E^\times,\ \beta \in E^1,$$

with respect to the fixed basis $\{e, w, e'\}$ of W. Then $h \in B_3 \cap P$ if and only if $\beta = 1$, i.e., *h acts as the identity on the quotient $(Ee)^\perp / Ee$.*

We also need to compare the unipotent radicals N_3 of B_3, N_B of B, and N of P. Since $N_3 \subset N_B$ and $N_3 \cap N = Z_{N_3}$, we can identify the cosets

$$Z_{N_3} \backslash N_3 = N \backslash N_B. \tag{2.5}$$

By this identification we mean that a set of coset representatives of the left-hand side, when embedded into U_4, will be a set of coset representatives of the right-hand side. For a cuspform $\varphi \in V_\pi$, and a nontrivial character ψ of $F\backslash\mathbb{A}_F$, define the associated Whittaker function by

$$W^\varphi(g) = \int_{N_3(F)\backslash N_3(\mathbb{A}_F)} \varphi(n(x,y)g)\ \psi^{-1}(\mathrm{Tr}_{E/F}(x))\, dxdy.$$

The notation $n(x,y)$ was defined above in (2.2). The measure dx on $E\backslash\mathbb{A}_E$ is selfdual with respect to the character $\psi^{-1} \circ \mathrm{Tr}_{E/F}$, similarly the measure dy on $F\backslash\mathbb{A}_F$ is selfdual with respect to ψ. By assumption π is generic, which implies the existence of a character ψ such that the functions $W^\varphi(g)$ are nonzero. Thus we may assume ψ chosen such that this condition is satisfied. Consider the function

$$\varphi_0(g) = \int_{F\backslash\mathbb{A}_F} \varphi(n(0,y)g)\, dy.$$

Define $R := B_3 \cap P$. Then in view of Lemma 2.3, the Whittaker–Fourier expansion along N_3 has the form

$$\varphi_0(g) = \sum_{r \in N(F)\backslash R(F)} W^\varphi(rg). \tag{2.6}$$

With this in mind, we compute the integral, assuming $\Re(s)$ sufficiently large so that the manipulations are justified. We prefer to use the unnormalized Eisenstein

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series, in order not to have to carry around the additional Asai L-factor.

$$
\begin{aligned}
I(s,\varphi,E) &= \int_{U_3(F)\backslash U_3(\mathbb{A}_F)} \varphi(h)E(s,h,f)\,dh \\
&= \int_{U_3(F)\backslash U_3(\mathbb{A}_F)} \varphi(h) \sum_{\gamma\in P(F)\backslash U_4(F)} \delta(\gamma h)^{\frac{s+1}{2}} f(\gamma h)\,dh \\
&= \int_{U_3(F)\backslash U_3(\mathbb{A}_F)} \varphi(h) \sum_{\gamma\in R(F)\backslash U_3(F)} \delta(\gamma h)^{\frac{s+1}{2}} f(\gamma h)\,dh \quad \text{(by Lemma 2.2)} \\
&= \int_{R(F)\backslash U_3(\mathbb{A}_F)} \varphi(h)\delta(h)^{\frac{s+1}{2}} f(h)\,dh \\
&= \int_{R(F)Z_{N_3}(\mathbb{A}_F)\backslash U_3(\mathbb{A}_F)} \varphi_0(h)\delta(h)^{\frac{s+1}{2}} f(h)\,dh \quad (\text{since } Z_{N_3}\subset N) \\
&= \int_{N_3(F)Z_{N_3}(\mathbb{A}_F)\backslash U_3(\mathbb{A}_F)} W^{\varphi}(h)\delta(h)^{\frac{s+1}{2}} f(h)\,dh \quad \text{(by (2.6))} \\
&= \int_{N_3(\mathbb{A}_F)\backslash U_3(\mathbb{A}_F)} \int_{N_3(F)Z_{N_3}(\mathbb{A}_F)\backslash N_3(\mathbb{A}_F)} W^{\varphi}(nh)\delta(nh)^{\frac{s+1}{2}} f(nh)\,dndh \\
&= \int_{N_3(\mathbb{A}_F)\backslash U_3(\mathbb{A}_F)} W^{\varphi}(h) \int_{N(\mathbb{A}_F)N_B(F)\backslash N_B(\mathbb{A}_F)} \psi'(n)\ \delta(nh)^{\frac{s+1}{2}} f(nh)\,dndh \quad \text{(by (2.5))}.
\end{aligned}
$$

Here ψ' denotes the nondegenerate character on the quotient $N(\mathbb{A}_F)N_B(F)\backslash N_B(\mathbb{A}_F)$ that is induced by ψ under the identification (2.5). The inner integral will be given a name:

$$\widetilde{W}^{\tau}(s,g) := \int_{N_B(F)N(\mathbb{A}_F)\backslash N_B(\mathbb{A}_F)} \delta(ng)^{\frac{s+1}{2}} f(ng)\psi'(n)\,dn.$$

The notation is justified since the function $\widetilde{W}^{\tau}(s,g)$ is related to Whittaker functions in the space of τ. Under the isomorphism $M \approx \mathrm{Res}_{E/F}\, GL(2)$, $N_B\cap M$ corresponds to a unipotent radical U of a Borel subgroup of M. Moreover, ψ defines a nondegenerate character on $N_3(F)Z_{N_3}(\mathbb{A}_F)\backslash N_3(\mathbb{A}_F) \approx N_B(F)N(\mathbb{A}_F)\backslash N_B(\mathbb{A}_F)$, hence also of $U(F)\backslash U(\mathbb{A}_F)$. If we denote this character by ψ'', then we obtain

$$
\begin{aligned}
\widetilde{W}^{\tau}(s,g) &= \delta(g)^{\frac{s+1}{2}} \int_{N_B(F)N(\mathbb{A}_F)\backslash N_B(\mathbb{A}_F)} f(ng)\psi'(n)\,dn \\
&= \delta(g)^{\frac{s+1}{2}} \int_{U(F)\backslash U(\mathbb{A}_F)} f(m(u)g)\ \psi''(u)\,du \\
&= \delta(g)^{\frac{s+1}{2}} \int_{U(F)\backslash U(\mathbb{A}_F)} \tilde{f}(g)(u)\psi''(u)\,du.
\end{aligned}
$$

The last integral is nothing but the Whittaker function of $\tilde{f}(g)$ along U with respect to the character $(\psi'')^{-1}$, evaluated at the identity. It will be convenient to set

$W^\tau(s,g) = L(1+2s,\tau,\mathrm{Asai})\widetilde{W}^\tau(s,g)$. Then for suitable choice of data the functions $W^\tau(s,g)$ and $\widetilde{W}^\tau(s,g)$ are decomposable into a product of local functions. We denote the local components at a place v of F by $W_v^\tau(s,g)$ and $\widetilde{W}_v^\tau(s,g)$, respectively. This finishes the proof of the global Theorem 1.1. ■

Note that the computations in this section can be performed with either the unnormalized or the normalized Eisenstein series. All statements are correct for both of them. This is because they differ by a meromorphic function $L(1+2s,\tau,\mathrm{Asai})$, which has the two properties that it decomposes into an Euler product for large real part of s, and that it is bounded at infinity in vertical strips of finite width, and hence does not affect convergence questions.

Both Eisenstein series have advantages. The normalized one E^* has fewer poles and gives rise to a nicer formula for the local integrals in Theorem 1.2. The unnormalized one E is needed for the application in Theorem 1.3.

3 The Local Integral

3.1 The Local Integral at Nonsplit Nonarchimedean Places

Suppose v is a finite place of F which remains prime in E. Let w denote the place of E lying above v. Let $q = q_v$ denote the order of the residue field of F at v. We consider the integrals

$$I_v(s, W_v, W_v^\tau) = \int_{N_3(F_v)\backslash U_3(F_v)} W_v(g) W_v^\tau(s,g)\, dg.$$

Here W_v runs through the functions in the Whittaker model $\mathcal{W}(\pi_v,\psi_v)$ of π_v, and W_v^τ belongs to a space of functions defined as follows. We consider functions $u(s,g)$ for which there exists a compact open subgroup $K_v \subset U_4(F_v)$ such that $u(s,\,\cdot\,)$ has its support in $P(F_v)K_v$ and is right K_v invariant. Moreover, we require that there exist a function $W \in \mathcal{W}(\tau_v,\psi_v'')$ in the Whittaker model of τ_v, defined using the unipotent radical $U(F_v)$ and the character $(\psi'')_v^{-1}$, with the following property. Whenever $n_v \in N(F_v), k_v \in K_v, x_v \in GL(2,E_w)$ are such that $n_v m(x_v) k_v = g_v$ lies in the support of $u(s,\,\cdot\,)$, then

$$u(s,g_v) = L_v(1+2s,\tau_v,\mathrm{Asai})\delta(m(x_v))^{\frac{s+1}{2}} W(x_v). \tag{3.1}$$

We require that W_v^τ be a finite linear combination of such functions $u(s,g)$.

Proposition 3.1 (i) *The integrals $I_v(s,W_v,W_v^\tau)$ converge absolutely for $\Re(s)$ sufficiently large. They are rational functions of q^{-s} and therefore can be analytically continued as functions of s to the entire complex plane.*

(ii) *Suppose v is unramified in E and π_v and τ_v are both spherical. Let*

$$t(\pi_v \times \tau_v) = \begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} \times \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix} \times \mathbf{1}_2 \rtimes c$$

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be the Langlands parameter attached to them. Let W_v^0 be the normalized spherical Whittaker function of π_v, and $W_v^{\tau,0} = u(s,g)$ be right K_v-invariant, for K_v such that $K_v \cap M(F_v) = GL_2(\mathcal{O}_{E_w})$, $K_v \cap U_3(F_v) = U_3(\mathcal{O}_v)$, and moreover such that W (in (3.1)) is the normalized spherical Whittaker function of τ_v. Then

$$\begin{aligned}(3.2)\quad & I_v(s, W_v^0, W_v^{\tau,0}) \\ &= L_v(s+1/2, \tau_v \times \pi_v) = \det(\mathbf{1}_{12} - q^{-s}\rho(t(\pi_v \times \tau_v)))^{-1} \\ &= \prod_{i=1,2}(1-\beta_i q^{-2(s+1/2)})(1-\beta_i\alpha q^{-2(s+1/2)})(1-\beta_i\alpha^{-1}q^{-2(s+1/2)}).\end{aligned}$$

We remark that the Euler factor can also be interpreted through the standard base change $BC(\pi_v)$ of π_v to $GL(3, E_w)$. Namely, it is the Rankin–Selberg convolution L-factor of $BC(\pi_v) \times \tau_v$ of degree 6 over E_w, as defined in [9]. The group $U_3(\mathcal{O}_v)$ denotes a fixed hyperspecial maximal compact subgroup of $U_3(F_v)$, (which exists by our choice of U_3) to be quasisplit and v to be unramified in E.

Proof Since the functions under the integral are smooth in the algebraic sense, we see that, given W_v, W_v^τ as above, there exists a finite number of matrices $k_i \in U_3(\mathcal{O}_v)$, $1 \le i \le n$, such that

$$(3.3)\qquad I_v(s, W_v, W_v^\tau) = \sum_{i=1}^n \int_{E_w^\times} W_v(m_3(\alpha)k_i)W_v^\tau(m_3(\alpha)k_i)\,|\alpha|_{F_v}^{-2}\,d^\times\alpha.$$

Now the Whittaker functions in the space of π_v and τ_v can be uniformly bounded by gauges. More precisely there exists a positive real number r such that for any $W \in W(\pi_v, \psi_v)$ there exists a Bruhat–Schwartz function Φ on E_w such that

$$\forall \alpha \in E_w^\times, n \in N_3(F_v), k \in K_v:\ |W(nm_3(\alpha)k)|_{\mathbb{C}} \le |\alpha|_{E_w}^r\,\Phi(\alpha).$$

With that bound, together with a similar well-known one for $GL_2(E_w)$, the integrals $I_v(s, W_v, W_v^\tau)$ can be majorized in absolute value by a finite sum of integrals of the form

$$\int_{E_w^\times} \Phi'(\alpha)\,|\alpha|_{E_w}^{r'+s}\,d^\times\alpha,$$

for a constant real number r' that only depends on π_v and τ_v, and a Bruhat–Schwartz function Φ' that depends on the particular ingredients. In any case, this Tate-type integral converges for $\Re(s)+r' > 0$. The assertion that the local integrals are rational functions in q^{-s} follows from (3.3), by taking into account the fact that Whittaker functions restricted to the diagonal $\{m_3(\alpha), \alpha \in E_w^\times\}$, are finite sums of products of Bruhat–Schwartz functions on E_w with finite functions on $E_w^\times$.

The result (ii) rests upon the fact that since W_v^0 and $W_v^{\tau,0}$ are invariant under the maximal open compact subgroup $U_3(\mathcal{O}_v)$, the integral in question becomes essentially a sum. More precisely, using the Iwasawa decomposition

$$U_3(F_v) = N(F_v)A(F_v)U_3(\mathcal{O}_v)$$

with A being a maximal F-split torus of U_3, we obtain

$$\begin{aligned} I_v(s, W^0, W_v^{\tau,0}) &= \sum_{n\in\mathbb{Z}} W^0(m_3(\varpi_v^n))\, W_v^{\tau,0}\big(s, \operatorname{diag}(\varpi_v^n, 1, 1, \varpi_v{}^{-n})\big)\ \ |\varpi_v^n|_{F_v}^{-4} \\ &= L_v(1+2s, \tau_v, \mathrm{Asai}) \\ &\quad \times \sum_{n\in\mathbb{Z}} W^0(m_3(\varpi_v^n))\, |\varpi_v^n|_{F_v}^{2(s+1)}\, W\big(\operatorname{diag}(\varpi_v^n, 1)\big)\ |\varpi_v^n|_{F_v}^{-4}. \end{aligned}$$

The absolute value factor on the right appears because of the expression of the Haar measure with respect to the Iwasawa decomposition. The formula (3.2) now follows from the standard formulas for the spherical Whittaker functions, which are well known for GL_2. For U_3 they are given in [1, Theorem 5.4]. Namely, the integral $I_v(s, W^0, W_v^{\tau,0})$ equals $L_v(2s+1, \tau_v, \mathrm{Asai})$ times

$$\begin{aligned} &\sum_{n\geq 0} |\varpi_v|_{E_w}^n \,\frac{\alpha^{n+1} - \alpha^{-n-1}}{\alpha - \alpha^{-1}}\, |\varpi_v|_{F_v}^{2n(s+1)+2\frac{n}{2}} \sum_{k+l=n} (\beta_1^k \beta_2^l)\, |\varpi_v^n|_{F_v}^{-4} \\ &\quad = \frac{1}{\alpha - \alpha^{-1}} \sum_{k,l\geq 0} [\alpha(\alpha\beta_1)^k(\alpha\beta_2)^l - \alpha^{-1}(\alpha^{-1}\beta_1)^k(\alpha^{-1}\beta_2)^l] q^{-2(k+l)(s+1/2)} \\ &\quad = (1 - \beta_1\beta_2 q^{-2(2s+1)}) \times \prod_{i=1,2} \Big(1 - \frac{\alpha\beta_i}{q^{2s+1}}\Big)^{-1} \Big(1 - \frac{\alpha^{-1}\beta_i}{q^{2s+1}}\Big)^{-1} \end{aligned}$$

Since $L_v(2s+1, \tau_v, \mathrm{Asai}) = (1-\beta_1 q^{-(2s+1)})^{-1}(1-\beta_2 q^{-(2s+1)})^{-1}(1-\beta_1\beta_2 q^{-2(2s+1)})^{-1}$, this finishes the proof in the nonsplit nonarchimedean case. ■

3.2 The Local Integral at Split Nonarchimedean Places

Suppose v is a finite place of F which splits in E. In this section, $|\cdot|$ denotes the normalized absolute value on $F_v^\times$. Let $q = q_v$ be the order of the residue field of F at v. Denote the two places of E lying above v by w_1, w_2. Choosing one place w_1 is equivalent to fixing an embedding of E into F_v. Supposing we have made this (non-canonical) choice, there are isomorphisms

$$GL_4(E \otimes_F F_v) \approx GL_4(F_v) \times GL_4(F_v), \quad g \otimes 1 \longmapsto (g, \overline{g}). \tag{3.4}$$

Composing the natural embedding of $U_4(F \otimes_F F_v) \hookrightarrow GL_4(E \otimes_F F_v)$ with the projection onto the first factor defines an isomorphism $U_4(F_v) \approx GL_4(F_v)$. Similarly $U_3(F_v) \approx GL_3(F_v)$. We remark here that in the computations that follow we actually must check that the result we obtain does not depend on our initial choice of a place w_1. This fact translates into a symmetry condition among the Euler factors. It is satisfied by all our results.

Now π_v is an irreducible representation of $GL_3(F_v)$, and $\tau_v = \tau_{w_1} \otimes \tau_{w_2}$ is an irreducible representation of $GL_2(E_{w_1}) \times GL_2(E_{w_2}) \approx GL_2(F_v) \times GL_2(F_v)$. The local zeta integrals we are considering are

$$I_v(W_v, W_v^\tau) = \int_{N_3(F_v)\backslash GL_3(F_v)} W_v(g) W_v^\tau(s,g)\, dg.$$

Here W_v denotes a function in the Whittaker model $\mathcal{W}(\pi_v, \psi_v)$ of π_v. Under the identification $U_3(F_v) \approx GL_3(F_v)$, $N_3(F_v)$ gets identified with the unipotent upper triangular matrices in $GL_3(F_v)$, and ψ_v is a nondegenerate character of this group.

The functions $W_v^\tau(s,g)$ are defined analogously to the inert case, but let us make some of the differences more precise. Again W_v^τ is a finite linear combination of functions $u(s,g)$ which are smooth in the algebraic sense and satisfy the following condition. There exists a compact open subgroup $K_v \subset U_4(F_v)$ such that $u(s,\cdot)$ has its support in $P(F_v)K_v$ and is right K_v invariant. Moreover we require that there exists a function $W \in \mathcal{W}(\tau_v, \psi_v)$ in the Whittaker model of τ_v, with the following property. Whenever $n_v \in N(F_v)$, $k_v \in K_v$, $x_v \in GL(2, E\otimes F_v)$ are such that $n_v m(x_v) k_v = g_v$ lies in the support of $u(s,\cdot)$, then

$$u(s, g_v) = L_v(1+2s, \tau_v, \mathrm{Asai})\delta(m(x_v))^{\frac{s+1}{2}} W(m(x_v)).$$

Now the Whittaker model is a tensor product

$$\mathcal{W}(\tau_v, \psi_v) = \mathcal{W}(\tau_{w_1}, \psi_{w_1}) \otimes \mathcal{W}(\tau_{w_2}, \psi_{w_2}).$$

Assume for simplicity that W corresponds to a pure tensor $W_1 \otimes W_2$ under this isomorphism. Then if an element $m_v \in M(F_v)$ corresponds to the pair (m_1, m_2) under the identification $M(F_v) = GL(2, E\otimes F_v) = GL(2, E_{w_1}) \times GL(2, E_{w_2})$, the following identities hold.

$$\delta(m_v) = |\det(m_1)|_{E_{w_1}}^2 \, |\det(m_2)|_{E_{w_2}}^2 = |\det(m_1 m_2)|^{2,} \tag{3.5}$$

$$W(m_v) = W_1(m_1) W_2(m_2). \tag{3.6}$$

On the other hand, when we identify $U_4(F_v)$ with $GL_4(F_v)$ using the place w_1 and the basis $\{e, l, l', e'\}$, then $M(F_v)$ maps onto the 2×2 block diagonal matrices. Suppose the element $m_v \mapsto \mathrm{diag}(A, D), A, D \in GL_2(F_v)$ under this identification. Now $M(F)$ consists of matrices (with respect to the same basis) of the form $\mathrm{diag}(X, \tilde{X}), X \in GL_2(E)$ and $\tilde{X} = \kappa^{-1\,t}\bar{X}^{-1}\kappa$. Recall that $\kappa = \left(\begin{smallmatrix} 0 & 1 \\ -2d & 0 \end{smallmatrix}\right)$. If we compare this with the identification (3.4) above, we find that

$$m_1 = A, \qquad m_2 = {}^t(\kappa D \kappa^{-1})^{-1}.$$

Using (3.5) and (3.6), one obtains the following formulas, which will be used in the computations below.

$$W(\mathrm{diag}(A, D)) = W_1(A) W_2({}^t(\kappa D\kappa^{-1})^{-1}), \tag{3.7}$$

$$\delta(\mathrm{diag}(A, D)) = \left|\det(AD^{-1})\right|^2. \tag{3.8}$$

In this context we call $W_v^\tau(s,g)$ unramified if it is right invariant by $GL_4(\mathcal{O}_v)$ and the functions W_1 and W_2 are normalized spherical Whittaker functions.

Proposition 3.2 (i) *The integrals $I_v(W_v, W_v^\tau)$ converge absolutely for $\Re(s)$ sufficiently large. They are rational functions of q^{-s} and therefore can be analytically continued to the entire complex plane.*

(ii) *Suppose τ_{w_1}, τ_{w_2}, and π_v are spherical. Let the Langlands parameters be*

$$t(\pi_v \times \tau_v) = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix} \times \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix} \times \begin{pmatrix} \gamma_1 & \\ & \gamma_2 \end{pmatrix} \rtimes 1.$$

Suppose W_v^0 is the normalized spherical Whittaker function of π_v, and $W_v^{\tau,0}$ is unramified in the sense described above. Then

$$\begin{aligned} I_v(s, W_v^0, W_v^{\tau,0}) &= L_v(s+1/2, \tau_v \times \pi_v) \\ &= \det(\mathbf{1}_{12} - q^{-s}\rho(t(\pi_v \times \tau_v)))^{-1} \\ &= \prod_{i=1}^{3}\prod_{j=1}^{2} (1 - \alpha_i\beta_j q^{-(s+\frac{1}{2})})^{-1}\,(1-\alpha_i^{-1}\gamma_j q^{-(s+\frac{1}{2})})^{-1}. \end{aligned} \tag{3.9}$$

Again the main Euler factor may be interpreted in terms of the Rankin–Selberg convolution of $\tau_v = \tau_{w_1} \otimes \tau_{w_2}$ and the standard base change of π_v to $GL(3, E\otimes_F F_v) = GL(3, E_{w_1}) \times GL(3, E_{w_2})$.

Proof Let us note some general facts about these integrals. First, due to the Iwasawa decomposition $GL_3(F_v) = N_3(F_v)A(F_v)\,GL_3(\mathcal{O}_v)$, the domain of the integral in question consists of the last two factors. On the other hand, for convergence questions we may assume that the support of W_v^τ is contained in $P(F_v)K_v \subset GL_4(F_v)$. So one needs to know the Iwasawa decomposition in U_4 of an element in the image of U_3. Since for most practical purposes, *i.e.,* convergence and computation of the unramified case, it suffices to consider diagonal matrices, we introduce the following notation: $d(a,b,c)$ will denote the element of $U_3(F_v) \approx GL_3(F_v)$ given by a diagonal matrix with entries $a, b, c \in F_v^\times$. We then have (*cf.* (2.1))

$$\iota(d(a,b,c)) = \begin{pmatrix} a & & & \\ & \frac{b+1}{2} & \frac{b-1}{2} & \\ & \frac{b-1}{2} & \frac{b+1}{2} & \\ & & & c \end{pmatrix} \in U_4(F_v) \approx GL_4(F_v).$$

If v does not divide 2, then the Iwasawa decomposition $U_4(F_v) = P(F_v)U_4(\mathcal{O}_v)$ of the matrix on the right-hand side is given as follows:

$$|b| = 1: \ \iota(d(a,b,c)) = \begin{pmatrix} a & & \\ & \mathbf{1}_2 & \\ & & c \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \frac{b+1}{2} & \frac{b-1}{2} & \\ & \frac{b-1}{2} & \frac{b+1}{2} & \\ & & & 1 \end{pmatrix},$$

$$|b| \neq 1: \ \iota(d(a,b,c)) = \begin{pmatrix} a & & & \\ & \frac{2b}{b-1} & \frac{b+1}{2} & \\ & 0 & \frac{b-1}{2} & \\ & & & c \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & -1 & \\ & 1 & \frac{b+1}{b-1} & \\ & & & 1 \end{pmatrix}.$$

If v divides 2, there are more cases for $|b| = 1$. Again for the purpose of showing convergence we may suppose that given W_v^τ, there exist two Whittaker functions $W_i \in \mathcal{W}(\tau_{w_i}, \psi''_{w_i})$, $i = 1, 2$, such that the following equality holds (*cf.* (3.7)).

$$W_v^\tau\left(s, \left(\begin{smallmatrix} A & 0_2 \\ 0_2 & D \end{smallmatrix}\right)\right) = L_v(2s+1, \tau_v, \mathrm{Asai}) W_1(A) W_2(\kappa{}^tD^{-1}\kappa^{-1}) \left|\det(AD^{-1})\right|^{s+1}.$$

Now we write the local integral as a sum of three terms and show for each one that it converges absolutely for $\Re(s)$ sufficiently large. The first integral will be over the domain where b has absolute value 1. Since $(\mathcal{O}_{F_v})^\times$ is compact, and the functions involved K_v-finite, we may as well assume, for the purpose of showing convergence, that $b = 1$. The second term will be over $|b| < 1$ and the third one over $|b| > 1$. We need to use in all cases one basic result concerning bounds on Whittaker functions. It follows directly from [9, Proposition 2.2, p. 181]. This result has already been used in the nonsplit case, but it is recalled only here because in the split case one must be more careful about convergence questions.

Lemma 3.3 *Let σ be an irreducible generic representation of $GL_n(F_v)$. Suppose it is realized in its Whittaker model $\mathcal{W}$ with respect to some nondegenerate character. Then there exists a positive number r which only depends on τ, such that for a given Whittaker function $W \in \mathcal{W}$ there exists a positive Bruhat–Schwartz function $\Phi \in \mathcal{S}(F_v^{n-1})$ with the following property.*

$$|W(\mathrm{diag}(a_1, a_2, \dots, a_n))|_{\mathbb{C}} \leq \Phi\left(\frac{a_1}{a_2}, \dots, \frac{a_{n-1}}{a_n}\right) |a_1 a_2 \dots a_n|_{F_v}^r.$$

Using this result we now show that each of the three terms converges. The first one, where we set $b = 1$, simply becomes

$$\begin{aligned} \int_{(F_v^\times)^2} & W_v(d(a,1,c)) W_v^\tau(s, \iota(d(a,1,c))) \left|\frac{a}{c}\right|^{-2} d^\times a\, d^\times c \\ &= \int_{(F_v^\times)^2} W_v(d(a,1,c)) W_1\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} W_2\begin{pmatrix} c^{-1} & 0 \\ 0 & 1 \end{pmatrix} \left|\frac{a}{c}\right|^{s-1} d^\times a\, d^\times c. \end{aligned}$$

Using the lemma and making a change of variables $c \mapsto c^{-1}$, we can bound this expression in absolute value by

$$\int_{(F_v^\times)^2} \Phi(a,c)\, |ac|^{r+s-1}\, d^\times a\, d^\times c.$$

Again r does not depend on the particular Whittaker functions, and $\Phi \in \mathcal{S}(F_v^2)$. This is a local integral of Tate-type, hence it converges for $\Re(s) > 1 - r$.

For the second and third term, we need to use the above matrix identities. For simplicity of exposition we also assume that v does not divide 2. Then in the case

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$|b| < 1$ we need to bound an integral of the form

$$\int_{\substack{(F_v^\times)^3\\|b|<1}} W_v\big(d(a,b,c)\big)\, W_v^\tau\big(s, \iota(d(a,b,c))\big)\, \Big|\frac{a}{c}\Big|^{-2}\, d^\times a\, d^\times b\, d^\times c =$$

$$\int_{\substack{(F_v^\times)^3\\|b|<1}} W_v\big(d(a,b,c)\big)\, W_1\begin{pmatrix} a & 0\\ 0 & \frac{2b}{b-1}\end{pmatrix} W_2\begin{pmatrix} c^{-1} & 0\\ 0 & \frac{2}{b-1}\end{pmatrix}$$

$$\times \Big|\frac{4ab}{(b-1)^2c}\Big|^{s+1}\, \Big|\frac{a}{c}\Big|^{-2}\, d^\times a\, d^\times b\, d^\times c.$$

First we note that $|b-1| = 1$. Then, using Lemma 3.3, we can bound this expression by

$$\int_{\substack{(F_v^\times)^3\\|b|<1}} \Phi_1\Big(\frac{a}{b}, \frac{b}{c}\Big)\, \Phi_2\Big(\frac{a(b-1)}{2b}\Big)\, \Phi_3\Big(\frac{b-1}{2c}\Big)\, \Big|\frac{4ab}{c}\Big|^{r'+s+1}\, \Big|\frac{a}{c}\Big|^{-2}\, d^\times a\, d^\times b\, d^\times c.$$

Here again the Φ_i's are appropriate Bruhat–Schwartz functions. Notice that they combine to essentially one Bruhat–Schwartz function on F_v^4 evaluated at the variables $(a/b, b/c, 1/c, a)$. Thus, changing c to c^{-1} once again, we see that we are essentially dealing with Tate-type integrals and hence obtain absolute convergence for $\Re(s) > 1 - r'$.

The third integral for $|b| > 1$ works similarly to the second, except that now $|b-1| = |b|$, and we need to make an extra change of variables $b \mapsto b^{-1}$. This establishes the absolute convergence. It is then a direct consequence of the known behavior of Whittaker functions on the diagonal that the resulting integral is a rational function of q^{-s}.

We now calculate the integrals in the spherical case. Suppose $\iota(U_3(\mathcal{O}_v)) \subset U_4(\mathcal{O}_v)$ and we are in the situation of Proposition 3.2(ii). Then the integral reduces to an integral over the diagonal $A(F_v) \approx (F_v^\times)^3$.

$$I_v(s, W_v^0, W_v^{\tau,0}) = \int_{(F_v^\times)^3} W_v^0\big(d(a,b,c)\big)\, W_v^{\tau,0}\big(\iota(d(a,b,c))\big)\, \big|ac^{-1}\big|^{-2}\, d^\times a\, d^\times b\, d^\times c.$$

This in turn reduces to a triple infinite sum. Because of the absolute convergence for $\Re(s)$ large that we just established, we can rearrange the summands as we like without changing the result.

For $(n,m,r) \in \mathbb{Z}^3, n \ge m \ge r$ (respectively $(n,m) \in \mathbb{Z}^2, n \ge m$) we denote by $\rho_{(n,m,r)}$ (respectively $\rho_{(n,m)}$) the irreducible finite dimensional representation of $GL_3(\mathbb{C})$ of highest weight (n,m,r) (respectively of $GL_2(\mathbb{C})$ of highest weight (n,m)). We write $\chi_{(n,m,r)}(A)$ for the trace of $\rho_{(n,m,r)}$ evaluated at a matrix $A \in GL_3(\mathbb{C})$, and define similarly $\chi_{(n,m)}(B), B \in GL_2(\mathbb{C})$.

We split $I_v(s, W_v^0, W_v^{\tau,0})$ into three parts as before. The term corresponding to $|b| = 1$ will contribute $L(1+2s, \tau_v, \text{Asai})$ times the expression

$$\sum_{\substack{n,m\in\mathbb{Z}\\ a=\varpi^n, c=\varpi^m}} W_v^0(d(\varpi^n, 1, \varpi^m))\, W_1\begin{pmatrix} \varpi^n & \\ & 1\end{pmatrix} W_2\begin{pmatrix} \varpi^{-m} & \\ & 1\end{pmatrix} \big|\varpi^{n-m}\big|^{s-1}.$$

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Clearly the nonzero contributions arise only when $n \geq 0 \geq m$. Changing m to $-m$, this equals

$$(3.10)\qquad \sum_{n,m=0}^{\infty} |\varpi|^{n+m} \chi_{(n,0,-m)}(A)\, |\varpi|^{n/2} \chi_{(n,0)}(B)\, |\varpi|^{m/2} \chi_{(m,0)}(C)\, |\varpi|^{(n+m)(s-1)}.$$

Here the notation means

$$A = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3), \quad B = \operatorname{diag}(\beta_1, \beta_2), \quad C = \operatorname{diag}(\gamma_1, \gamma_2).$$

Next we investigate the contributions from $|b| < 1$. Now we obtain $L(1+2s, \tau_v, \text{Asai})$ times a triple sum which looks as follows (since $|2b/(b-1)| = |b|$, $|(b-1)/2| = 1$):

$$\sum_{\substack{n,m\in\mathbb{Z}\\ a=\varpi^n, c=\varpi^m}} \sum_{\substack{r=1\\ b=\varpi^r}}^{\infty} W_v^0(d(\varpi^n, \varpi^r, \varpi^m)) W_1 \begin{pmatrix} \varpi^n & \\ & \varpi^r \end{pmatrix} \times W_2 \begin{pmatrix} \varpi^{-m} & \\ & 1 \end{pmatrix} |\varpi^{n+r-m}|^{s+1} |\varpi^{n-m}|^{-2}.$$

Making the shift $a = \varpi^{r+n}$, $b = \varpi^r$, $c = \varpi^{-m}$, we see that the nonzero terms add up to

$$(3.11)\qquad \sum_{n,m=0}^{\infty} \sum_{r=1}^{\infty} |\varpi|^{n+m+r} \chi_{(n+r,r,-m)}(A)\, |\varpi|^{n/2} \chi_{(n+r,r)}(B) \times |\varpi|^{m/2} \chi_{(m,0)}(C)\, |\varpi|^{(n+m+2r)(s+1)} |\varpi|^{-2(n+m+r)}.$$

Finally the same analysis as above shows that the integral over $|b| > 1$ contributes the Asai L-factor times the following infinite sum (use $|2b/(b-1)| = 1$, $|(b-1)/2| = |b|$, and set $a = \varpi^n$, $b = \varpi^{-r}$, $c = \varpi^{-m-r}$):

$$(3.12)\qquad \sum_{n,m=0}^{\infty} \sum_{r=1}^{\infty} |\varpi|^{n+m+r} \chi_{(n,-r,-m-r)}(A)\, |\varpi|^{n/2} \chi_{(n,0)}(B) \times |\varpi|^{m/2} \chi_{(m+r,r)}(C)\, |\varpi|^{(n+m+2r)(s+1)} |\varpi|^{-2(n+m+r)}.$$

Combining the three separate contributions (3.10), (3.11), and (3.12), the unramified local integral $I_v(s, W_v^0, W_v^{\tau,0})$ equals $L(1+2s, \tau_v, \text{Asai})$ times

$$(3.13)\qquad \sum_{n,m=0}^{\infty} q^{-(s+1/2)(n+m)} \chi_{(n,0)}(B) \chi_{(m,0)}(C) \times \Big(\chi_{(n,0,-m)}(A) + \sum_{r=1}^{\infty} q^{-(s+1/2)2r} \chi_{(n+m+r,m+r,0)}(A) (\beta_1\beta_2)^r (\alpha_1\alpha_2\alpha_3)^{-m} + \sum_{r=1}^{\infty} q^{-(s+1/2)2r} \chi_{(n+m+r,n+r,0)}(A^{-1}) (\gamma_1\gamma_2)^r (\alpha_1\alpha_2\alpha_3)^n \Big).$$

In the last line, we used the fact that the contragredient representation of $\rho_{(a,b,c)}$ is $\rho_{(-c,-b,-a)}$, and that the corresponding characters satisfy $\chi_\rho(A) = \chi_{\check{\rho}}(A^{-1})$. As written above, it is clear that the local integral has the expected symmetry.

Next note that, for $a \geq b \geq 0$,

$$\chi_{(a,b,0)}(A) = \frac{1}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)} \sum_{\tau \in S_3} \operatorname{sgn}(\tau) \alpha_{\tau(1)}^{a+2} \alpha_{\tau(2)}^{b+1}.$$

Let us set $\epsilon = ((\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3))^{-1}$, and $X = q^{-s-1/2}$. Using this, we compute that the sum of the second and third line in (3.13) equals

$$\epsilon \sum_{\tau \in S_3} \operatorname{sgn}(\tau) \alpha_{\tau(1)}^{n+2} \alpha_{\tau(2)} \alpha_{\tau(3)}^{-m} \left(\frac{1}{1 - \beta_1\beta_2\alpha_{\tau(1)}\alpha_{\tau(2)}X^2} + \frac{\gamma_1\gamma_2\alpha_{\tau(2)}^{-1}\alpha_{\tau(3)}^{-1}X^2}{1 - \gamma_1\gamma_2\alpha_{\tau(2)}^{-1}\alpha_{\tau(3)}^{-1}X^2} \right).$$

Combining this with the sum over n and m, the unramified local integral $I_v(s, W_v^0, W_v^{\tau,0})$ equals $L(1 + 2s, \tau_v, \mathrm{Asai})$ times the sum over $\tau \in S_3$ of

$$\frac{\epsilon \operatorname{sgn}(\tau) \alpha_{\tau(1)}^2 \alpha_{\tau(2)} (1 - \alpha_{\tau(1)}\alpha_{\tau(3)}^{-1}\beta_1\beta_2\gamma_1\gamma_2X^4)}{(1 - \beta_1\beta_2\alpha_{\tau(1)}\alpha_{\tau(2)}X^2)(1 - \gamma_1\gamma_2\alpha_{\tau(2)}^{-1}\alpha_{\tau(3)}^{-1}X^2)} \prod_{i=1}^{2} (1 - \beta_i\alpha_{\tau(1)}X)^{-1}(1 - \gamma_i\alpha_{\tau(3)}^{-1}X)^{-1}.$$

This implies that $I_v(s, W_v^0, W_v^{\tau,0})$ equals $L(1 + 2s, \tau_v, \mathrm{Asai})$ times $L(s + 1/2, \pi_v \times \tau_v)$ times

$$\epsilon \sum_{\tau \in S_3} \operatorname{sgn}(\tau) \alpha_{\tau(1)}^2 \alpha_{\tau(2)} \frac{1 - \alpha_{\tau(1)}\alpha_{\tau(3)}^{-1}\beta_1\beta_2\gamma_1\gamma_2X^4}{(1 - \beta_1\beta_2\alpha_{\tau(1)}\alpha_{\tau(2)}X^2)(1 - \gamma_1\gamma_2\alpha_{\tau(2)}^{-1}\alpha_{\tau(3)}^{-1}X^2)}$$
$$\times \prod_{i=1}^{2} (1 - \beta_i\alpha_{\tau(2)}X)(1 - \beta_i\alpha_{\tau(3)}X)(1 - \gamma_i\alpha_{\tau(1)}^{-1}X)(1 - \gamma_i\alpha_{\tau(2)}^{-1}X).$$

At this point, proving the local formula (3.9) is equivalent to showing the equality of the following two expressions, which we may view as polynomials of degree 20 in X.

$$\begin{aligned}
&\sum_{\tau \in S_3} \operatorname{sgn}(\tau) \alpha_{\tau(1)}^2 \alpha_{\tau(2)} (1 - \alpha_{\tau(1)}\alpha_{\tau(3)}^{-1}\beta_1\beta_2\gamma_1\gamma_2X^4)(1 - \beta_1\beta_2\alpha_{\tau(1)}\alpha_{\tau(3)}X^2) \\
&\quad \times (1 - \beta_1\beta_2\alpha_{\tau(2)}\alpha_{\tau(3)}X^2)(1 - \gamma_1\gamma_2\alpha_{\tau(1)}^{-1}\alpha_{\tau(2)}^{-1}X^2)(1 - \gamma_1\gamma_2\alpha_{\tau(1)}^{-1}\alpha_{\tau(3)}^{-1}X^2) \\
&\quad \times \prod_{i=1}^{2} (1 - \beta_i\alpha_{\tau(2)}X)(1 - \beta_i\alpha_{\tau(3)}X)(1 - \gamma_i\alpha_{\tau(1)}^{-1}X)(1 - \gamma_i\alpha_{\tau(2)}^{-1}X) \\
&\overset{!}{=} \epsilon^{-1} \prod_{i,j=1}^{2} (1 - \beta_i\gamma_jX^2) \prod_{i=1}^{3} (1 - \alpha_1\alpha_2\alpha_3\beta_1\beta_2\alpha_i^{-1}X^2)(1 - \alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1}\gamma_1\gamma_2\alpha_iX^2).
\end{aligned}$$

It turns out that the terms of odd degree in X on the left-hand side all vanish, and that the two expressions are equal. The verification is lengthy, but straightforward. This finishes the proof of Proposition 3.2. ■

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3.3 The Local Integral at Archimedean Places

Let v be a real place of F, so that $F_v \approx \mathbb{R}, E_v \approx \mathbb{C}$. We consider local integrals of the form

$$I_v(s, W_v, W_v^\tau) = \int_{N_3(F_v)\backslash U_3(F_v)} W_v(g) W_v^\tau(s, g)\, dg.$$

Here W_v denotes a function in the smooth Whittaker model $\mathcal{W}(\pi_v, \psi_v)$ of π_v, as described in [12, §5.3]. Let $\mathcal{W}(\tau_v, \psi_v)$ denote the space of smooth functions in the Whittaker model of τ_v. Let $K_v \subset U_4(F_v)$ be a maximal compact subgroup. Then we require of $W_v^\tau(s, g)$ that it be smooth in g, right K_v-finite, left $N(F_v)$-invariant, and that for any g the function on $M(F_v) \approx GL(2, \mathbb{C})$ defined by

$$m_v \mapsto L(1 + 2s, \tau_v, \mathrm{Asai})^{-1} \delta(m_v)^{-\frac{s+1}{2}} W_v^\tau(s, m_v g)$$

is independent of s and lies in $\mathcal{W}(\tau_v, \psi_v)$ (for values of s at which there is no pole).

Lemma 3.4 *There exists a positive real number r such that for $\Re(s) > r$ and any two functions W_v, W_v^τ, the integrals $I_v(s, W_v, W_v^\tau)$ converge absolutely and normally in s. Moreover, they can be analytically continued to meromorphic functions of $s \in \mathbb{C}$.*

Proof By the Iwasawa decomposition it suffices to show the convergence and analytic continuation of integrals of the form

$$(3.14)\quad \int_{F_v^\times} W_v(\mathrm{diag}(t, 1, t^{-1})) W_v^\tau(s, \mathrm{diag}(t, 1, 1, t^{-1}))\, |t|_{F_v}^{-2}\, d^\times t$$
$$= L(1 + 2s, \tau_v, \mathrm{Asai}) \int_{F_v^\times} W_v(\mathrm{diag}(t, 1, t^{-1})) W \begin{pmatrix} t & \\ & 1 \end{pmatrix} |t|_{F_v}^{2s}\, d^\times t.$$

Here $W \in \mathcal{W}(\tau_v, \psi_v)$ is a Whittaker function on $GL(2, \mathbb{C})$ associated with τ_v. For such a function there is a well-known bound due to Jacquet and Shalika. The following lemma recalls their result [10, Proposition 4.].

Lemma 3.5 *There exists a finite set X of finite functions on $\mathbb{C}^\times$ such that for all $V \in \mathcal{W}(\tau_v, \psi_v)$ there exist Bruhat–Schwartz functions $\phi_x \in \mathcal{S}(\mathbb{C}), x \in X$ such that*

$$V \begin{pmatrix} t & \\ & 1 \end{pmatrix} = \sum_{x \in X} \phi_x(t) \cdot x(t).$$

A similar result holds for the functions W_v on $U_3(F_v)$ when restricted to the largest split torus. It was proved by Soudry for quasisplit SO_{2l+1} in [15, Proposition 3.3], and can be obtained for U_3 in a similar manner (see also Watanabe [17, (4.2)]). Taking into account how finite functions on $\mathbb{R}^\times$ look, the integral (3.14) therefore reduces to a finite sum of terms of the form

$$L(1 + 2s, \tau_v, \mathrm{Asai}) \int_{F_v^\times} \phi(t)\, |t|_{\mathbb{R}}^{2s+n} \log |t|^m d^\times t.$$

In the above equation, n is a real number, m a nonnegative integer, and $\phi \in \mathcal{S}(\mathbb{C})$. It is known that such integrals have the desired convergence and analytic continuation properties, hence the same is true for $I_v(s, W_v, W_v^\tau)$. This finishes the proof of Lemma 3.4. ■

The precise determination of archimedean Zeta integrals is quite subtle, as can be seen in the work of Koseki and Oda [12]. In the present case an additional difficulty is given by the fact that these local integrals also involve Whittaker functions on $GL_2(\mathbb{C})$ coming from the Levi factor M of the parabolic subgroup $P \subset U_4$.

However we would like to point out one important property that these integrals have. Namely, if we quotient out by the Asai L-factor and set

$$I_v^*(s, W_v, W_v^\tau) = I_v(s, W_v, W_v^\tau) \cdot L_v(1 + 2s, \tau, \mathrm{Asai})^{-1},$$

then the meromorphic properties do not change. Moreover, the following is true.

Lemma 3.6 *For any $s_0 \in \mathbb{C}$ there exist finitely many data $W_{v,i}, W_{v,i}^\tau$, $1 \le i \le r$, such that the sum $\sum_{i=1}^r I_v^*(s, W_{v,i}, W_{v,i}^\tau)$ is holomorphic and nonzero at $s = s_0$.*

Proof We present a sketch of the argument. If $\Re(s)$ is sufficiently large, the function $t \mapsto W_v(\mathrm{diag}(t, 1, t^{-1}))\,|t|^{2s}$ belongs to $L^2(F_v^\times)$, and it is known that the functions $W \in \mathcal{W}(\tau_v, \psi_v)$, when restricted to matrices of the form $\mathrm{diag}(t, 1)$, are dense in this space. Therefore we can choose, for any given W_v, a function W_v^τ such that the function

$$k \mapsto F(k) := \int_{F_v^\times} W_v(\mathrm{diag}(t, 1, t^{-1})k) W_v^\tau(s, \iota(\mathrm{diag}(t, 1, t^{-1})k))\,|t|_{F_v}^{-2}\, d^\times t$$

is not identically zero. Here $k \in K \subset U_3(F_v)$ is a maximal compact subgroup such that the Iwasawa decomposition $U_3(F_v) = N_3(F_v)\,\mathrm{diag}(t, 1, t^{-1})K, t \in F_v^\times$, holds. Then $I_v^*(s, W_v, W_v^\tau) = \int_K F(k)\, dk$. Since $F \ne 0$, and since we are free to replace W_v by $\pi_v(\phi)W_v$ for any smooth function ϕ on K, this proves the lemma in the case $\Re(s_0) \gg 0$.

To pass to the case of an arbitrary complex number, one proceeds analogously to [15, Proposition 7.2]. (See also [17, Lemma 6].) One first shows that linear combinations of integrals $I_v^*(s, W_v, W_v^\tau)$, viewed as meromorphic functions of s, contain the space of archimedean Rankin–Selberg integrals on suitable general linear groups. But for these, which are studied in [10], the desired nonvanishing result is known. Note that contrary to the case of large real part, in general we can only assure that a finite linear combination is holomorphic and nonzero, and we have to include functions which are not necessarily K-finite. ■

4 Proof of the Application

First let us restate the content of the two main theorems in a way that is more suitable for certain applications, including the one we have in mind here.

Theorem 4.1 *Let π be an irreducible generic unitary cuspidal automorphic representation of U_3. Let τ be an irreducible cuspidal automorphic representation of $GL(2,E)$. Let $\varphi \in \pi$ denote a cuspform, and let $E(s,g,f)$ denote an Eisenstein series on U_4, defined as in* (2.3), i.e., *unnormalized with respect to the functional equation. Then there exists a finite set S of places of F, including all the archimedean ones, and a meromorphic function $C(s)$, defined as a finite product of local integrals, such that for $\Re(s) \gg 0$,*

$$\int_{U_3(F)\backslash U_3(\mathbb{A}_F)} \varphi(g)E(s,g,f)\,dg = C(s)\cdot\frac{L^S(s+1/2,\pi\times\tau)}{L^S(1+2s,\tau,\mathrm{Asai})}. \tag{4.1}$$

It is clear from the proofs that a little more can be said about the function $C(s)$. For us, what is most important, is that it can be analytically continued to a meromorphic function of $s \in \mathbb{C}$, and that for a fixed number s_0 one can find suitable choices for φ and E such that the resulting $C(s)$ is holomorphic and nonzero at $s = s_0$.

Next we recall a description of the part of the residual spectrum of U_4 that comes from the Siegel parabolic P. The following result was proved in [11, Theorem 4.4].

Theorem 4.2 **(T. Kon-No)** *Let τ be an irreducible unitary cuspidal automorphic representation of $GL_2(E)$ whose central character ω_τ has trivial restriction to $\mathbb{A}_F^\times$. Moreover assume that $L(s,\tau,\mathrm{Asai})$ has a (necessarily simple) pole at $s = 1$. Then the global Langlands quotient of $\mathrm{Ind}_{P(\mathbb{A}_F)}^{U_4(\mathbb{A}_F)}(\tau\otimes|\det(\cdot)|_{\mathbb{A}_E}^{1/2})$ appears in the residual spectrum of U_4. These representations appear with multiplicity one, and they, with their direct sum (over τ), comprise all of the residual spectrum that arises from cuspidal data from the parabolic P.*

We remark that in rephrasing Kon-No's theorem we also used the fact that the pole condition of the Asai L-function is equivalent to the nonvanishing of certain period integrals (see [8, §3.13], or the appendix of [11]).

Now fix a τ that satisfies the conditions of the above theorem. The pole of the Asai L-function implies that the representation τ is isomorphic to its Galois conjugate τ', which is defined by $\tau'(g) = \tau(\bar{g})$. Moreover, by [2, Theorem 1], the pole of the Asai L-function implies that τ is the image of a stable L-packet on U_2 under the unstable base change lift.

The unstable base change lift is defined via the homomorphism ξ_1 of L-groups (1.3). So let τ_0 denote the unique stable cuspidal global L-packet on U_2 whose unstable base change is τ. We remark that if τ_0 has central character ω_0, that is, if one and hence all representations in the packet have this central character, then the central character of τ satisfies $\omega_\tau(z) = \omega_0(z/\bar{z})\mu(z)^2$.

Now let $E(s,g,f)$ be an Eisenstein series on U_4, as defined in (2.3), corresponding to τ. Then, due to our definition, $E(s,g,f)$ has a simple pole at $s = 1/2$, and the space of functions $g \mapsto \mathrm{Res}_{s=1/2} E(s,g,f)$ occurs in the residual spectrum of U_4. Let $V_{\sigma(\tau)}$ denote the space of functions on U_4 so obtained, and $\sigma(\tau)$ the representation of U_4 on this space. Consider now the restriction of functions in $V_{\sigma(\tau)}$ to U_3.

Let π be a unitary irreducible cuspidal automorphic representation of U_3. If π is not generic, then the global integral (1.1) is identically zero, so π does not occur in any Siegel induced automorphic representation of U_4. So let us assume that π is generic. Recall the two facts that $L(s, \tau, \mathrm{Asai})$ is holomorphic and nonzero at $s = 2$ and that the data on the left-hand side of the equality (4.1) can be chosen so that $C(1/2) \neq 0$. From this it follows that π and the restriction of $\sigma(\tau)$ to U_3 have a nonzero L^2-pairing if and only if $L^S(s, \pi \times \tau)$ has a simple pole at $s = 1$.

But this L-function equals the Rankin–Selberg convolution of τ and the standard (or stable) base change of π to $GL_3(E)$. Thus if, for example, π lies in a stable L-packet, then its base change is cuspidal and $L^S(s, \pi \times \tau)$ is entire. Hence the only possibility for a pole is if π is endoscopic. More precisely, the following is true. Since τ is the unstable base change of τ_0, the following equality of local L-factors holds, at least for almost all places v of F.

$$L_v(s, \tau) = L_v(s, BC_u(\tau_0)) = L_v(s, BC(\tau) \otimes \mu).$$

Here we denote by BC_u the unstable base change and by BC the stable base change. Therefore $L^S(s, \pi \times \tau) = L^S(s, BC(\pi) \times BC(\tau_0) \otimes \mu)$. Particularly, if the base change of π is cuspidal, then this standard Rankin–Selberg L-function has no poles. On the other hand, from the explicit description of the discrete spectrum of U_3 given in [14], it also follows for which π $L^S(s, \pi \times \tau)$ does have a pole. Namely, if π lies in a packet that is the endoscopic transfer with respect to ξ_2 ((1.4), which uses μ^{-1}) of an L-packet $\rho_2 \times \rho_1$ on $U_2 \times U_1$, then

$$L_v(s, \pi) = L_v(s, BC(\rho_2) \otimes \mu^{-1}) \cdot L_v(s, BC(\rho_1)).$$

Combining these facts, we see that in this case,

$$L^S(s, \pi \times \tau) = L^S(s, BC(\rho_2) \times BC(\tau_0)) \cdot L^S(s, BC(\rho_1) \otimes BC(\tau_0)).$$

The second factor has no poles and is nonzero at $s = 1$, since τ_0 is stable. The first has a simple pole at $s = 1$ precisely when $\rho_2 \approx \tilde{\tau}_0$. This proves Theorem 1.3.

The Corollary 1.4 to Theorem 1.3 follows directly from the arguments used in the above proof. For clarity, we emphasize again that $L^S(s, \tau, \mathrm{Asai})$ has a simple pole at $s = 1$ if and only if $E(s, g, f)$ has a simple pole at $s = 1/2$, for τ an irreducible unitary cuspidal automorphic representation of $\mathrm{Res}_{E/F}\, GL(2)$. All of these poles contribute to the residual spectrum of U_4. Also, if an irreducible cuspidal automorphic representation π of U_3 is not endoscopic, then it is stable, hence its base change to $GL(3)$ is cuspidal, and the Rankin–Selberg L-function $L(s, \pi \times \tau)$ is entire. In particular, it has no poles.

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