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CAUCHY PROBLEMS FOR SECOND ORDER HYPERBOLIC DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

by J. S. LOWNDES

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1. It is well known [1] that there is a one-to-one relation between solutions of the Darboux equation and the wave equation. The purpose of this paper is to show that some recent results in the fractional calculus can be used to obtain a similar connection between solutions of Darboux's equation and second order linear hyperbolic differential equations with constant coefficients.

If we denote by L_η the singular differential operator

$$L_\eta = t^{-(2\eta+1)} \frac{\partial}{\partial t} t^{2\eta+1} \frac{\partial}{\partial t}, \tag{1}$$

then two relevant results to be found in [1] are contained in the following two lemmas.

Lemma 1. *All homogeneous second order linear hyperbolic differential equations with constant coefficients can be reduced to the form*

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - (L_{-1/2} + k^2)u = 0 \tag{2}$$

where $u = u(x,t)$, $x = (x_1, x_2, \ldots, x_n)$ *and* k *is a constant.*

Lemma 2. *Darboux's differential equation*

$$\sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} - L_{\frac{n-2}{2}} v = 0 \tag{3}$$

with the initial conditions

$$v(x,0) = f(x), \quad v_t(x,0) = 0 \tag{4}$$

has the solution

$$v(x,t) = Q_n(x,t) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_S f(x + \beta t)\, d\beta \tag{5}$$

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which is the mean *value of f over the hypersphere S of radius t and centre $x=(x_1,x_2,\dots,x_n)$. In the integral $\beta=(\beta_1,\beta_2,\dots,\beta_n)$ is a unit vector, $d\beta$ is the surface element of the unit hypersphere and $t^{n-1}d\beta=dS$ is the surface element of S.*

2. The generalised Riemann–Liouville operators of fractional integration I_k^α and I_{ik}^α are defined in [3] by the expressions

$$I_\lambda^\alpha f(t)=2^\alpha k^{1-\alpha}\int_0^t \xi(t^2-\xi^2)^{(\alpha-1)/2}G_\lambda(k\sqrt{t^2-\xi^2})f(\xi)\,d\xi, \qquad \alpha>0, \tag{6}$$

$$=\mathscr{D}^m I_\lambda^{\alpha+m}f(t), \qquad \alpha<0, \tag{7}$$

where $\lambda=k$ or $\lambda=ik$, $k\geqq 0$, $G_k(z)=J_{\alpha-1}(z)$ is the Bessel function of the first kind, $G_{ik}(z)=I_{\alpha-1}(z)$ is the modified Bessel function of the first kind, m is a positive integer such that $0<\alpha+m\leqq 1$ when $\alpha<0$ and $\mathscr{D}$ denotes the differential operator

$$\mathscr{D}=\frac{\partial}{\partial t^2}=\frac{1}{2t}\frac{\partial}{\partial t}. \tag{8}$$

In terms of the above operators we can write the generalised Erdélyi–Kober operators [2] as

$$\mathfrak{I}_\lambda(\eta,\alpha)f(t)=t^{-2(\alpha+\eta)}I_\lambda^\alpha t^{2\eta}f(t), \tag{9}$$

whose inverses are given by

$$\mathfrak{I}_k^{-1}(\eta,\alpha)=\mathfrak{I}_{ik}(\eta+\alpha,-\alpha), \mathfrak{I}_{ik}^{-1}(\eta,\alpha)=\mathfrak{I}_k(\eta+\alpha,-\alpha). \tag{10}$$

A useful result connecting the operators L_η and $\mathfrak{I}_\lambda(\eta,\alpha)$ is described in the following lemma [2].

Lemma 3. *If $\alpha>0, f\in C^2(0,b)$ for some $b>0$, $t^{2\eta+1}f(t)$ is integrable at the origin and $t^{2\eta+1}f'(t)\to 0$ as $t\to 0$; then*

$$\mathfrak{I}_\lambda(\eta,\alpha)L_\eta f(t)=(L_{\eta+\alpha}+\lambda^2)\mathfrak{I}_\lambda(\eta,\alpha)f(t) \tag{11}$$

where $\lambda=k$ or $\lambda=ik$, $k\geqq 0$.

3. From Lemma 2 and the above results we can now obtain the solution to the corresponding Cauchy problem for the hyperbolic equation (2).

To do this we set

$$v(x,t)=\frac{\Gamma(n/2)}{\Gamma(1/2)}\mathfrak{I}_{ik}\left(-\tfrac{1}{2},\frac{n-1}{2}\right)u(x,t) \tag{12}$$

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in Darboux's equation (3) and use the result (11) to find that it becomes

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - (L_{-1/2} + k^2)u = 0 \tag{13}$$

and the initial conditions (4) become

$$\lim_{t \to 0} \frac{\Gamma(n/2)}{\Gamma(1/2)} \mathfrak{I}_{ik}\left(-\tfrac{1}{2}, \frac{n-1}{2}\right) u(x, t) = u(x, 0) = f(x), \tag{14}$$

$$u_t(x, 0) = 0. \tag{15}$$

Hence from equations (5) and (12) we see that the initial-value problem for the hyperbolic equation described by equations (13) to (15) has the solution

$$\begin{aligned} u(x, t) &= \frac{\Gamma(1/2)}{\Gamma(n/2)} \mathfrak{I}_{ik}^{-1}\left(-\tfrac{1}{2}, \frac{n-1}{2}\right) v(x, t) \\ &= \frac{\Gamma(1/2)}{\Gamma(n/2)} \mathfrak{I}_{k}\left(\frac{n-2}{2}, \frac{1-n}{2}\right) Q_n(x, t) \\ &= \frac{\Gamma(1/2)}{\Gamma(n/2)} t I_k^{(1-n)/2} t^{n-2} Q_n(x, t), \end{aligned} \tag{16}$$

where we have used the results (9) and (10).

The solution can further be simplified if we consider separately the cases when n is even or odd and we find that when n is even

$$\begin{aligned} u(x, t) &= \frac{\Gamma(1/2)}{\Gamma(n/2)} t \mathscr{D}^{n/2} I_k^{1/2} t^{n-2} Q_n(x, t) \\ &= \frac{2t}{\Gamma(n/2)} \mathscr{D}^{n/2} \int_0^t \xi^{n-1} \frac{\cos(k\sqrt{t^2 - \xi^2})}{\sqrt{t^2 - \xi^2}} Q_n(x, \xi)\, d\xi \end{aligned} \tag{17}$$

and when n is odd

$$\begin{aligned} u(x, t) &= \frac{\Gamma(1/2)}{\Gamma(n/2)} t \mathscr{D}^{(n+1)/2} I_k^{1} t^{n-2} Q_n(x, t) \\ &= \frac{2\Gamma(1/2)}{\Gamma(n/2)} t \mathscr{D}^{(n+1)/2} \int_0^t \xi^{n-1} J_0(k\sqrt{t^2 - \xi^2}) Q_n(x, \xi)\, d\xi \end{aligned} \tag{18}$$

where $\mathscr{D}$ is defined by equation (8).

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4. We can also use the above results to obtain the solution of the complementary Cauchy problem

$$\sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} - (L_{-\frac{1}{2}} + k^2)w = 0, \tag{19}$$

$$w(x, 0) = 0, \qquad w_t(x, 0) = f(x), \tag{20}$$

since it is known that its solution is

$$w(x, t) = \int_0^t u(x, t)\, dt, \tag{21}$$

where $u(x, t)$ is given by equations (17) and (18).

In this way we find that when n is even

$$w(x, t) = \frac{\Gamma(1/2)}{2\Gamma(n/2)} \int_0^t \frac{\partial}{\partial t} \{\mathscr{D}^{(n-2)/2} I_k^{1/2} t^{n-2} Q_n(x, t)\}\, dt$$

$$= \frac{\Gamma(1/2)}{2\Gamma(n/2)} \mathscr{D}^{(n-2)/2} I_k^{1/2} t^{n-2} Q_n(x, t) \tag{22}$$

and when n is odd

$$w(x, t) = \frac{\Gamma(1/2)}{2\Gamma(n/2)} \mathscr{D}^{(n-1)/2} I_k^{1} t^{n-2} Q_n(x, t) \tag{23}$$

As a simple example we see from equation (22) that when $n = 2$ the solution is

$$w(x, t) = \int_0^t \xi \frac{\cos(k\sqrt{t^2 - \xi^2})}{\sqrt{t^2 - \xi^2}} Q_2(x, \xi)\, d\xi$$

where

$$Q_2(x, \xi) = \frac{1}{2\pi} \int_0^{2\pi} f(x_1 + \xi \cos\theta, x_2 + \xi \sin\theta)\, d\theta$$

and this agrees with the result given in [1] which was obtained by a different method.

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Department of Mathematics
University of Strathclyde
Glasgow, G1 1XH

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