On the Set of Common Differences in van der Waerden's Theorem on Arithmetic Progressions

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Abstract. Analogues of van der Waerden's theorem on arithmetic progressions are considered where the family of all arithmetic progressions, AP, is replaced by some subfamily of AP. Specifically, we want to know for which sets A, of positive integers, the following statement holds: for all positive integers r and k, there exists a positive integer $n = w'(k, r)$ such that for every r-coloring of $[1, n]$ there exists a monochromatic k-term arithmetic progression whose common difference belongs to A. We will call any subset of the positive integers that has the above property *large*. A set having this property for a specific fixed r will be called *r-large*. We give some necessary conditions for a set to be large, including the fact that every large set must contain an infinite number of multiples of each positive integer. Also, no large set $\{a_n : n = 1, 2, \dots\}$ can have $\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} > 1$. Sufficient conditions for a set to be large are also given. We show that any set containing n-cubes for arbitrarily large n, is a large set. Results involving the connection between the notions of "large" and "2-large" are given. Several open questions and a conjecture are presented.

1 Introduction

An arithmetic progression of length k is a set $P = \{x + id : i = 0, \dots, k - 1\}$ where x and d are integers, $d > 0$. We call d the *common difference* of P. Van der Waerden's classic theorem on arithmetic progressions [13] says that, for each positive integer r, if the set of positive integers, Z^+, is partitioned into r classes, then at least one of the classes will contain arbitrarily long arithmetic progressions. An alternate (and equivalent) form of van der Waerden's theorem says that for all positive integers k and r, there exists a least positive integer $w(k, r)$ such that for every partition of the interval $[1, w(k, r)] = \{1, 2, \dots, w(k, r)\}$ into r classes, at least one of the classes will contain a k-term arithmetic progression. A partition of a set into r classes is often referred to as an *r-coloring* of the set. So van der Waerden's theorem can be stated: for all positive integers r and k, there exists a positive integer $w(k, r)$ so that for every r-coloring of $[1, w(k, r)]$ there exists a monochromatic k-term arithmetic progression.

Analogues of van der Waerden's theorem may be considered, where the family of arithmetic progressions, AP, is replaced by some other family of integer sequences. That is, if r is a positive integer and T is a family of integer sequences, we can ask whether for every r-coloring of the positive integers there are arbitrarily long monochromatic members of T. If T is a family that does have this property, we say that T has the *r-Ramsey property*. If T has the r-Ramsey property for all r, we simply say that T has the *Ramsey property*.

By van der Waerden's theorem, if T includes all the arithmetic progressions, then T has the Ramsey property. The Ramsey functions associated with T (*i.e.*, the functions $w'(k, r)$,

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analogous to $w(k, r)$, but where members of T are sought rather than members of AP) have been studied for a variety of such T (see [4], [8], [10]).

In this paper, we wish to consider the Ramsey property for collections T that are not supersets of AP, but rather *subsets* of AP. This is of interest because if T is a proper subset of AP, and T has the Ramsey property, the conclusion to van der Waerden's theorem is strengthened.

Of course, if $T \subseteq$ AP is too small, it will not have the Ramsey property. For example, it is well known that if F is any finite set of positive integers, then it is possible to 2-color the positive integers in such way that there do not exist arbitrarily long monochromatic arithmetic progressions with common differences belonging to F (one proof can be found in [2]; this fact is also a consequence of Theorem 2.1 below).

On the other hand, one simple consequence (and strengthening) of van der Waerden's theorem is that if F is a fixed finite set of positive integers, then the family of all arithmetic progressions having common differences in $Z^+ - F$ has the Ramsey property. In fact, it is easy to see that if m is a fixed positive integer, then the family of all arithmetic progressions having common differences in the set mZ^+ has the Ramsey property (by van der Waerden's theorem, every r-coloring of $\{m, 2m, \ldots, (w(k, r))m\}$ produces a k-term monochromatic arithmetic progression, and this progression has common difference in mZ^+).

The examples just mentioned lead us to ask the general question: which subcollections of AP have the Ramsey property?

For this paper, we consider those subcollections of AP which consist of all arithmetic progressions having common differences in a given set A.

In this paper, we will call a subset A of the positive integers *large* if the collection T, consisting of all arithmetic progressions having common differences in A, has the Ramsey property. A subset of the positive integers that is not large is called *small*. Similarly, if T has the r-Ramsey property, we will say that A is *r-large*. Thus, we seek an answer to the question: which sets of positive integers are large?

Furstenberg, using dynamical systems methods, showed in [5] that every infinite cube (see definition below) is large.

Definition   Let $\{a_1, a_2, \ldots, a_n\}$ be any set of positive integers. The *n-cube* $Q(a_1, \ldots, a_n)$ is the set of all linear combinations $c_1 a_1 + \cdots + c_n a_n$ such that $c_i \in \{0, 1\}$ for all i, but where not all the c_i are 0. Similarly, the *infinite cube* $Q(a_1, a_2, \ldots)$ consists of all finite linear combinations of $a_1, a_2, \ldots$, with coefficients in $\{0, 1\}$, except for 0. (Note: in the literature, an "*n*-cube" often refers to any translate of our $Q(a_1, \ldots, a_n)$; also, 0 is often included in the definition, and then our $Q(a_1, \ldots, a_n)$ is called a "punctured *n*-cube").

More recently Bergelson and Leibman [2, Corollary 1.9] showed, using measure-preserving systems methods and ergodic theory, that if C is any infinite cube, and $p(x)$ is any polynomial with integer coefficients, positive leading coefficient, and $p(0) = 0$, then $\{p(x) : x \in C\} \cap Z^+$ is large (in particular, $\{p(x) : x \geq 1\} \cap Z^+$ is large). In fact, they showed that if S is any subset of Z^+ with positive upper density, then S contains arbitrarily long arithmetic progressions with common differences of the form $p(x)$, $x \in C$.

For an excellent exposition of some of these results and methods, see [1] and [7].

Here we give some results on large sets using completely elementary methods. In particular, we strengthen Furstenberg's result mentioned above by showing (Theorem 3.2 below)

that if the set A contains an n-cube for arbitrarily large n, then A is large. Our Corollary 2.1 and Theorem 2.4 below are similar to results of Furstenberg [6, Theorem 3.2 and Lemma 3.3] which deal with recurrence in measure-preserving systems.

2 Necessary Conditions

We begin with some conditions that are necesssary for a set to be large. Of course any condition that is necessary for 2-large sets is also necessary for large sets.

For convenience, we will often use the term "d-a.p." to refer to an arithmetic progression whose common difference is d. Further, if A is a set of positive integers and if P is a d-a.p. where $d \in A$, we say that P is an "A-a.p."

Theorem 2.1   *If A is 2-large, then for each positive integer m, A contains an infinite number of multiples of m.*

Proof   It suffices to show that for every positive integer m, A contains some multiple of m. By way of contradiction, assume A contains no multiples of the positive integer n. Color the positive integers with the coloring χ represented by the sequence $11 \ldots 1\, 00 \ldots 0\, 11 \ldots 1\, 00 \ldots 0 \ldots$, where each block of 0's or 1's has length n.

Let $d \in A$ and let $X = \{x_1, \ldots, x_{n+1}\}$ be a d-a.p. Since $d \equiv i \pmod{2n}$ for some $1 \leq i \leq 2n-1$, we see that X must contain some element, a, of the form $2k_1 n + j_1$, $1 \leq j_1 \leq n$, as well as some element, b, of the form $2k_2 n + j_2$, $n + 1 \leq j_2 \leq 2n$. Then $\chi(a) \neq \chi(b)$, so X is not monochromatic. Hence under χ there do not exist monochromatic $(n + 1)$-term arithmetic progressions having common difference in A. Hence A is not 2-large. ∎

Before stating the next theorem we adopt some notation and terminology. If I and J are intervals of the same size having opposite color patterns (*i.e.*, whenever x is in position i of I, and y is in position i of J, then $\chi(x) \neq \chi(y)$), then if C is a string of 0's and 1's representing the color pattern of I, we say that J has color pattern $\overline{C}$. Also, if χ is a 2-coloring and I and J are intervals of the same size with color patterns C and D, respectively, such that either $C = D$ or $C = \overline{D}$, then we say that I and J *imitate* each other under χ.

Theorem 2.2   *Let $A = \{a_k\}_{k=1}^{\infty}$ be a sequence of positive integers where either*

(1) $$a_k \geq 3a_{k-1} \quad \text{for } k \geq 2$$

or

(2) $$a_1 = 1, \quad a_2 = 2, \quad \text{and} \quad a_k \geq 3a_{k-1} \quad \text{when } k \geq 3.$$

Then A is not 2-large.

Proof   We will prove that $\{a_k\}_{k=1}^{\infty}$ is not 2-large whenever

(3) $$a_1 = 1, \quad a_2 \geq 2, \quad \text{and, for } k \geq 3, \quad a_k \geq 3a_{k-1}.$$

This is sufficient for the theorem since (2) is a special case of (3), and any A satisfying (1) is a subset of some A' satisfying (3).

Let $A = \{a_k\}$ satisfy (3). Then for each $k \geq 3$, a_k can be expressed uniquely in the form

$$(4) \qquad\qquad a_k = \sum_{j=1}^{k-1} c_j^{(k)} a_j,$$

where the $c_j^{(k)}$ are defined recursively as follows: (i) $c_{k-1}^{(k)}$ is the largest positive integer such that $a_k \geq (c_{k-1}^{(k)} + 1) a_{k-1}$; (ii) if $k \geq 4$, then for each $j = 2, \ldots, k - 2$, once $c_{k-1}^{(k)}, c_{k-2}^{(k)}, \ldots, c_{j+1}^{(k)}$ have been defined, define $c_j^{(k)}$ to be the largest integer such that

$$a_k \geq \left(\sum_{i=j+1}^{k-1} c_i^{(k)} a_i \right) + (c_j^{(k)} + 1) a_j;$$

(iii) finally, $c_1^{(k)}$ is defined so that $a_k = \sum_{i=1}^{k-1} c_i^{(k)} a_i$. For $k = 2$, set $c_1^{(k)} = c_1^{(2)} = a_2$. It then follows from (3) that $c_i^{(k)} \geq 2$ for all $k \geq 2$ and all $1 \leq i \leq k - 1$. Thus, for each $k \geq 2$, we can partition $[1, a_k]$ into subintervals:

$$B^{(k)}(k-1, 1) \ldots B^{(k)}(k-1, c_{k-1}^{(k)}) B^{(k)}(k-2, 1) \ldots B^{(k)}(k-2, c_{k-2}^{(k)}) \ldots$$

$$\ldots B^{(k)}(1, 1) \ldots B^{(k)}(1, c_1^{(k)}),$$

where $|B^{(k)}(i, j)| = a_i$ for $1 \leq i \leq k - 1$, $1 \leq j \leq c_i^{(k)}$, and where the subintervals are listed such that subinterval B precedes subinterval B' when the members of B are less than the members of B'.

Let χ be the 2-coloring of the positive integers defined by: (i) $\chi(1) = 1$; (ii) once $[1, a_1], \ldots, [1, a_{k-1}]$ have been colored, color $[1, a_k]$ by coloring the $B^{(k)}(i, j)$ as follows:

$$\chi\big(B^{(k)}(i, j)\big) = \begin{cases} C_i & \text{if } j \text{ is odd} \\ \overline{C_i} & \text{if } j \text{ is even} \end{cases}$$

where C_i denotes the color pattern for $[1, a_i]$.

We show that under χ there is no 5-term monochromatic a_m-a.p. for any m. Let $m > 0$ be fixed and let $\{x_1, \ldots, x_5\}$ be a 5-term a_m-a.p. Let k be the smallest positive integer such that $k > m$ and $x_1 \in [1, a_k]$. We consider three cases.

Case 1

$x_1 \in B^{(k)}(i, j)$ where $1 \leq i \leq m - 1$ and $1 \leq j \leq c_i^{(k)}$. Let

$$S = \bigcup \{ B^{(k)}(i, j) : 1 \leq i \leq m - 1, 1 \leq j \leq c_i^{(k)} \}.$$

Then $|S| = \sum_{i=1}^{m-1} c_i^{(k)} a_i$. By the way $c_m^{(k)}$ is defined, we have

$$a_k \leq \left(\sum_{i=m+1}^{k-1} c_i^{(k)} a_i \right) + (c_m^{(k)} + 2) a_m.$$

Hence, using (4), $2a_m \geq |S|$. Therefore $x_3 \notin S$.

We consider two subcases. In case $x_2 \in S$, then x_3 and x_4 both belong to $[a_k + 1, a_k + 2a_m]$. Hence, by the definition of k it follows that x_3 and x_4 both belong to an interval that imitates $B^{(m+1)}(m, 1) \cup B^{(m+1)}(m, 2)$. Then from the definition of χ, $\chi(x_3) \neq \chi(x_4)$. In case $x_2 \notin S$, then by the same reasoning we have that $\chi(x_2) \neq \chi(x_3)$. Thus, in either subcase, $\{x_1, x_2, x_3, x_4\}$ is not a 4-term monochromatic a_m-a.p.

Case 2

$x_1 \in B^{(k)}(m, j)$ for some j, $1 \leq j \leq c_m^{(k)}$. Now $a_m \leq a_k - \sum_{i=m}^{k-1} c_i^{(k)} a_i$, so that $a_m \leq |S|$. Therefore $x_2 \in S$, so by Case 1 $\{x_2, x_3, x_4, x_5\}$ cannot be monochromatic.

Case 3

$x_1 \in B^{(k)}(i, j)$ where $m + 1 \leq i \leq k - 1$ and $1 \leq j \leq c_i^{(k)}$. The block $B^{(k)}(i, j)$ imitates $[1, a_i]$ under χ, *i.e.*, it imitates

$$B^{(i)}(i - 1, 1) \ldots B^{(i)}(i - 1, c_{i-1}^{(i)}) \quad B^{(i)}(i - 2, 1) \ldots B^{(i)}(i - 2, c_{i-2}^{(i)}) \ldots$$
$$\ldots B^{(i)}(1, 1) \ldots B^{(i)}(1, c_1^{(i)}).$$

Hence by Cases 1 and 2 we may assume x_1 belongs to a sub-block of $B^{(k)}(i, j)$ that imitates $B^{(i)}(r, j)$ where $m < r \leq i - 1$. If $r > m + 1$, then we can repeat this argument, so that by a simple induction proof we may assume that x_1 belongs to a block B^* that imitates $[1, a_{m+1}]$ under χ. This means that we can assume $k = m + 1$, and we're done by Case 1 and Case 2.

We have shown that in all cases, there is no 5-term monochromatic a_m-a.p. for any m, so that A is not 2-large. ∎

The next theorem shows that we can weaken the hypothesis of Theorem 2.2 from $a_i \geq 3a_{i-1}$ to $a_i \geq 2a_{i-1}$, if we make the additional assumption that a_{i-1} divides a_i for all i.

Theorem 2.3 *If $A = \{a_i\}_{i=1}^{\infty}$ is an increasing sequence of positive integers where a_i divides a_{i+1} for all i, then A is not 2-large.*

Proof Define a 2-coloring χ on the set of positive integers recursively, as follows. First let $\chi(x) = 1$ for all $x \in [1, a_1]$. Once χ has been defined on $[1, a_i]$, we define χ on $[1, a_{i+1}]$ by $\chi(x) \neq \chi(x - a_i)$ for each $x \in [a_i + 1, a_{i+1}]$.

First note that, from the way χ is defined on $[a_i + 1, a_{i+1}]$, for each $i \geq 1$ there is no 2-term monochromatic a_i-a.p. contained in $[1, a_{i+1}]$.

Now assume $j > i + 1$ and that $\{x_1, x_2, x_3\}$ is a monochromatic a_i-a.p. that is contained in $[1, a_j]$. Since a_{i+1} divides a_j, we see that every subinterval of $[1, a_j]$ of the form $[ka_{i+1} + 1, (k + 1)a_{i+1}]$, $k \geq 1$, imitates $[1, a_{i+1}]$. Hence, since there is no 2-term monochromatic a_i-a.p. in $[1, a_{i+1}]$, neither of the pairs $\{x_1, x_2\}$ and $\{x_2, x_3\}$ could be in any one interval $[ka_{i+1} + 1, (k + 1)a_{i+1}]$. This implies that $x_3 - x_1 > a_{i+1} \geq 2a_i$, a contradiction.

We have shown that for each $i \geq 1$ and each $j \geq 1$, there is no 3-term monochromatic a_i-a.p. with respect to χ, which is contained in $[1, a_j]$, which proves the theorem. ∎

The next theorem, important in proving many subsequent results, shows that finite unions of small sets cannot be large.

Theorem 2.4   *If $A = A_1 \cup \cdots \cup A_n$ and A is large, then some A_i is large.*

Proof   By an obvious induction argument, it suffices to prove the result for $n = 2$. So let $A = B \cup C$, and assume neither B nor C is large. Since B is not large, there exist positive integers r and k_1, and some r-coloring ρ of Z^+, under which there is no monochromatic k_1-term B-a.p. Likewise, there exist positive integers s and k_2, and an s-coloring σ of Z^+, under which there is no monochromatic k_2-term C-a.p.

Define χ to be the *rs*-coloring of Z^+ given by $\chi(n) = \big(\rho(n), \sigma(n)\big)$. Let $k = \max\{k_1, k_2\}$. Let X be any k-term a.p. that is monochromatic with respect to χ. Then X must also be monochromatic with respect to each of the the colorings ρ and σ. Hence X must have common difference lying outside $B \cup C$. Hence $B \cup C$ is not large.   ∎

Corollary 2.1   *Let $c > 1$ be a fixed real number. Let $A = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that $a_i \geq ca_{i-1}$ for all but a finite number of $i \geq 2$. Then A is not large.*

Proof   Consider first the case in which $a_i \geq ca_{i-1}$ for all $i \geq 2$. Let n be such that $c^n \geq 3$. For each $i = 1, \ldots, n$, let $A_i = \{a_{jn+i} : j = 0, 1, 2, \ldots\}$. For each i and each $j \geq 1$, $a_{jn+i} \geq 3a_{(j-1)n+i}$. Hence by Theorem 2.2 each A_i is not 2-large. Since $A = A_1 \cup \cdots \cup A_n$, by Theorem 2.4 A is not large (by the proof of Theorem 2.4, A is not 2^n-large).

To complete the proof, let $m \geq 2$ be such that $a_i \geq ca_{i-1}$ for all $i \geq m$. By the above case, $\{a_{m-1}, a_m, \ldots\}$ is not large. By Theorem 2.1, $\{a_1, \ldots, a_{m-2}\}$ is not large. Hence, by Theorem 2.4, A is not large (by the proof of Theorem 2.4, it is not 2^{n+1}-large).   ∎

Remarks   By Theorem 2.4 it is obvious that Corollary 2.1 can be extended to any set $A = B_0 \cup \cdots \cup B_m$ having the property that for each $k = 0, \ldots, m$, there exists a $c_k > 1$ such that $B_k = \{b_{k,i} : i = 1, 2, \ldots\}$ with $b_{k,i} \geq c_k b_{k,i-1}$ for $i \geq 2$. For example, let $\{f_n : n \geq 1\}$ be the set of Fibonacci numbers. It is easy to show that for each $k \geq 0$ there exists a $c_k > 1$ such that for all $n \geq 2$, $f_n + k \geq c_k(f_{n-1} + k)$ (when $k = 0$ we can take $c_k = 3/2$). Hence if m is a fixed nonnegative integer, the set $\bigcup_{n=1}^{\infty} [f_n, f_n + m]$ is not large.

Although the complement of a small set is large (by Theorem 2.4), the complement of a large set need not be small. For example, by the work of Bergelson and Leibman, $A = \{n^2\}$ and $B = \{2n^2\}$ are both large sets, but since $B \subseteq Z^+ - A$, we have that $Z^+ - A$ is large.

We note that if A and B are small sets, then the proof of Theorem 2.4 does not necessarily give the "best" (*i.e.*, the least) value of n such that $A \cup B$ is not n-large. For example, let $m \geq 3$ be odd and let $A = \{a_i\}$ be an increasing sequence of positive multiples of m such that a_{i-1} divides a_i for all $i \geq 2$. By Theorem 2.2 (or Theorem 2.3), A is not 2-large. Now let B be the set of all positive integers n such that $m \nmid n$. By Theorem 2.1, B is not 2-large. Hence, according to Theorem 2.4, $A \cup B$ is not 4-large. However, by the following result, we can make the stonger statement that $A \cup B$ is not 3-large.

Before proceeding we define, for $m > 1$, a k-term a.p. (mod m) to be an increasing sequence of positive integers $\{x_i\}_{i=1}^{k}$ such that for some $d \in \{1, 2, \ldots, m-1\}$, $x_i - x_{i-1} \equiv d$

(mod m) for all $i = 2, \ldots, k$. Denote by $(AP)_m$ the family of all a.p.'s (mod m). In [11] it was shown that if m is odd and A is a finite set of multiples of m, then the family $A^* \cup (AP)_m$ does not have the 3-Ramsey property, where A^* is the family of all A-a.p.'s. The next proposition extends this result to certain cases in which A contains an infinite number of multiples of m.

Proposition 2.1   *Let $m \geq 3$ be odd and let $A = \{a_i\}$ be a sequence of positive integers such that $m \mid a_1$ and $a_{i-1} \mid a_i$ for $i \geq 2$. Let A^* be the family of all A-a.p.'s. Then $A^* \cup (AP)_m$ does not have the 3-Ramsey property.*

Proof   We give a 3-coloring χ of the positive integers, and show that under χ there is no monochromatic m-term a.p. (mod m) and no monochromatic 3-term a_i-a.p. for all i.

Let $S_1 = \left[1, \lceil m/3 \rceil\right]$, $S_2 = \left[\lceil m/3 \rceil + 1, \lceil 2m/3 \rceil\right]$, and $S_3 = \left[\lceil 2m/3 \rceil + 1, m\right]$. Denote by C_1 the string of length m representing the coloring defined by

$$C_1(x) = \begin{cases} 1 & \text{if } x \in S_1 \\ 2 & \text{if } x \in S_2 \\ 3 & \text{if } x \in S_3. \end{cases}$$

Denote by C_2 the string of length m representing the coloring defined by

$$C_2(x) = \begin{cases} 3 & \text{if } x \in S_1 \\ 1 & \text{if } x \in S_2 \\ 2 & \text{if } x \in S_3. \end{cases}$$

If I is an interval of length m and χ is a coloring such that $\chi(I) = C_1$, define $\overline{\chi}(I)$ to be the coloring C_2; and if $\chi(I) = C_2$, define $\overline{\chi}(I)$ to be the coloring C_1.

We now define the coloring χ recursively as follows: (i) for each $x \in [1, a_1]$, let x' be the element of $[1, m]$ such that $x \equiv x' \pmod{m}$, and let $\chi(x) = C_1(x')$; (ii) once $[1, a_{i-1}]$, $i \geq 2$, has been colored, we color $[a_{i-1} + 1, a_i]$ as follows:

$$\chi\big([ka_{i-1} + (j-1)m + 1, ka_{i-1} + jm]\big)$$
$$= \overline{\chi}\big([(k-1)a_{i-1} + (j-1)m + 1, (k-1)a_{i-1} + jm]\big),$$

for $1 \leq k \leq (a_i/a_{i-1}) - 1$ and $1 \leq j \leq a_{i-1}/m$.

Note that, from elementary group theory, since $m \geq 3$, any m-term a.p. (mod m) contains at least one element from each of S_1, S_2, and S_3. Hence there is no m-term monochromatic a.p. (mod m).

Now assume that $\{x, y\}$ is monochromatic with $y - x = a_i$. Consider the partition of the positive integers $Z^+ = \bigcup_{j=1}^{\infty} B_j$, where $B_j = [(j-1)a_{i+1} + 1, ja_{i+1}]$. Note that, by the way χ is defined, x and y cannot be monochromatic and lie in the same B_j. Thus, y and $y + a_i$ do lie in the same B_j, and hence $\chi(y) \neq \chi(y + a_i)$. That is, there is no 3-term monochromatic a_i-a.p., and the proof is complete. ∎

Corollary 2.2   *Let $m \geq 3$ be odd and let B be the set of all positive integers not divisible by m. Let $A = \{a_i\}$ be an increasing sequence of positive multiples of m such that $a_{i-1} \mid a_i$ for $i \geq 2$. Then $A \cup B$ is not 3-large.*

Proof   This is immediate from Proposition 2.1 since each B-a.p. is a member of $(AP)_m$. ∎

Remark   Corollary 2.2 follows from Proposition 2.1 by the fact that $A^*(m)$, the set of all arithmetic progressions having common differences which are not multiples of m, is a subset of the set $(AP)_m$. On the other hand, there are examples showing that we cannot always replace the collection $(AP)_m$ by the collection $A^*(m)$ and expect the Ramsey properties to be unaffected. For example, in [12] it was shown that if D is the set of all arithmetic progressions with common difference 2, then the family $(AP)_2 \cup D$ has the 3-Ramsey property. In contrast, by Theorem 2.1, we see that $A^*(2) \cup D$ does not even have the 2-Ramsey property, *i.e.*, $A = \{2n-1 : n \in Z^+\} \cup \{2\}$ is not 2-large. In fact, $B = \{2n-1 : n \in Z^+\} \cup \{2^n : n \geq 0\}$ is not 2-large, since B contains no multiples of six.

3   Some Positive Results

In this section we give some sufficient conditions for a set to be large.

Lemma 1   *For each positive integer r, if A is r-large and F is finite, then $A - F$ is r-large.*

Proof   It suffices to show that $A - \{a\}$ is r-large for each $a \in A$. Assume this is not the case. Then there is an $a_0 \in A$, an r-coloring f of Z^+, and a $k \in Z^+$ such that there is no monochromatic k-term $(A - \{a_0\})$-a.p. Hence, under f, there are arbitrarily long monochromatic a_0-a.p.'s. By Theorem 2.1, $ma_0 \in A$ for some $m > 1$. Under f, there are arbitrarily long monochromatic ma_0-a.p.'s, a contradiction. ∎

Theorem 3.1   *Let $p(x)$ be a polynomial with integer coefficients and leading coefficient positive. If $x + a$ divides $p(x)$ for some integer a, then $\big(\text{range}(p) = \{p(x) : x = 1, 2, \dots\}\big) \cap Z^+$ is large.*

Proof   Let $p(x) = (x + a)s(x)$. Let $q(x) = p(x - a)$. So $q(x) = xs(x - a)$. By the result of Bergelson and Leibman mentioned in Section 1, $\text{range}(q) \cap Z^+$ is large. If $a \leq 0$, then $\text{range}(q) \subseteq \text{range}(p)$, so $\text{range}(p) \cap Z^+$ is large. If $a > 0$, then $\text{range}(p) = \text{range}(q) - F$, where F is finite, so by Lemma 3.1 $\text{range}(p) \cap Z^+$ is large. ∎

Remarks   Does the converse of Theorem 3.1 hold? That is, if no $x+a$ divides the polynomial $p(x)$, does it follow that $\{p(n) : n \in Z^+\}$ is not large? It is easy to see (by Theorem 2.1) that the answer is yes if $p(x)$ has degree one, for if $p(x) = ax + b$, where b is not a multiple of a, then the range of $p(x)$ contains no multiples of a. We do not know the answer for the general case. Another question: is it true that whenever the range of a polynomial $p(x)$ is not large, then the range fails to contain multiples of some positive integer m? It would be

interesting to know whether the range of the polynomial $f(x) = (x^2 - 13)(x^2 - 17)(x^2 - 221)$ is large, because although $f(x)$ has no linear factors, its range contains an infinite number of multiples of every positive integer m (using the properties of the Legendre symbol, one can show that the congruence $f(x) \equiv 0 \pmod{m}$ is solvable for all m).

Theorem 3.2   *If A is a set of positive integers containing n-cubes for arbitrarily large n, then A is large.*

Proof   The proof makes use of the Hales-Jewett theorem [9], for which we need some notation. Let k be any fixed positive integer, and let $B = \{0, 1, \ldots, k - 1\}$. For a positive integer n, we consider the set B^n of n-tuples with entries from B, and the set $(B \cup \lambda)^n$ of n-tuples with entries from $B \cup \lambda$, where λ is an indeterminate.

Let $w(\lambda)$ denote an element of $(B \cup \lambda)^n$ in which at least one λ occurs. For each i, $0 \le i \le k - 1$, let $w(i)$ denote the result of replacing each occurrence of λ in $w(\lambda)$ by i.

A *combinatorial line* in B^n is any set L having the form $L = \{w(0), w(1), \ldots, w(k - 1)\}$ for some n-tuple $w(\lambda)$ in $(B \cup \lambda)^n$, where $w(\lambda)$ has at least one occurrence of λ.

The Hales-Jewett theorem asserts that for any given positive integers k and r, with $B = \{0, 1, \ldots k - 1\}$, there exists a positive integer n such that for every r-coloring of B^n there is a monochromatic combinatorial line.

Now let r be a positive integer and let χ be any r-coloring of the positive integers. We assume that A contains arbitrarily large cubes, and we wish to show that for each k there are monochromatic k-term A-a.p.'s. Let k be given, and choose n according to the Hales-Jewett theorem so that every r-coloring of B^n, where $B = \{0, \ldots, k - 1\}$, has a monochromatic combinatorial line. Since A contains arbitrarily large cubes, A contains an n-cube, say $Q(a_1, \ldots, a_n)$.

Now define an r-coloring σ, of B^n, as follows: for each $(x_1, \ldots, x_n)$ in B^n, let

$$\sigma(x_1, \ldots, x_n) = \chi(x_1 a_1 + x_2 a_2 + \cdots + x_n a_n).$$

By the choice of n, there exists a combinatorial line $L = \{w(0), \ldots, w(k - 1)\}$ that is monochromatic with respect to σ. To simplify our notation, we may assume that $w(\lambda) = (x_1, x_2, \ldots, x_s, \lambda, \lambda, \ldots, \lambda)$, where $s \le n - 1$, all the x_i's belong to B, and there are $n - s$ occurrences of λ. Then for $0 \le i \le k - 1$,

$$\sigma(w(i)) = \sigma(x_1, x_2, \ldots, x_s, i, i, \ldots, i) = \chi\big(x_1 a_1 + \cdots x_s a_s + i(a_{s+1} + \cdots a_n)\big).$$

Writing $a = x_1 a_1 + \cdots x_s a_s$ and $d = a_{s+1} + \cdots + a_n$, we have that $d \in Q(a_1, \ldots, a_n) \subseteq A$, and χ is constant on the k-term arithmetic progression $P = \{a + id : 0 \le i \le k - 1\}$. Hence we have shown that for each $r > 0$ and each $k > 0$, every r-coloring of the positive integers contains a monochromatic k-term A-a.p. ∎

In the next corollary, the symbol $\{r\}$ denotes the fractional part of the real number r, *i.e.*, $\{r\} = r - \lfloor r \rfloor$.

Corollary 3.1

(a) *Let $\alpha > 0$ be irrational and let $\epsilon > 0$. Then $A = \{i \in Z^+ : \{i\alpha\} < \epsilon\}$ is large.*

(b) Let $\alpha > 0$. Then $A = \{\lfloor n\alpha \rfloor : n \in Z^+\}$ is large.
(c) If A is a set of positive integers containing arbitrarily long intervals, then A is large.

Proof (a) Since $\{\{i\alpha\} : i \in Z^+\}$ is dense in the unit interval, for each $k \geq 1$ we may choose $a_k \in A$ such that $\{a_k\alpha\} < \epsilon/2^k$. Let n be a given positive integer. Then $\{(a_1 + \cdots + a_n)\alpha\} < \epsilon$, so A contains the infinite cube $Q(a_1, a_2, \dots)$. By Theorem 3.2, A is large.

(b) The proof is essentially the same as the proof of (a).

(c) The set must contain an infinite cube (see [5, p. 171]).   ∎

By Theorem 3.1, the range of any polynomial with integer coefficients, that is divisible by $x + a$ for some a, is large. However, it is easy to find a large set A with the property that for each polynomial $p(x)$ with integer coefficients, A does not contain all but finitely many elements of range(p) (we say "all but finitely many" because of Lemma 3.1). This follows easily from the following more general result.

Proposition 3.1 *If $R_1, R_2, \dots$ is a sequence of infinite subsets of Z^+, then there exists a large set A such that the complement of A contains infinitely many elements of every R_i.*

Proof We define $B = \{b_1, b_2, \dots\}$ as follows. Let b_1 and b_2 be arbitrary elements of R_1 such that $b_2 > b_1$.

Now assume $j \geq 3$. Once $b_1, \dots, b_{j-1}$ have been defined, choose b_j such that $b_j \in R_i$ and $b_j - b_{j-1} > b_{j-1} - b_{j-2}$. Then the sequence $\{b_j : j = 1, 2, \dots\}$ contains infinitely many members of each R_i. Also, $b_j - b_{j-1}$ goes to infinity with j. Hence, if $A = Z^+ - \{b_j : j = 1, 2, \dots\}$, A contains arbitrarily long intervals, hence is large by Corollary 3.1(c). ∎

The following theorem provides some simple ways of obtaining large sets from other large sets. The proofs are relatively straightforward and are omitted.

Theorem 3.3

(a) If A is large and m is a positive integer, then mA is large.
(b) If A is large and m is a positive integer, then $A - \{x : m \nmid x\}$ is large.
(c) If A is r-large, and if all elements of A are multiples of the positive integer m, then $\frac{1}{m}A$ is r-large (hence, A large implies $\frac{1}{m}A$ large).

We have yet to see an example of a set A that is r-large for some $r \geq 2$, but that is not large. We make the following conjecture.

Conjecture If A is 2-large, then A is large.

We have some partial results concerning the above conjecture. In the following theorem, we use the symbol A^n, where A is a set of positive integers, to denote the set of products $\{x_1 x_2 \dots x_n : x_i \in A\}$.

Theorem 3.4 *If A is 2-large, then A^n is 2^n-large.*

Sketch of Proof We prove this by induction on n. For the induction step, let χ be a 2^{n+1}-coloring of the positive integers, using the colors $1, 2, 3, \ldots, 2^{n+1}$. Define the 2-coloring λ on Z^+ by

$$\lambda(x) = \begin{cases} a & \text{if } \chi(x) \in [1, 2^n] \\ b & \text{if } \chi(x) \in [2^n + 1, 2^{n+1}]. \end{cases} \qquad \blacksquare$$

Corollary 3.2 *If A is 2-large and is closed under multiplication, then A is large.*

4 Remarks and Questions

We do not know if Theorem 2.4 is still true if we replace "large" with "r-large." In particular, if $A \cup B$ is 2-large, must it follow that at least one of A or B is 2-large? We remarked in Section 2 that, by Theorem 2.1, the set $\{2n - 1 : n \geq 1\} \cup \{2^n : n \geq 0\}$ is not 2-large. We would like to know if $\{2n - 1 : n \geq 1\} \cup \{n! : n \geq 1\}$ (an example not covered by Theorem 2.1) is 2-large (it is not 4-large by Theorem 2.2 and the proof of Theorem 2.4).

We also ask: which sets A have the property that some translation of A is large, *i.e.*, for which A does there exist an integer x such that $x + A = \{x + a : a \in A\}$ is large? By the result of Bergelson and Leibman, the range of any polynomial $p(x)$ has this property, since $p(x) - p(0)$ sends 0 into 0. Also, it follows from Theorem 2.4 that any set A with bounded gaps has this property since then $Z^+ = \bigcup_{i=0}^{s} (A + i)$ for some s. It would be interesting to know if some translation of the set of primes is large (by Theorem 2.1, it would have to be an odd translation).

Let $p(x)$ be any polynomial with integer coefficients, positive leading coefficient, and $p(0) = 0$, and let A be a large set. Must $\{p(x) : x \in A\}$ be large? In particular, must $\{x^2 : x \in A\}$ be large?

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