Compositio Math. 143 (2007) 1493–1510
doi:10.1112/S0010437X0700276X

Higher Nash blowups

Takehiko Yasuda

Abstract

For each non-negative integer n we define the nth Nash blowup of an algebraic variety, and call them all higher Nash blowups. When $n = 1$, it coincides with the classical Nash blowup. We study higher Nash blowups of curves in detail and prove that any curve in characteristic zero can be desingularized by its nth Nash blowup with n large enough. Moreover, we completely determine for which n the nth Nash blowup of an analytically irreducible curve singularity in characteristic zero is normal, in terms of the associated numerical monoid.

Introduction

The classical Nash blowup of an algebraic variety is the parameter space of the tangent spaces of smooth points and their limits, and the normalized Nash blowup is the Nash blowup followed by the normalization. It is natural to ask whether the iteration of Nash blowups or normalized Nash blowups leads to a smooth variety. There are works on this question by Nobile [Nob75], Rebassoo [Reb77], González-Sprinberg [Gon82], Hironaka [Hir83] and Spivakovsky [Spi90]. If the answer is affirmative, we obtain a canonical way to resolve singularities.

In this paper, we take a similar but different approach to a resolution of singularities. Let X be an algebraic variety of dimension d over an algebraically closed field k. For a point $x \in X$, we denote by $x^{(n)}$ its nth infinitesimal neighborhood, that is, if $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is the local ring at x, the closed subscheme $\operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1} \subseteq X$. If x is a smooth point, being an Artinian subscheme of length $\binom{n+d}{d}$, $x^{(n)}$ corresponds to a point $[x^{(n)}]$ of the Hilbert scheme $\mathbf{Hilb}_{\binom{n+d}{d}}(X)$ of $\binom{n+d}{d}$ points of X. We define the *nth Nash blowup* of X, denoted $\mathbf{Nash}_n(X)$, to be the closure of the set

$$\{(x, [x^{(n)}]) \mid x \text{ smooth point of } X\}$$

in $X \times_k \mathbf{Hilb}_{\binom{n+d}{d}}(X)$. We also call it a *higher Nash blowup* of X. The first projection restricted to $\mathbf{Nash}_n(X)$

$$\pi_n : \mathbf{Nash}_n(X) \to X$$

is a projective birational morphism which is an isomorphism over the smooth locus of X. The first Nash blowup is canonically isomorphic to the classical Nash blowup (see Proposition 1.8). Every point of $\mathbf{Nash}_n(X)$ corresponds to an Artinian subscheme Z of X which is set-theoretically a single point.

If $\mathbf{Nash}'_n(X)$ is the closure of $\{[x^{(n)}] \mid x \text{ smooth point of } X\}$ in $\mathbf{Hilb}_{\binom{n+d}{d}}(X)$, then there exists a natural morphism $\mathbf{Nash}_n(X) \to \mathbf{Nash}'_n(X)$, $(x, [Z]) \mapsto [Z]$, which is bijective and, in characteristic zero, even an isomorphism. Thus, $\mathbf{Nash}_n(X)$ is identified with the set of the nth infinitesimal neighborhoods of smooth points and their limits. We can also construct higher Nash blowups by

Received 7 October 2006, accepted in final form 24 November 2006, published online 1 October 2007.
2000 Mathematics Subject Classification 14E15, 14B12.
Keywords: Nash blowup, resolution of singularities, Hilbert scheme of points.
Financial support has been provided by the Japan Society for the Promotion of Science.

using the relative Hilbert scheme or the Grassmaniann schemes of coherent sheaves. The last construction is essentially the same as a special case of Oneto and Zatini's Nash blowup associated to a coherent sheaf [OZ91].

An interesting problem is whether, when n is sufficiently large, $\mathbf{Nash}_n(X)$ is smooth. We will obtain several results concerning this question, behind which we become aware of the following general principle. It is interesting to ask when the principle holds.

Principle 0.1. Let $\mathbf{P}$ be a local property for points of varieties, such as smoothness, normality, analytic irreducibility and so on. Then there exists a suitable closed subscheme $A \subsetneq X$ (depending only on X and $\mathbf{P}$) whose support contains the non-$\mathbf{P}$ locus, and if $[Z] \in \mathbf{Nash}_n(X)$ with $Z \not\subseteq A$, then $[Z] \in \mathbf{Nash}_n(X)$ satisfies $\mathbf{P}$.

We will show that for every $A \subseteq X$ with $\dim A < \dim X$, there exists n_0 such that for every $n \geqslant n_0$ and for every $[Z] \in \mathbf{Nash}_n(X)$, $Z \not\subseteq A$ (Proposition 2.8). Thus, when the principle holds, then for large n, all points of $\mathbf{Nash}_n(X)$ satisfy $\mathbf{P}$.

We select a special case of the principle as a conjecture: recall that the *Jacobian ideal sheaf* $\mathrm{j}_X \subseteq \mathcal{O}_X$ is the ideal sheaf locally generated by suitable minors of the Jacobian matrix associated to defining equations of X, and that the support of $\mathcal{O}_X/\mathrm{j}_X$ is exactly the singular locus of X.

Conjecture 0.2. Suppose that k has characteristic zero. Let X be a variety of dimension d and let be $J^{(d-1)}$ the $(d-1)$th neighborhood of the Jacobian subscheme $J \subseteq X$ (that is, the closed subscheme defined by j_X^d). Let $[Z] \in \mathbf{Nash}_n(X)$ with $Z \not\subseteq J^{(d-1)}$. Then $\mathbf{Nash}_n(X)$ is smooth at $[Z]$.

If the conjecture is true, we obtain a canonical way to resolve singularities in *one step*.

Remark 0.3. We make the following remarks.

(i) At this point, there is little evidence for Conjecture 0.2 in higher dimensions. It is, perhaps, safer to replace $J^{(d-1)}$ with $J^{(a_d)}$, where a_d is a positive integer depending only on d. The principle and the conjecture are based on the idea that Artinian subschemes protruding much from the singular locus behave well. A similar idea for jets appears in the theory of motivic integration for singular varieties (see [DL99]), in which the Jacobian ideal also plays an important role.

(ii) The conjecture fails if we replace $J^{(d-1)}$ with $J = J^{(0)}$. Let
$$X := (x^2 + y^2 + z^{n+1} = 0) \subseteq \mathbb{A}^3_{\mathbb{C}}$$
be a surface with an A_n-singularity. Its Jacobian ideal is $(x, y, z^n) \subseteq \mathbb{C}[x, y, z]/(x^2 + y^2 + z^{n+1})$. Let $A \subseteq X$ be the subscheme defined by the Jacobian ideal, which is isomorphic to $\operatorname{Spec}\mathbb{C}[z]/(z^n)$. For any $[Z] \in \mathbf{Nash}_1(X)$, $Z \cong \operatorname{Spec}\mathbb{C}[s,t]/(s,t)^2$ and $Z \not\subseteq A$. However, the classical Nash blowup of X is not generally smooth (see [Gon82, § 5.2]).

(iii) The conjecture fails also in positive characteristic at least in dimension 1. Let X be an analytically irreducible curve in characteristic $p > 0$. Then $\mathbf{Nash}_{p^e-1}(X) \cong X$ for $e \gg 0$ (Proposition 3.8). If k is of characteristic either two or three and if $X = \operatorname{Spec} k[[x^2, x^3]]$, then $\mathbf{Nash}_n(X) \cong X$ for every n (Proposition 3.9).

Our first step towards proving the conjecture is a separation of analytic branches. Let $\hat{X} := \operatorname{Spec}\hat{\mathcal{O}}_{X,x}$ be the completion of a variety X at $x \in X$, and $\hat{X}_i$, $i = 1, \dots, l$, its irreducible components. Then we can define higher Nash blowups of $\hat{X}$ and $\hat{X}_i$, and obtain
$$\mathbf{Nash}_n(X) \times_X \hat{X} \cong \mathbf{Nash}_n(\hat{X}) = \bigcup_{i=1}^{l} \mathbf{Nash}_n(\hat{X}_i).$$

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

Let $\nu : \tilde{X} \to X$ be the normalization. The *conductor ideal sheaf* is the annihilator ideal sheaf of the coherent sheaf $\nu_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X$. The *conductor subscheme* $C \subseteq X$ is the closed subscheme defined by the conductor ideal sheaf.

Proposition 0.4 (Proposition 2.6). *Let $[Z] \in \mathbf{Nash}_n(X)$ with $Z \nsubseteq C$. Then Z is contained in a unique analytic branch of X.*

If $x \in X$ is the support of Z and $\hat{X}_i$ are as above, then the proposition states that $\mathbf{Nash}_n(\hat{X}_i)$ are disjoint around $[Z]$. Therefore, the study of $\mathbf{Nash}_n(X)$ is reduced to that of $\mathbf{Nash}_n(\hat{X}_i)$.

We study the case of curves in more detail. Let R be a local complete Noetherian domain of dimension 1 with coefficient field k and $X := \operatorname{Spec} R$. The integral closure of R is (isomorphic to) $k[[x]]$. Then we define a numerical monoid $S := \{i \mid \exists f \in R,\ \operatorname{ord} f = i\}$ and write

$$S = \{0 = s_{-1} < s_0 < s_1 < \cdots\}.$$

In characteristic zero, we can completely determine when $\mathbf{Nash}_n(X)$ is normal in terms of S.

Theorem 0.5 (Theorem 3.3). *Let X and S be as above. Suppose that k has characteristic zero. Then $\mathbf{Nash}_n(X)$ is normal if and only if $s_n - 1 \in S$.*

As a corollary, we prove the following, which implies Conjecture 0.2 in dimension 1.

Corollary 0.6 (Corollary 3.7). *Let X be a variety of dimension 1 over k, C its conductor subscheme and $[Z] \in \mathbf{Nash}_n(X)$. Suppose that k has characteristic zero and that $Z \nsubseteq C$. Then $\mathbf{Nash}_n(X)$ is normal at $[Z]$.*

In contrast to the iteration of classical Nash blowups, each higher Nash blowup is directly constructed from the given variety. There is no direct relation between $\mathbf{Nash}_{n+1}(X)$ and $\mathbf{Nash}_n(X)$. In fact, from Theorem 0.5, we see that even if $\mathbf{Nash}_n(X)$ is smooth, $\mathbf{Nash}_{n+1}(X)$ is not generally smooth. So there is no birational morphism $\mathbf{Nash}_{n+1}(X) \to \mathbf{Nash}_n(X)$ (See Example 3.5).

Nakamura's G-Hilbert scheme is also a kind of blowup constructed by using a Hilbert scheme of points. For an algebraic variety M with an effective action of a finite group G, its G-Hilbert scheme G-$\mathbf{Hilb}(M)$ parameterizes the free orbits and their limits in the Hilbert scheme of points of M, and there exists a projective birational morphism G-$\mathbf{Hilb}(M) \to M/G$. Replacing free orbits with their nth infinitesimal neighborhoods, we can define a higher version of G-Hilbert scheme, although the author does not know whether it is interesting.

We can easily generalize the higher Nash blowup to generically smooth morphisms, that is, to the relative setting, and even more generally to foliations. The latter was actually what the author first thought of. There should be other similar constructions of blowups. If such a construction is a resolution of singularities (in different senses in different situations), it is likely to be a good choice of resolutions, because it is a moduli space of some objects on the original variety.

In § 1, we give the definition of higher Nash blowup and several alternative constructions. In § 2, we prove basic properties of higher Nash blowups. In § 3, we study the case of curves.

Conventions

We work in the category of schemes over an algebraically closed field k. A *point* means a k-point. A *variety* means an integral separated scheme of finite type over k. For a closed subscheme $Z \subseteq X$ defined by an ideal $\mathcal{I} \subseteq \mathcal{O}_X$, we denote by $Z^{(n)}$ its *nth infinitesimal neighborhoods*, that is, the closed subscheme defined by $\mathcal{I}^{n+1}$. We denote by $\mathbb{N}$ the set $\{1, 2, \dots\}$ of positive integers and by $\mathbb{N}_0$ the set $\{0, 1, 2, \dots\}$ of non-negative integers.

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

1. Definition and several constructions

1.1 Definition

Let X be a variety of dimension d, $x \in X$ and $x^{(n)} := \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ its nth infinitesimal neighborhood. If X is smooth at x, then $x^{(n)}$ is an Artinian subscheme of X of length $\binom{d+n}{n}$. Therefore, it corresponds to a point

$$[x^{(n)}] \in \mathbf{Hilb}_{\binom{d+n}{n}}(X),$$

where $\mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is the Hilbert scheme of $\binom{d+n}{n}$ points of X. If X_{sm} denotes the smooth locus of X, then we have a map

$$\sigma_n : X_{\mathrm{sm}} \to \mathbf{Hilb}_{\binom{d+n}{n}}(X), x \mapsto [x^{(n)}].$$

Lemma 1.1. *We have that σ_n is a morphism of schemes.*

Proof. Let $\Delta \subseteq X_{\mathrm{sm}} \times_k X_{\mathrm{sm}}$ be the diagonal and $\Delta^{(n)} \subseteq X_{\mathrm{sm}} \times_k X_{\mathrm{sm}}$ its nth infinitesimal neighborhood. Consider the following diagram of the projections restricted to $\Delta^{(n)}$.

$$\begin{tikzcd} \Delta^{(n)} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X_{\mathrm{sm}} \\ X_{\mathrm{sm}} & \end{tikzcd}$$

For $x \in X_{\mathrm{sm}}$,

$$\mathrm{pr}_2(\mathrm{pr}_1^{-1}(x)) = x^{(n)}.$$

Therefore, by the definition of a Hilbert scheme, there exists a morphism

$$X_{\mathrm{sm}} \to \mathbf{Hilb}_{\binom{d+n}{n}}(X)$$

corresponding to the diagram above. It is identical to σ_n. □

The graph $\Gamma_{\sigma_n} \subseteq X_{\mathrm{sm}} \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$ of σ_n is canonically isomorphic to X_{sm}.

Definition 1.2. We define *the nth Nash blowup of X*, denoted by $\mathbf{Nash}_n(X)$, to be the closure of Γ_{σ_n} with reduced scheme structure in $X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$.

The first projection restricted $\mathbf{Nash}_n(X)$,

$$\pi_n : \mathbf{Nash}_n(X) \to X,$$

is projective and birational. Moreover, it is an isomorphism over X_{sm}.

Let $\mathbf{Nash}'_n(X)$ be the closure of $\sigma_n(X_{\mathrm{sm}})$ in $\mathbf{Hilb}_{\binom{d+n}{n}}(X)$. Then the second projection $X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$ induces a morphism

$$\psi_n : \mathbf{Nash}_n(X) \to \mathbf{Nash}'_n(X).$$

This bijectively sends $(x, [Z])$ to $[Z]$. Thus, $\mathbf{Nash}_n(X)$ is set-theoretically identified with $\mathbf{Nash}'_n(X)$, the set of the nth infinitesimal neighborhoods of smooth points and their limits. Hereafter we abbreviate $(x, [Z]) \in \mathbf{Nash}_n(X)$ as $[Z] \in \mathbf{Nash}_n(X)$.

1.2 In characteristic zero ψ_n is an isomorphism

Let $S^m X$ denote the mth symmetric product of X. The Hilbert–Chow morphism of [Fog68] is a morphism

$$(\mathbf{Hilb}_m(X))_{\mathrm{red}} \to S^m X$$

which assigns a closed subscheme $Z \subseteq X$ the associated 0-cycle.

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

In characteristic zero, X is embedded into $S^m X$ as the small diagonal, $\{(x,\dots,x) \mid x \in X\} \subseteq S^m X$. (In positive characteristic, the diagonal morphism $X \to S^m X$ is not generally a closed embedding.) When $m = \binom{d+n}{n}$, the Hilbert–Chow morphism restricted to $\mathbf{Nash}'_n(X)$,

$$\pi'_n : \mathbf{Nash}'_n(X) \to X,$$

is a morphism onto X.

Proposition 1.3. *Suppose that k has characteristic zero. Then ψ_n is an isomorphism and $\pi_n = \pi'_n \circ \psi_n$.*

Proof. The graph $\Gamma_{\pi'_n} \subseteq X \times_k \mathbf{Nash}'_n(X)$ of π'_n is identical to $\mathbf{Nash}_n(X)$. Therefore, ψ_n is an isomorphism. Now the equality $\pi_n = \pi'_n \circ \psi_n$ is obvious. □

Remark 1.4. In positive characteristic, ψ_n is not generally an isomorphism. For instance, let $X := \operatorname{Spec} k[x]$. Since X is smooth, ψ_n is isomorphic to $\sigma_n : X \to \mathbf{Nash}'_n(X) \subseteq \mathbf{Hilb}_{\binom{n+d}{d}}(X)$. Suppose that k has characteristic $p > 0$ and that p divides $n+1$. Then $\Delta^{(n)} \times_X \operatorname{Spec} k[x]/(x^2)$ is a trivial embedded deformation over $\operatorname{Spec} k[x]/(x^2)$. So the corresponding morphism $\operatorname{Spec} k[x]/(x^2) \to X \to \mathbf{Nash}'_n(X)$ factors as $\operatorname{Spec} k[x]/(x^2) \to \operatorname{Spec} k \to \mathbf{Nash}'_n(X)$. It follows that ψ_n is not an isomorphism.

1.3 Construction with the relative Hilbert scheme

We can construct higher Nash blowups also by using the *relative* Hilbert scheme. Let X be a variety and $\Delta^{(n)} \subseteq X \times_k X$ the nth infinitesimal neighborhood of the diagonal. Then the restricted first projection

$$\mathrm{pr}_1 : \Delta^{(n)} \to X$$

is a finite morphism. Its relative Hilbert scheme

$$\mathbf{Hilb}_{\binom{d+n}{n}}(\mathrm{pr}_1 : \Delta^{(n)} \to X)$$

for a constant Hilbert polynomial $\binom{d+n}{n}$ is a projective X-scheme. It is easy to see that

$$\mathbf{Hilb}_{\binom{d+n}{n}}(\mathrm{pr}_1 : \Delta^{(n)} \to X) \times_X X_{\mathrm{sm}} \cong X_{\mathrm{sm}}.$$

Proposition 1.5. *The irreducible component of $\mathbf{Hilb}_{\binom{d+n}{n}}(\mathrm{pr}_1 : \Delta^{(n)} \to X)$ dominating X is canonically isomorphic to $\mathbf{Nash}_n(X)$.*

Proof. A closed embedding $\Delta^{(n)} \hookrightarrow X \times_k X$ induces a closed embedding

$$\mathbf{Hilb}_{\binom{n+d}{d}}(\mathrm{pr}_1 : \Delta^{(n)} \to X) \hookrightarrow \mathbf{Hilb}_{\binom{n+d}{d}}(\mathrm{pr}_1 : X \times_k X \to X).$$

We also have a closed embedding

$$\mathbf{Nash}_n(X) \hookrightarrow X \times_k \mathbf{Hilb}_{\binom{n+d}{d}}(X) = \mathbf{Hilb}_{\binom{n+d}{d}}(\mathrm{pr}_1 : X \times_k X \to X).$$

Then $\mathbf{Nash}_n(X)$ and the irreducible component of $\mathbf{Hilb}_{\binom{n+d}{d}}(\mathrm{pr}_1 : \Delta^{(n)} \to X)$ dominating X determines the same closed subscheme of $\mathbf{Hilb}_{\binom{n+d}{d}}(\mathrm{pr}_1 : X \times_k X \to X)$. This proves the assertion. □

Corollary 1.6. *Let $[Z] \in \mathbf{Nash}_n(X)$ such that the support of Z is x (that is, $\pi_n([Z]) = x$). Then $Z \subseteq x^{(n)}$.*

Proof. The subscheme $Z \subseteq X$ is contained in the fiber of $\mathrm{pr}_1 : \Delta^{(n)} \to X$ over x, which is exactly $x^{(n)}$. □

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

1.4 The Nash blowup associated to a coherent sheaf

Let X be a reduced Noetherian scheme, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module locally free of constant rank r on an open dense subscheme $U \subseteq X$, and $\mathbf{Grass}_r(\mathcal{M})$ the Grassmaniann of $\mathcal{M}$ of rank r, which is a projective X-scheme. Then the fiber product $\mathbf{Grass}_r(\mathcal{M}) \times_X U$ is isomorphic to U by the projection.

Definition 1.7. The closure of $\mathbf{Grass}_r(\mathcal{M}) \times_X U$ is called the *Nash blowup of X associated to $\mathcal{M}$* and denoted by $\mathbf{Nash}(X, \mathcal{M})$ (see [OZ91]).

Then the natural morphism $\pi_{\mathcal{M}} : \mathbf{Nash}(X, \mathcal{M}) \to X$ is projective and birational. When X is a variety and $\mathcal{M} = \Omega_{X/k}$, then $\mathbf{Nash}(X, \Omega_{X/k})$ is the classical Nash blowup of X.

If *tors* $\subseteq \pi^*_{\mathcal{M}}\mathcal{M}$ denotes the torsion part, then by definition, $(\pi^*_{\mathcal{M}}\mathcal{M})/tors$ is locally free. Moreover, $\mathbf{Nash}(X, \mathcal{M})$ has the following universal property. If $f : Y \to X$ is a modification with $(f^*\mathcal{M})/tors$ locally free, then there exists a unique morphism $g : Y \to \mathbf{Nash}(X, \mathcal{M})$ with $\pi_{\mathcal{M}} \circ g = f$.

Let $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X\times_k X}$ be the ideal sheaf defining the diagonal $\Delta \subseteq X \times_k X$. Put $\mathcal{P}^n_X := \mathcal{O}_{X\times_k X}/\mathcal{I}^{n+1}_\Delta$ and $\mathcal{P}^n_{X,+} := \mathcal{I}_\Delta/\mathcal{I}^{n+1}_\Delta$, $n \in \mathbb{N}$. The $\mathcal{P}^n_X$ is the structure sheaf of $\Delta^{(n)}$ and called the *sheaf of principal parts of order n of X* (see [Gro67, Definition 16.3.1]). We regard $\mathcal{P}^n_X$ and $\mathcal{P}^n_{X,+}$ as $\mathcal{O}_X$-modules through the first projection. When X is a variety, these are coherent sheaves.

Proposition 1.8. *For every variety X and every $n \in \mathbb{N}_0$, we have canonical isomorphisms,*

$$\mathbf{Nash}_n(X) \cong \mathbf{Nash}(X, \mathcal{P}^n_X) \cong \mathbf{Nash}(X, \mathcal{P}^n_{X,+}).$$

In particular, $\mathbf{Nash}_1(X)$ is canonically isomorphic to the classical Nash blowup of X.

Proof. Because of the universal property, if $\mathcal{N}$ is locally free, then we have a canonical isomorphism $\mathbf{Nash}(X, \mathcal{M} \oplus \mathcal{N}) \cong \mathbf{Nash}(X, \mathcal{M})$. In particular, since $\mathcal{P}^n_X \cong \mathcal{O}_X \oplus \mathcal{P}^n_{X,+}$, we have $\mathbf{Nash}(X, \mathcal{P}^n_X) \cong \mathbf{Nash}(X, \mathcal{P}^n_{X,+})$.

The moduli schemes $\mathbf{Hilb}_{\binom{d+n}{n}}(\mathrm{pr}_1 : \Delta^{(n)} \to X)$ and $\mathbf{Grass}_{\binom{d+n}{n}}(\mathcal{P}^n_X)$ represent equivalent functors. Hence, they are canonically isomorphic. It follows that $\mathbf{Nash}_n(X) \cong \mathbf{Nash}(X, \mathcal{P}^n_X)$. □

Corollary 1.9. *Let X be a variety of dimension d, $n \in \mathbb{N}_0$, and $r := \binom{n+d}{d}$.*

(i) *We have $\mathbf{Nash}_n(X) \cong \mathbf{Nash}(X, \bigwedge^r \mathcal{P}^n_X)$.*
(ii) *Let $\mathcal{K}(X)$ be the constant sheaf of rational functions. Fix an isomorphism $\bigwedge^r \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{K}(X) \to \mathcal{K}(X)$ and define a homomorphism*

$$\psi : \bigwedge^r \mathcal{P}^n_X \to \bigwedge^r \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{K}(X) \to \mathcal{K}(X).$$

Then $\mathbf{Nash}_n(X)$ is isomorphic to the blowup of X with respect to a fractional ideal $\psi(\bigwedge^r \mathcal{P}^n_X)$.

Proof. These are results due to Oneto and Zatini [OZ91] restricted to the case where $\mathcal{M} = \mathcal{P}^n_X$. □

1.5 Formal completion

For a complete local Noetherian ring S with coefficient field k, the module $\Omega_{S/k}$ of Kähler differentials is not generally finitely generated over S, while its completion $\hat{\Omega}_{S/k}$ is. The latter is usually the suitable one to handle. We show the appropriate analogues of the above facts about the completion $\hat{P}^n_S$ that are required in applications to higher Nash blowups.

Let $k[\mathbf{x}] := k[x_1, \dots, x_r]$ be a polynomial ring in r variables and $R = k[\mathbf{x}]/\mathfrak{a}$ its quotient ring by an ideal $\mathfrak{a}$. We define the ideal I_R of $R \otimes_k R$,

$$I_R := (x_i \otimes 1 - 1 \otimes x_i; i = 1, \dots, r)(R \otimes_k R).$$

Then we put

$$P_R^n := R \otimes_k R / I_R^{n+1}$$

and regard it as an R-module via the map

$$R \to R \otimes_k R, \quad a \mapsto a \otimes 1.$$

This module is finitely generated over R. If $X := \operatorname{Spec} R$, then the $\mathcal{O}_X$-module $\mathcal{P}_X^n$ defined above is identified with the sheaf $\widetilde{P_R^n}$ associated to the R-module P_R^n.

Let $k[[\mathbf{x}]] := k[[x_1, \dots, x_r]]$ be a formal power series ring in r variables and $S := k[[\mathbf{x}]]/\mathfrak{b}$ its quotient ring. Similarly we define an ideal $\hat{I}_S$ of $S \hat{\otimes}_k S$,

$$\hat{I}_S := (x_i \otimes 1 - 1 \otimes x_i;\ i = 1, \dots, r)(S \hat{\otimes}_k S).$$

Then we put

$$\hat{P}_S^n := S \hat{\otimes}_k S / \hat{I}_S^{n+1}$$

and regard it as an S-module via the map

$$S \to S \hat{\otimes}_k S, \quad a \mapsto a \otimes 1.$$

This module is finitely generated over S. For the affine scheme $Y = \operatorname{Spec} S$, we define a coherent $\mathcal{O}_Y$-module $\hat{\mathcal{P}}_Y^n$ to be the sheaf $\widetilde{\hat{P}_S^n}$ associated to $\hat{P}_S^n$.

Definition 1.10. Suppose that Y is reduced and of pure dimension d, and that $\hat{\mathcal{P}}_Y^n$ is locally free of constant rank $\binom{n+d}{d}$ on an open dense subset of Y. Then we define the *nth Nash blowup* of Y, denoted $\mathbf{Nash}_n(Y)$, to be $\mathbf{Nash}(Y, \hat{\mathcal{P}}_Y^n)$.

Let $\hat{\Delta}_Y^{(n)} := \operatorname{Spec} \hat{P}_S^n$. Then $\mathbf{Nash}_n(X)$ is identified with the union of the irreducible components of $\mathbf{Hilb}_{\binom{d+n}{d}}(\mathrm{pr}_1 : \hat{\Delta}_Y^{(n)} \to Y)$ that dominate irreducible components of Y.

The condition that $\hat{\mathcal{P}}_Y^n$ is locally free of constant rank $\binom{n+d}{d}$ on an open dense subset is perhaps superfluous. From the following lemma, when Y is the completion of a variety at a point or its irreducible component, the condition is indeed satisfied.

Lemma 1.11. *Let $R = k[\mathbf{x}]/\mathfrak{a}$ and $\hat{R} := k[[\mathbf{x}]]/\mathfrak{a}k[[\mathbf{x}]]$. Then there exists a natural isomorphism*

$$\hat{P}_{\hat{R}}^n \cong P_R^n \otimes_R \hat{R}.$$

Proof. Let us view $k[\mathbf{x}] \otimes_k k[\mathbf{x}]$ (respectively, $k[[\mathbf{x}]] \hat{\otimes}_k k[[\mathbf{x}]]$) as a $k[\mathbf{x}]$-algebra (respectively, a $k[[\mathbf{x}]]$-algebra) by the map $x \mapsto x \otimes 1$. We have an isomorphism of $k[\mathbf{x}]$-algebras,

$$\begin{gathered}\phi : k[\mathbf{x}] \otimes_k k[\mathbf{x}] \to k[\mathbf{x}, \mathbf{y}] := k[\mathbf{x}, y_1, \dots, y_r] \\ 1 \otimes x_i \mapsto x_i - y_i\end{gathered}$$

and an isomorphism of $k[[\mathbf{x}]]$-algebras,

$$\begin{gathered}\hat{\phi} : k[[\mathbf{x}]] \hat{\otimes}_k k[[\mathbf{x}]] \to k[[\mathbf{x}, \mathbf{y}]] := k[[\mathbf{x}, y_1, \dots, y_r]] \\ 1 \otimes x_i \mapsto x_i - y_i.\end{gathered}$$

Then $R \otimes_k R \cong k[\mathbf{x}, \mathbf{y}]/\phi(\mathfrak{a} \otimes_k k[\mathbf{x}] + k[\mathbf{x}] \otimes_k \mathfrak{a})$ and

$$P_R^n \cong k[\mathbf{x}, \mathbf{y}]/(\phi(\mathfrak{a} \otimes_k k[\mathbf{x}] + k[\mathbf{x}] \otimes_k \mathfrak{a}) + (y_1, \dots, y_r)^{n+1}).$$

Similarly, if $\hat{\mathfrak{a}}$ denotes $\mathfrak{a}k[[\mathbf{x}]]$, then

$$\hat{P}_{\hat{R}}^n \cong k[[\mathbf{x}, \mathbf{y}]]/(\hat{\phi}(\hat{\mathfrak{a}} \hat{\otimes}_k k[[\mathbf{x}]] + k[[\mathbf{x}]] \hat{\otimes}_k \hat{\mathfrak{a}}) + (y_1, \dots, y_r)^{n+1}).$$

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

We have

$$\begin{aligned}\hat{R} \otimes_R P^n_R &\cong k[[\mathbf{x}]][\mathbf{y}]/(\phi(\mathfrak{a} \otimes_k k[\mathbf{x}] + k[\mathbf{x}] \otimes_k \mathfrak{a}) + (y_1, \dots, y_r)^{n+1}) \\ &\cong k[[\mathbf{x}, \mathbf{y}]]/(\hat{\phi}(\hat{\mathfrak{a}} \hat{\otimes}_k k[[\mathbf{x}]] + k[[\mathbf{x}]] \hat{\otimes}_k \hat{\mathfrak{a}}) + (y_1, \dots, y_r)^{n+1}) \\ &\cong \hat{P}^n_{\hat{R}}.\end{aligned}$$

□

Corollary 1.12. *Let X be a variety, $x \in X$ and $\hat{X} := \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$. Then there exists a natural isomorphism*

$$\mathbf{Nash}_n(\hat{X}) \cong \mathbf{Nash}_n(X) \times_X \hat{X}.$$

Proof. Let $f : \hat{X} \to X$ be the natural morphism. From Lemma 1.11, $\hat{\mathcal{P}}^n_{\hat{X}} \cong f^* \mathcal{P}^n_X$, which implies the corollary. □

2. General properties

2.1 Compatibility with étale morphisms

Theorem 2.1. *Let $Y \to X$ be an étale morphism of varieties. Then for every n, there exists a canonical isomorphism*

$$\mathbf{Nash}_n(Y) \cong \mathbf{Nash}_n(X) \times_X Y.$$

Proof. Let Δ_X and Δ_Y be the diagonals in $X \times_k X$ and $Y \times_k Y$, respectively. Then the natural morphism

$$\Delta_Y^{(n)} \to \Delta_X^{(n)} \times_X Y$$

is an isomorphism. This induces an isomorphism

$$\mathbf{Hilb}_{\binom{d+n}{n}}(\mathrm{pr}_1 : \Delta_Y^{(n)} \to Y) \cong \mathbf{Hilb}_{\binom{d+n}{n}}(\mathrm{pr}_1 : \Delta_X^{(n)} \to X) \times_X Y$$

and the isomorphism of the assertion. □

2.2 Group actions

Let X be a variety of dimension d and G an algebraic group over k acting on X. For each $l \in \mathbb{N}$, we have a natural action of G on $X \times_k \mathbf{Hilb}_l(X)$,

$$\begin{aligned} G \times_k X \times_k \mathbf{Hilb}_l(X) &\to X \times_k \mathbf{Hilb}_l(X) \\ (g, x, [Z]) &\mapsto (gx, [gZ]). \end{aligned}$$

When $l = \binom{d+n}{n}$, the subscheme $\mathbf{Nash}_n(X) \subseteq X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is stable under this action. Thus, the G-action on X naturally lifts to $\mathbf{Nash}_n(X)$ and the morphism $\pi_n : \mathbf{Nash}_n(X) \to X$ is G-equivariant.

2.3 Conductor and Jacobian ideals

We now recall the conductor and Jacobian ideals, and their relation. The conductor ideal plays an important role in what follows, while the Jacobian ideal appears in Conjecture 0.2.

Let R be either a finitely generated k-algebra or a local complete Noetherian ring with coefficient field k. Suppose that R is reduced and of pure dimension d. Let $\tilde{R}$ be the integral closure of R in the total ring of fractions.

Definition 2.2. The *conductor ideal* of R, denoted by $\mathfrak{c}_R$, is the annihilator of the R-module $\tilde{R}/R$.

The conductor ideal is characterized as the largest ideal of R that is also an ideal of $\tilde{R}$.

Definition 2.3. When R is a finitely generated k-algebra (respectively, a complete local Noetherian ring with coefficient field k), then the *Jacobian ideal* of R, denoted $\mathfrak{j}_R$, is the dth fitting ideal of the module of Kähler differentials $\Omega_{R/k}$ (respectively, the complete module of Kähler differentials $\hat{\Omega}_{R/k}$).

If R is represented as

$$R = k[x_1, \dots, x_m]/(f_1, \dots, f_r) \quad \text{or} \quad R = k[[x_1, \dots, x_m]]/(f_1, \dots, f_r),$$

then $\mathfrak{j}_R$ is generated by the $(m-d) \times (m-d)$-minors of the Jacobian matrix $(\partial f_i/\partial x_j)_{i,j}$.

The conductor and Jacobian ideals commute with localizations. Therefore, they define ideal sheaves on varieties. More directly, if X is a variety of dimension d and $\nu : \tilde{X} \to X$ is the normalization, then the *conductor ideal sheaf* $\mathfrak{c}_X \subseteq \mathcal{O}_X$ is defined to be the annihilator ideal sheaf of a coherent $\mathcal{O}_X$-module $\nu_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X$. The *Jacobian ideal sheaf* $\mathfrak{j}_X \subseteq \mathcal{O}_X$ is defined to be the dth fitting ideal sheaf of the sheaf of Kähler differentials $\Omega_{X/k}$. We call the closed subscheme $C_X \subseteq X$ defined by $\mathfrak{c}_X$ the *conductor subscheme* and the closed subscheme $J_X \subseteq X$ defined by $\mathfrak{j}_X$ the *Jacobian subscheme*. Similarly, when $X = \operatorname{Spec} R$ with R a complete local Noetherian ring with coefficient field k, then the conductor subscheme $C_X \subseteq X$ and the Jacobian subscheme $J_X \subseteq X$ are defined to be the subschemes defined by $\mathfrak{c}_R$ and $\mathfrak{j}_R$, respectively.

The conductor and Jacobian ideals commute also with completion. Let R be a finitely generated k-algebra, $\mathfrak{m} \subseteq R$ a maximal ideal and $\hat{R}$ the $\mathfrak{m}$-adic completion of R. Then $\mathfrak{j}_{\hat{R}} = \mathfrak{j}_R \hat{R}$ and $\mathfrak{c}_{\hat{R}} = \mathfrak{c}_R \hat{R}$.

The relation of the conductor and Jacobian ideals is as follows.

Theorem 2.4. *Let R be either a finitely generated k-algebra or a local complete Noetherian ring with coefficient field k. Suppose that R is reduced and of pure dimension d. Then $\mathfrak{j}_R \subseteq \mathfrak{c}_R$.*

Proof. We prove only the case where R is a finitely generated k-algebra. The proof of the other case is parallel.

From the Noether normalization theorem, there exists a k-homomorphism $\phi : k[x_1, \dots, x_d] \to R$ which makes R generically étale over $k[x_1, \dots, x_d]$. We can represent R as

$$R = k[x_1, \dots, x_d][x_{d+1}, \dots, x_m]/(f_1, \dots, f_r) = k[x_1, \dots, x_m]/(f_1, \dots, f_r).$$

Then the Lipman–Sathaye theorem [LS81, Theorem 2] implies that every $(m-d) \times (m-d)$-minor of the matrix $(\partial f_i/\partial x_j)_{\substack{1 \leqslant i \leqslant r \\ d+1 \leqslant j \leqslant m}}$ is contained in $\mathfrak{c}_R$. (For the case where R is not a domain, see [Hoc02, Theorem 3.1].)

For a suitable choice of variables $x_1, \dots, x_m$ and for every subset $\{j_1, \dots, j_d\} \subseteq \{1, \dots, m\}$ of d elements, R is generically étale over $k[x_{j_1}, \dots, x_{j_d}]$. Then $\mathfrak{c}_R$ contains every $(m-d) \times (m-d)$-minor of the matrix $(\partial f_i/\partial x_j)_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant m}}$. Namely $\mathfrak{j}_R \subseteq \mathfrak{c}_R$. □

The following proposition is required in the next section.

Proposition 2.5. *Let R be as above, $X := \operatorname{Spec} R$ and $X_1, \dots, X_l$ be the irreducible components of X.*

(i) *For $1 \leqslant l' \leqslant l$ and for $n \in \mathbb{N}_0$, if we put $X' := X_1 \cup \cdots \cup X_{l'}$, we have*

$$C_{X'} \subseteq C_X \cap X' \quad \text{and} \quad J_{X'}^{(n)} \subseteq J_X^{(n)} \cap X'.$$

Here $\subseteq, \cap, \cup$ are all scheme-theoretic.

(ii) *The following inclusions hold*

$$J_X \supseteq C_X \supseteq X_1 \cap X_2.$$

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

Proof. (i) The first inclusion follows from the inclusion

$$\widetilde{X'} \subseteq \tilde{X} \times_X X'.$$

To show the second inclusion, it suffices to show $J_{X'} \subseteq J_X \cap X'$. If R is finitely generated over k and represented as

$$R = k[x_1, \dots, x_m]/(f_1, \dots, f_r),$$

and if

$$R' = k[x_1, \dots, x_m]/(f_1, \dots, f_r, f_{r+1}, \dots, f_{r'})$$

is the coordinate ring of X', then $\mathrm{j}_R R'$ is generated by the $(m-d)\times(m-d)$-minors of $(\partial f_i/\partial x_j)_{\substack{1\leqslant i\leqslant r\\ 1\leqslant j\leqslant m}}$, while $\mathrm{j}_{R'}$ is generated by the $(m-d)\times(m-d)$-minors of $(\partial f_i/\partial x_j)_{\substack{1\leqslant i\leqslant r'\\ 1\leqslant j\leqslant m}}$. This shows the second inclusion of the assertion in this case. The formal complete case is parallel.

(ii) The inclusion $J_X \supseteq C_X$ is equivalent to Theorem 2.4. Concerning the other inclusion, from assertion (i), we may suppose that X_1 and X_2 are the only irreducible components of X. Let $R_i = R/I_i$, $i = 1, 2$, be the coefficient rings of X_1 and X_2, respectively. Since $R \subseteq R_1 \times R_2 \subseteq \tilde{R}$, we have

$$\mathfrak{c}_R \subseteq \mathrm{ann}(R_1 \times R_2/R) = I_1 + I_2.$$

This proves the assertion. □

2.4 Separation of analytic branches

Let X be a variety of dimension $d > 0$, $\hat{X} := \mathrm{Spec}\,\hat{\mathcal{O}}_{X,x}$ the completion of X at a point $x \in X$ and $\hat{X}_i$, $i = 1, \dots, l$, its irreducible components. Then we have

$$\mathbf{Nash}_n(X) \times_X \hat{X} \cong \mathbf{Nash}_n(\hat{X}) \cong \bigcup_{i=1}^{l} \mathbf{Nash}_n(\hat{X}_i).$$

Let $[Z] \in \mathbf{Nash}_n(X)$ with $\pi_n([Z]) = x$. Then we can regard $[Z]$ also as a (k-)point of $\mathbf{Nash}_n(\hat{X})$ and of $\mathbf{Nash}_n(\hat{X}_{i_0})$ for some $0 \leqslant i_0 \leqslant l$. Then Z is a closed subscheme of $\hat{X}_{i_0}$. Moreover, from Corollary 1.6, $Z \subseteq \hat{X}_{i_0} \cap x^{(n)}$.

Proposition 2.6. *Let $[Z] \in \mathbf{Nash}_n(X)$ with support x and $Z \nsubseteq C_X$. Then Z is contained in a unique analytic branch $\hat{X}_i$. Equivalently, $[Z]$ is contained in $\mathbf{Nash}_n(\hat{X}_i)$ for a unique i.*

Proof. From Proposition 2.5, Z cannot be contained simultaneously in two irreducible components. This proves the proposition. □

Let X, x, Z be as above. If $Z \nsubseteq J_X^{(d-1)}$, then $Z \nsubseteq J_X$ and hence, by Theorem 2.4, $Z \nsubseteq C_X$. Hence, $[Z] \in \mathbf{Nash}_n(\hat{X}_i)$ for a unique i, say i_0. Moreover, from Proposition 2.5, $Z \nsubseteq J_{\hat{X}_{i_0}}^{(d-1)}$. As a consequence, Conjecture 0.2 is reduced to the following conjecture.

Conjecture 2.7. Let R be a local complete Noetherian domain with coefficient field k and $X := \mathrm{Spec}\,R$. Then for every $n \in \mathbb{N}_0$, the nth Nash blowup $\mathbf{Nash}_n(X)$ is well-defined even if X is not algebraizable (see Definition 1.10). Moreover, if $[Z] \in \mathbf{Nash}_n(X)$ with $Z \nsubseteq J_X^{(d-1)}$, then $\mathbf{Nash}_n(X)$ is regular at $[Z]$.

The following proposition assures that a condition such as $Z \nsubseteq C_X$ or $Z \nsubseteq J_X^{(d-1)}$ holds for all $[Z] \in \mathbf{Nash}_n(X)$ if n is sufficiently large.

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

Proposition 2.8. *Let X be a variety of dimension d and $A \subseteq X$ a closed subscheme of dimension less than d. Then there exists $n_0 \in \mathbb{N}_0$ such that for every $n \geqslant n_0$ and for every $[Z] \in \mathbf{Nash}_n(X)$, $Z \nsubseteq A$.*

Proof. Since A is of dimension less than d, for every $a \in A$, the Hilbert function of $\mathcal{O}_{A,a}$ is a polynomial of degree $< d$ for $n \gg 0$. It follows that for $n \gg 0$,

$$\operatorname{length} \mathcal{O}_{A,a}/\mathfrak{m}_{A,a}^{n+1} < \binom{n+d}{d}.$$

Because of the semi-continuity of Hilbert functions proved by Bennett [Ben70], for $n \gg 0$, the inequality holds simultaneously for all $a \in A$.

Let $[Z] \in \mathbf{Nash}_n(X)$ and let a be its support. From Corollary 1.6, $Z \subseteq a^{(n)}$. Since $\operatorname{length} \mathcal{O}_Z = \binom{n+d}{d}$, if the inequality holds, then $Z \nsubseteq A \cap a^{(n)}$ and hence $Z \nsubseteq A$. □

3. Higher Nash blowups of curves

3.1 A deformation-theoretic criterion for normality

Let $R \subseteq k[[x]]$ be a complete k-subalgebra such that $k[[x]]$ is the integral closure of R (in the quotient field of R), $X := \operatorname{Spec} R$ and $\nu : \tilde{X} \to X$ its normalization. Since X is algebraizable, we can define higher Nash blowups of X. To make the computations below simpler, we fix the identification

$$\tilde{X} = \operatorname{Spec} k[[y]]$$

such that the ring homomorphism $\nu^* : R \to k[[y]]$ is the composite of the inclusion $R \hookrightarrow k[[x]]$ and the map $k[[x]] \to k[[y]]$, $x \mapsto -y$. Then the complete fiber product of X and $\tilde{X}$ is represented as

$$X \hat{\times}_k \tilde{X} := \operatorname{Spec} R \hat{\otimes}_k k[[y]] = \operatorname{Spec} R[[y]].$$

The graph $\Gamma_\nu \subseteq X \hat{\times}_k \tilde{X}$ of ν is generically defined by $(x + y)$. To be precise, if $I \subseteq R[[y]]$ is the defining ideal of Γ_ν, then

$$IR[[y]]_I = (x + y).$$

Here $R[[y]]_I$ is the localization of $R[[y]]$ with respect to the prime ideal I. Let $\mathcal{Z}_n \subseteq X \hat{\times}_k \tilde{X}$ be the closed subscheme defined by the $(n + 1)$th symbolic power of I,

$$I^{(n+1)} := R[[y]] \cap I^{n+1} R[[y]]_I.$$

Since the projection

$$q_n : \mathcal{Z}_n \to X$$

is flat, we obtain the corresponding birational morphism

$$\phi_n : \tilde{X} \to \mathbf{Nash}_n(X)$$

such that $\pi_n \circ \phi_n = \nu$.

Let $o \in \tilde{X}$ be the closed point and $Z_n := q_n^{-1}(o) \subseteq X$, the subscheme corresponding to $\phi_n(o) \in \mathbf{Nash}_n(X)$. Consider the natural morphism

$$\epsilon : \operatorname{Spec} k[y]/(y^2) \to \operatorname{Spec} k[[y]] = \tilde{X},$$

which is a nonzero tangent vector of $\tilde{X}$ at o. The fiber product

$$\mathcal{Z}_{n,\epsilon} := \mathcal{Z}_n \times_{q_n, \tilde{X}, \epsilon} \operatorname{Spec} k[y]/(y^2) \subseteq X \times_k \operatorname{Spec} k[y]/(y^2)$$

is the first-order embedded deformation of $Z_n \subseteq X$ corresponding to

$$\phi_n \circ \epsilon : \operatorname{Spec} k[y]/(y^2) \to \mathbf{Nash}_n(X).$$

Let $\mathfrak{a}_n \subseteq R$ be the defining ideal of Z_n, which is identical to $I^{(n+1)}$ modulo (y).

Theorem 3.1. *Suppose that X is non-normal. Then the following are equivalent:*

(i) *$\mathbf{Nash}_n(X)$ is normal;*

(ii) *ϕ_n is an isomorphism;*

(iii) *$\mathcal{Z}_{n,\epsilon}$ is not the trivial embedded deformation of Z_n;*

(iv) *there exists an element $g \in I^{(n+1)} \subseteq R[[y]]$ such that if we write*

$$g = g_0 + g_1 y + g_2 y^2 + \cdots, \quad g_i \in R,$$

then $g_1 \notin \mathfrak{a}_n$.

Proof. (i) $\Leftrightarrow$ (ii). This is obvious.

(ii) $\Leftrightarrow$ (iii). The morphism $\phi_n \circ \epsilon$ corresponding to the pair $(\nu \circ \epsilon, \mathcal{Z}_{n,\epsilon})$. From the assumption, $\nu \circ \epsilon$ is the zero tangent vector, that is, factors as $\operatorname{Spec} k[y]/(y^2) \to \operatorname{Spec} k \to X$. Hence, $\phi_n \circ \epsilon$ is the zero tangent vector if and only if $\mathcal{Z}_{n,\epsilon}$ is trivial. This shows the equivalence (ii) $\Leftrightarrow$ (iii).

(iii) $\Leftrightarrow$ (iv). If the defining ideal of $\mathcal{Z}_{n,\epsilon}$ in $R[y]/(y^2)$ is generated by

$$g_{j0} + g_{j1} y, \quad g_{j0}, g_{j1} \in R, j = 1, \ldots, m,$$

then $\mathcal{Z}_{n,\epsilon}$ corresponds to the homomorphism

$$\mathfrak{a}_n \to R/\mathfrak{a}_n, \quad g_{j0} \mapsto g_{j1}.$$

In particular, saying that $\mathcal{Z}_{n,\epsilon}$ is trivial is equivalent to saying that the homomorphism is the zero map. Hence, (iii) $\Leftrightarrow$ (iv). $\square$

Remark 3.2. In the theorem above, the assumption that X is non-normal is necessary. For example, in characteristic $p > 0$, if X is normal, then $\mathcal{Z}_{pm-1,\epsilon}$, $m \in \mathbb{N}_0$, are trivial.

3.2 The associated numerical monoid

A *numerical monoid* is by definition a submonoid S of the (additive) monoid $\mathbb{N}_0$ with $\sharp(\mathbb{N}_0 \setminus S) < \infty$.

To $R \subseteq k[[x]]$ as above, we associate a numerical monoid

$$S := \{i \in \mathbb{N}_0 \mid \exists f \in R, \operatorname{ord} f = i\} = \{0 = s_{-1} < s_0 < s_1 < \cdots\}.$$

Theorem 3.3. *Let $X := \operatorname{Spec} R$. Suppose that k has characteristic zero. Then $\mathbf{Nash}_n(X)$ is normal if and only if $s_n - 1 \in S$.*

Lemma 3.4. *Let $\mathbf{a} := \{a_1 < a_2 < \cdots < a_e\} \subseteq \mathbb{N}$ and define a $(e \times e)$-matrix*

$$M(n; \mathbf{a}) := \begin{pmatrix} \binom{n}{a_1} & \binom{n}{a_1 - 1} & \cdots & \binom{n}{a_1 - e + 1} \\ \binom{n}{a_2} & \binom{n}{a_2 - 1} & \cdots & \binom{n}{a_2 - e + 1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{a_e} & \binom{n}{a_e - 1} & \cdots & \binom{n}{a_e - e + 1} \end{pmatrix}$$

with entries in an algebraically closed field k of characteristic 0. Here $\binom{a}{b} := 0$ if either $b > a$ or $b < 0$. Then

$$\det M(n; \mathbf{a}) = \frac{\prod_{i<j}(a_j - a_i) \prod_{i=1}^{e}((n + e - i)(n + e - i - 1) \cdots (n + e - a_i))}{\prod_{i=1}^{e} a_i!}.$$

In particular, if $n + e - a_e > 0$, then $\det M(n; \mathbf{a}) \neq 0$, and the matrix $M(n; \mathbf{a})$ is regular.

Proof. This matrix appears also in [ACGH07, p. 353].

Without changing the determinant, we can replace the first column with the sum of the first and the second, and the second with the sum of the second and the third, and so on. The resulting matrix is

$$\begin{pmatrix} \binom{n+1}{a_1} & \binom{n+1}{a_1-1} & \cdots & \binom{n+1}{a_1-e+2} & \binom{n}{a_1-e+1} \\ \binom{n+1}{a_2} & \binom{n+1}{a_2-1} & \cdots & \binom{n+1}{a_2-e+2} & \binom{n}{a_2-e+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n+1}{a_e} & \binom{n+1}{a_e-1} & \cdots & \binom{n+1}{a_e-e+2} & \binom{n}{a_e-e+1} \end{pmatrix}.$$

Again we replace the first column with the sum of the second and the third, and so on. We obtain

$$\begin{pmatrix} \binom{n+2}{a_1} & \binom{n+2}{a_1-1} & \cdots & \binom{n+2}{a_1-e+3} & \binom{n+1}{a_1-e+2} & \binom{n}{a_1-e+1} \\ \binom{n+2}{a_2} & \binom{n+2}{a_2-1} & \cdots & \binom{n+2}{a_2-e+3} & \binom{n+1}{a_2-e+2} & \binom{n}{a_2-e+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \binom{n+2}{a_e} & \binom{n+2}{a_e-1} & \cdots & \binom{n+2}{a_e-e+3} & \binom{n+1}{a_e-e+2} & \binom{n}{a_e-e+1} \end{pmatrix}.$$

Repeating this, we finally arrive at

$$\begin{pmatrix} \binom{n+e-1}{a_1} & \binom{n+e-2}{a_1-1} & \cdots & \binom{n+1}{a_1-e+2} & \binom{n}{a_1-e+1} \\ \binom{n+e-1}{a_2} & \binom{n+e-2}{a_2-1} & \cdots & \binom{n+1}{a_2-e+2} & \binom{n}{a_2-e+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n+e-1}{a_e} & \binom{n+e-2}{a_e-1} & \cdots & \binom{n+1}{a_e-e+2} & \binom{n}{a_e-e+1} \end{pmatrix}.$$

(Check that this transformation makes sense even if the matrix $M(n;\mathbf{a})$ contains zero entries.) Then we have

$$\begin{aligned} \det M(n;\mathbf{a}) &= \det \begin{pmatrix} \binom{n+e-1}{a_1} & \binom{n+e-2}{a_1-1} & \cdots & \binom{n+1}{a_1-e+2} & \binom{n}{a_1-e+1} \\ \binom{n+e-1}{a_2} & \binom{n+e-2}{a_2-1} & \cdots & \binom{n+1}{a_2-e+2} & \binom{n}{a_2-e+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n+e-1}{a_e} & \binom{n+e-2}{a_e-1} & \cdots & \binom{n+1}{a_e-e+2} & \binom{n}{a_e-e+1} \end{pmatrix} \\ &= \prod_{i=1}^{e} ((n+e-i)(n+e-i-1)\cdots(n+e-a_i)) \\ &\quad \times \det \begin{pmatrix} (a_1!)^{-1} & ((a_1-1)!)^{-1} & \cdots & ((a_1-e+1)!)^{-1} \\ (a_2!)^{-1} & ((a_2-1)!)^{-1} & \cdots & ((a_2-e+1)!)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (a_e!)^{-1} & ((a_e-1)!)^{-1} & \cdots & ((a_e-e+1)!)^{-1} \end{pmatrix} \\ &= \frac{\prod_{i=1}^{e}((n+e-i)(n+e-i-1)\cdots(n+e-a_i))}{\prod_{i=1}^{e} a_i!} \\ &\quad \times \det \begin{pmatrix} 1 & a_1 & a_1(a_1-1) & \cdots & a_1(a_1-1)\cdots(a_1-e+2) \\ 1 & a_2 & a_2(a_2-1) & \cdots & a_2(a_2-1)\cdots(a_2-e+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_e & a_e(a_e-1) & \cdots & a_e(a_e-1)\cdots(a_e-e+2) \end{pmatrix} \end{aligned}$$

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

$$= \frac{\prod_{i=1}^{e}((n+e-i)(n+e-i-1)\cdots(n+e-a_i))}{\prod_{i=1}^{e} a_i!}$$

$$\times \det \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{e-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{e-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_e & a_e^2 & \cdots & a_e^{e-1} \end{pmatrix}$$

$$= \frac{\prod_{i=1}^{e}((n+e-i)(n+e-i-1)\cdots(n+e-a_i))}{\prod_{i=1}^{e} a_i!} \prod_{i<j}(a_j - a_i)$$

(Vandermonde's determinant). □

Proof of Theorem 3.3. Put $T := \mathbb{N}_0 \setminus S = \{t_1 < t_2 < \cdots < t_l\}$. Let $\mathbf{t}_{n,0} := \{t \in T \mid t < s_n\} = \{t_1 < t_2 < \cdots < t_{l_n}\}$, where $l_n := \sharp \mathbf{t}_{n,0}$, and $\mathbf{u}_{n,0} := \mathbf{t}_{n,0} \cup \{s_n\}$. Then $s_n = l_n + n + 1$. From Lemma 3.4, the matrix $M(n+1; \mathbf{u}_{n,0})$ is regular. We define $r_{n,i} \in k$, $i = 1, \ldots, l_n$, by the equation

$$M(n+1; \mathbf{u}_{n,0}) \begin{pmatrix} r_{n,0} \\ \vdots \\ r_{n,l_n-1} \\ r_{n,l_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then we define a homogeneous polynomial of degree s_n,

$$f_{n,0} := (r_{n,0} y^{l_n} + r_{n,1} x y^{l_n - 1} + \cdots + r_{n,l_n-1} x^{l_n-1} y + r_{n,l_n} x^{l_n})(x+y)^{n+1} \in k[[x,y]].$$

For $1 \leqslant i \leqslant l_n$, the coefficient of $x^{t_i} y^{s_n - t_i}$ in $f_{n,0}$ is

$$r_{n,0}\binom{n+1}{t_i} + r_{n,1}\binom{n+1}{t_i - 1} + \cdots + r_{t,l_n}\binom{n+1}{t_i - l_n} = 0$$

and the coefficient of $x^{s_n} = x^{n+l_n+1}$ is 1.

For $j \in \mathbb{N}$, we put $\mathbf{t}_{n,j} := \{t \in T \mid t \leqslant s_n + j\}$. If $m_{n,j} := l_n + j - \sharp \mathbf{t}_{n,j} \geqslant 0$, then we put $\mathbf{s}_{n,j} := \{s_0 < s_1 < \cdots < s_{m_{n,j}}\}$ and $\mathbf{u}_{n,j} := \mathbf{t}_{n,j} \cup \mathbf{s}_{n,j}$. Then $\sharp \mathbf{u}_{n,j} = l_n + j + 1$. Since

$$(n+1) + \sharp \mathbf{u}_{n,j} - \max \mathbf{u}_{n,j} \geqslant (n+1) + l_n + j + 1 - (s_n + j) = 1,$$

from Lemma 3.4, $M(n+1, \mathbf{u}_{n,j})$ is regular. Therefore, from the same argument as above, for every $(d_i; i \in \mathbf{u}_{n,j}) \in k^{\mathbf{u}_{n,j}}$, there exists a unique homogeneous polynomial $g \in k[[x,y]]$ such that:

(i) g has degree $(l+j) + (n+1) = s_n + j$;
(ii) g is divided by $(x+y)^{n+1}$; and
(iii) for each $i \in \mathbf{u}_{n,j}$, the coefficient of the term $x^i y^{s_n + j - i}$ is d_i.

Now we inductively choose homogeneous polynomials $f_{n,j}$, $j \in \mathbb{N}$, of degree $s_n + j$ divisible by $(x+y)^{n+1}$ as follows. For each $i \in \mathbb{N}_0$, we can take an element

$$h_i = \sum_{j \geqslant 0} h_{i,j} x^{i+j} \in R, \quad h_{i,j} \in k$$

such that:

(i) $h_0 = 1$;
(ii) for $i \in S$, $h_{i,0} = 1$;
(iii) for $i \in T$, $h_i = 0$; and
(iv) if $j > 0$ and $i + j \in S$, then $h_{i,j} = 0$. (In particular, if $i > t_l$ and $j > 0$, then $h_{i,j} = 0$.)

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

Suppose that we have chosen $f_{n,0}, f_{n,1}, \ldots, f_{n,j-1}$. Let $c_{i,j'}$, $i \leqslant s_n + j'$, $0 \leqslant j' < j$, be the coefficient of $x^i y^{s_n+j'-i}$ in $f_{n,j'}$. By convention, we put $c_{i,j'} := 0$ for $i < 0$ or for $j' < 0$. For $i \in \mathbf{s}_{n,j}$, put $c_{i,j} := 0$. For $i \in \mathbf{t}_{n,j}$, put

$$c_{i,j} := \sum_{a=1}^{j} c_{i-a,j-a} h_{i-a,a}.$$

Then we choose $f_{n,j}$ such that for every $i \in \mathbf{u}_{n,j}$, the coefficient of $x^i y^{s_n+j-i}$ is $c_{i,j}$.

We claim that for $j \gg 0$, $f_{n,j} = 0$. To see this, we first observe that for $j \gg 0$, the coefficients of $x^i y^{s_n+j-i}$, $i \in \{s \in S \mid s < t_l\} \subseteq \mathbf{s}_{n,j}$, are all 0. Then, if necessary, replacing j with a still larger integer, we obtain that $f_{n,j-1}, f_{n,j-2}, \ldots, f_{n,j-t_{l_n}}$ all have this property. Then for every $i \in \mathbf{t}_{n,j}$, $c_{i,j} = 0$. From the uniqueness, $f_{n,j} = 0$.

Define $f_n := \sum_{j=0}^{\infty} f_{n,j}$. Then f_n is divided by $(x+y)^{n+1}$. Moreover,

$$\begin{aligned} f_n &= \sum_{i,j} c_{i,j} x^i y^{s_n+j-i} \\ &= \sum_{\substack{i,j \\ i\in S}} c_{i,j} x^i y^{s_n+j-i} + \sum_{\substack{i,j \\ i\in T}} \left(\sum_{a=1}^{j} c_{i-a,j-a} h_{i-a,a} \right) x^i y^{s_n+j-i} \\ &= \sum_{\substack{i,j \\ i\in S}} c_{i,j} x^i y^{s_n+j-i} + \sum_{\substack{i,j,a \\ a>0, i+a\in T}} c_{i,j} h_{i,a} x^{i+a} y^{s_n+j-i} \\ &= \sum_{i,j} c_{i,j} h_i y^{s_n+j-i}. \end{aligned}$$

Thus, $f_n \in R[[y]]$ and so $f_n \in I^{(n+1)}$. By construction,

$$f_n(x,0) = x^{s_n} + (\text{higher terms}) \in \mathfrak{a}_n.$$

Similarly, for every $n' \geqslant n$, $f_{n'} \in I^{(n+1)}$, and

$$f_{n'}(x,0) = x^{s_{n'}} + (\text{higher terms}) \in \mathfrak{a}_n.$$

Since

$$\operatorname{length} R/(f_{n'}(x,0); n' \geqslant n) = n+1,$$

$\mathfrak{a}_n$ is, in fact, generated by $f_{n'}(x,0)$, $n' \geqslant n$, and identical to $\{f \in R \mid \operatorname{ord} f \geqslant s_n\}$. It follows that $I^{(n+1)}$ is generated by $f_{n'}$, $n' \geqslant n$.

Write

$$f_n \equiv f_n(x,0) + g_n y \mod (y^2), \quad g_n \in R.$$

From Theorem 3.1, $\mathbf{Nash}_n(X)$ is normal if and only if for some $n' \geqslant n$, $g_{n'} \notin \mathfrak{a}_n$. For every $n' > n$, $g_{n'}$ has order at least s_n, and so $g_{n'} \in \mathfrak{a}_n$.

Now the normality of $\mathbf{Nash}_n(X)$ is equivalent to saying that g_n has order $s_n - 1$ or, equivalently, $c_{s_n-1,0} \neq 0$. If $s_n - 1 \in T$, then $s_n - 1 = t_{l_n}$ and, by the construction, $c_{s_n-1,0} = 0$. If $s_n - 1 \in S$, then put $\mathbf{u}'_{n,0} := \{t_1, \ldots, t_{l_n}, s_n - 1\}$. From Lemma 3.4, the matrix $M(n+1; \mathbf{u}'_{n,0})$ is regular. We have

$$M(n+1; \mathbf{u}'_{n,0}) \begin{pmatrix} r_{n,0} \\ \vdots \\ r_{n,l_n-1} \\ r_{n,l_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_{s_n-1,0} \end{pmatrix} \neq 0.$$

This completes the proof. ∎

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

Example 3.5. We note that for every numerical monoid S, there exists $R \subseteq k[[x]]$ whose associated monoid is S. Suppose that S is the numerical monoid generated by 5 and 7. Then

$$S = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, n; n \geqslant 24\}.$$

Theorem 3.3 now states that

$$\mathbf{Nash}_n(X) \text{ is } \begin{cases} \text{non-normal} & (n = 0, 1, 2, 3, 4, 6, 7, 11) \\ \text{normal} & (\text{otherwise}). \end{cases}$$

Example 3.6. If for some m, $S = \{0, m, m+1, m+2, \dots\}$, then for every $n > 0$, $\mathbf{Nash}_n(X)$ is normal.

3.3 Conjecture 0.2 for curves

Corollary 3.7. *Suppose that k has characteristic zero. Let X be either a variety of dimension 1 or $\operatorname{Spec} R$ with R a reduced local complete Noetherian ring of dimension 1 with coefficient field k. Let $C \subseteq X$ be the conductor subscheme and $[Z] \in \mathbf{Nash}_n(X)$ with $Z \nsubseteq C$. Then $\mathbf{Nash}_n(X)$ is normal at $[Z]$. In particular, Conjecture 0.2 is true in dimension 1.*

Proof. The second assertion is a consequence of the first and Theorem 2.4. We now prove the first assertion. From Corollary 1.12, we may suppose that $X = \operatorname{Spec} R$ with R a reduced local complete Noetherian ring with coefficient field k. From Proposition 2.5, Z is contained in a unique irreducible component of X, say X_0. If C_0 is the conductor subscheme of X_0, then from Proposition 2.5, we have $Z \nsubseteq C_0$. Hence, it suffices to prove only the case where R is a domain, the case as in Theorem 3.3. With the notation as in Theorem 3.3, the conductor ideal $\mathfrak{c}$ of R is $(x^i; i > t_l)$. If $s_n > t_l + 1$, then from Theorem 3.3, $\mathbf{Nash}_n(X)$ is smooth. If $s_n \leqslant t_l + 1$, then as we saw in the proof of Theorem 3.3, $\mathfrak{a}_n = \{f \in R \mid \operatorname{ord} f \geqslant s_n\} \supseteq \mathfrak{c}$ and the condition, $Z \nsubseteq C$, is not satisfied. This completes the proof. □

If $Z = C$, then $\mathbf{Nash}_n(X)$ is not generally smooth at $[Z]$. In fact, with $X = \operatorname{Spec} R$ as in Theorem 3.3, if $n = i_0 := \max\{i \mid s_i - 1 \notin R\}$, then $Z_n = C$ and $\mathbf{Nash}_n(X)$ is non-normal at $[Z_n]$. Therefore, we cannot replace C in the corollary with any smaller subscheme of X.

3.4 Positive characteristic

As the following propositions show, it is impossible to resolve curve singularities in positive characteristic via higher Nash blowups.

Proposition 3.8. *Let $X = \operatorname{Spec} R$ be as in Theorem 3.3. Suppose that k has characteristic $p > 0$. Then for $e \gg 0$,*

$$\mathbf{Nash}_{p^e-1}(X) \cong X.$$

Proof. For $e \gg 0$,

$$(x+y)^{p^e} = x^{p^e} + y^{p^e} \in R \hat{\otimes}_k R \subseteq k[[x, y]].$$

Let $\mathcal{W} \subseteq X \times_k X$ be the closed subscheme defined by the ideal $(x+y)^{p^e}$. If $q \in X$ is the image of the origin $o \in \tilde{X} = \operatorname{Spec} k[x]$, then the fiber of $\mathrm{pr}_2 : \mathcal{W} \to X$ over q is $\operatorname{Spec} R/x^{p^e} R$. From [Eis99, Lemma 11.12],

$$\operatorname{length} R/x^{p^e} R = \operatorname{length} k[x]/x^{p^e} k[x] = p^e.$$

From [Eis99, Example 20.13], $\mathrm{pr}_1 : \mathcal{W} \to X$ is flat. There exists a corresponding morphism

$$X \to \mathbf{Nash}_{p^e-1}(X),$$

which is the inverse of $\pi_{p^e-1} : \mathbf{Nash}_{p^e-1}(X) \to X$. We have proved the proposition. □

PROPOSITION 3.9. *Suppose that k has characteristic either two or three. Let $X := \operatorname{Spec} k[[x^2, x^3]]$. Then for every $n \in \mathbb{N}_0$, $\mathbf{Nash}_n(X) \cong X$.*

Proof. We first consider the case of characteristic two. For $n \in \mathbb{N}$,

$$
\begin{aligned}
(x+y)^n &= x^n + nx^{n-1}y + \left(\sum_{i=2}^{n-2}\binom{n}{i}x^i y^{n-i}\right) + nxy^{n-1} + y^n \\
&= \begin{cases} x^n + \left(\sum_{i=2}^{n-2}\binom{n}{i}x^i y^{n-i}\right) + y^n & (n \text{ even}), \\ x^n + x^{n-1}y + \left(\sum_{i=2}^{n-2}\binom{n}{i}x^i y^{n-i}\right) + xy^{n-1} + y^n & (n \text{ odd}). \end{cases}
\end{aligned}
$$

Thus, for odd n, $(x+y)^{n+1} \in R\hat{\otimes}_k R$. By the same argument with the proof of the last proposition, we see that $\mathbf{Nash}_n(X) \cong X$.

For even n, the coefficients of $x^{n+2}y$ and xy^{n+2} in $(x^2+xy+y^2)(x+y)^{n+1}$ are both zero. Therefore, $(x^2+xy+y^2)(x+y)^{n+1} \in R\hat{\otimes}_k R$. For an ideal

$$I := ((x+y)^{n+2}, (x^2+xy+y^2)(x+y)^{n+1}) \subseteq R\hat{\otimes}_k R,$$

we have

$$\operatorname{length} R/IR = \operatorname{length} R/(x^{n+2}, x^{n+3})R = n+1.$$

Again by the same argument, we can show the assertion in the case where n is even.

We next consider the case of characteristic three. Similarly we have

$$
\begin{gathered}
(x+y)^n \in R \otimes_k R (n \equiv 0 \mod 3), \\
(x-y)(x+y)^n = (x^2-y^2)(x+y)^{n-1} \in R \otimes_k R (n \equiv 1 \mod 3), \\
(x^2+xy+y^2)(x+y)^n \in R \otimes_k R (n \equiv 2 \mod 3).
\end{gathered}
$$

For each $n \in \mathbb{N}$, we define an ideal $I \subseteq R \otimes_k R$ as follows:

$$
I := \begin{cases} ((x-y)(x+y)^{n+1}, (x+y)^{n+3}) & (n \equiv 0 \mod 3) \\ ((x+y)^{n+2}, (x^2+xy+y^2)(x+y)^{n+1}) & (n \equiv 1 \mod 3) \\ ((x+y)^{n+1}) & (n \equiv 2 \mod 3). \end{cases}
$$

Then

$$
\begin{aligned}
\operatorname{length} R/IR &= \begin{cases} \operatorname{length} R/x^{n+1}R \text{ or} \\ \operatorname{length} R/(x^{n+2}, x^{n+3})R \end{cases} \\
&= n+1.
\end{aligned}
$$

We can similarly show the assertion. □

ACKNOWLEDGEMENTS

The author would like to thank Shigefumi Mori, Shigeru Mukai, Hisanori Ohashi, Shunsuke Takagi, Masataka Tomari and Kei-ichi Watanabe for many useful comments.

REFERENCES

ACGH07 E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves*, Vol. I, Grundlehren der Mathematischen Wissenschaften, vol. 267 (Springer, New York, 2007).

https://doi.org/10.1112/S0010437X0700276X Published online by Cambridge University Press

Ben70 B. M. Bennett, *On the characteristic function of a local ring*, Ann. of Math. (2) **91** (1970), 25–87.

DL99 J. Denef and F. Loeser, *Germs of arcs on singular varieties and motivic integration*, Invent. Math. **135** (1999), 201–232.

Eis99 D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150 (Springer, New York, 1999).

Fog68 J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521.

Gon82 G. González-Sprinberg, *Résolution de Nash des points doubles rationnels*, Ann. Inst. Fourier (Grenoble) **32** (1982), 111–178.

Gro67 A. Grothendieck, *Élément de géométrie algébrique IV*, in *Étude locale des schémas et des morphismes de schémas, Quatrième partie*, Publ. Math. Inst. Hautes Études Sci., vol. 32 (Springer, Heidelberg, 1967).

Hir83 H. Hironaka, *On Nash blowing-up*, in *Arithmetic and Geometry II* (Birkhäuser, Basel, 1983), 103–111.

Hoc02 M. Hochster, *Presentation depth and the Lipman–Sathaye Jacobian theorem*, Homology Homotopy Appl. **4** (2002), 295–314.

LS81 J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon–Skoda*, Michigan Math. J. **28** (1981), 199–222.

Nob75 A. Nobile, *Some properties of the Nash blowing-up*, Pacific J. Math. **60** (1975), 297–305.

OZ91 A. Oneto and E. Zatini, *Remarks on Nash blowing-up*, Rend. Sem. Mat. Univ. Politec. Torino **49** (1991), 71–82.

Reb77 V. Rebassoo, *Desingularization properties of the Nash blowing-up process*, PhD thesis, University of Washington (1977).

Spi90 M. Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Ann. of Math. (2) **131** (1990), 411–491.

Takehiko Yasuda takehiko@kurims.kyoto-u.ac.jp

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan