THE ITERATED EQUATION OF GENERALIZED AXIALLY SYMMETRIC POTENTIAL THEORY

V. GENERALIZED WEINSTEIN CORRESPONDENCE PRINCIPLE

J. C. BURNS

(Received 8 July 1968)

Communicated by K. C. Westfold

1. Introduction

Solutions of the iterated equation of generalized axially symmetric potential theory [1],

$$L_k^n(f) = 0, \tag{1}$$

where the operator L_k is defined by

$$L_k(f) \equiv \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + ky^{-1}\,\partial f/\partial y,$$

will be denoted by $f_k^{(n)}$, except that when $n = 1$, f_k will be written instead of $f_k^{(1)}$. It is easily shown [2, 3] that

$$f_k^{(n)} \leftrightarrow y^{1-k} f_{2-k}^{(n)}, \tag{2}$$

by which is meant that any function $f_k^{(n)}$ can be expressed in the form $y^{1-k} f_{2-k}^{(n)}$ and, conversely, for any function $f_{2-k}^{(n)}$, $y^{1-k} f_{2-k}^{(n)}$ is a solution of (1).

The particular case $n = 1$ of this relation:

$$f_k \leftrightarrow y^{1-k} f_{2-k} \tag{3}$$

is well known as Weinstein's correspondence principle [1] and it is for this reason that (2) was described in [3] as the generalized Weinstein correspondence principle.

Weinstein and others, notably Payne [4], have made great use of this relation in discussing physical problems, particularly in the field of the two-dimensional and axially symmetric flow of an inviscid, incompressible fluid. In such problems it is convenient to consider the pair of equations

$$L_p(\phi) = 0, \quad L_{-p}(\psi) = 0, \tag{4}$$

for the velocity potential ϕ and the stream function ψ, the parameter p taking values 0 and 1 for two-dimensional and axially symmetric flow respectively. If solutions of (4) are denoted by ϕ_p and ψ_p, the correspondence principle (3) becomes

$$\psi_p \leftrightarrow y^{1+p}\phi_{p+2}. \tag{5}$$

The problem of determining the flow about a given symmetrical body when the flow is uniform at infinity can be simplified by the use of this principle (5) which, in Weinstein's words [1], 'reduces the determination of ψ_p to the determination of the electrostatic potential ϕ_{p+2} of the body with the same meridian profile but in a space of two more dimensions.' The procedure is set out by Weinstein [1] and by Payne [4] who applies the method to find the stream function for bodies such as a spindle, lens, spheroid set in a stream which is uniform at infinity. The correspondence principle provides a powerful and systematic method for dealing with all of these problems.

The aim of this paper is to show how the generalized correspondence principle (2) can similarly be used to systematize the treatment of problems involving the solution of the iterated equation (1). It will be shown that in several classes of such problems the generalized correspondence principle, together with an appropriate choice of general solution of (1), can be used to set up a general procedure which leads automatically to a successful analysis. The method will be illustrated in some problems in the Stokes flow of a viscous fluid which are analogous to the problems in inviscid flow mentioned above; in these problems the stream function ψ satisfies the equation $L^2_{-p}(\psi) = 0$. Payne and Pell [5, 6, 7] have discussed these Stokes flow problems and have carried out the difficult analysis required to produce the solutions. As would be expected, essentially the same detailed calculations are required however the problems are approached but the new procedure proposed here sets these calculations in a unified context and provides a natural approach to them.

In one simple case, when the boundary in the $x-y$ plane is the circle $r = a$, a formal solution is obtained for equations (1) for a general value of n. This case is interesting in that it reveals explicitly the need for appropriate restrictions on p for given n for the solution to satisfy both the conditions at infinity and those on the boundary of the circle. In particular, the impossibility of a solution for two-dimensional Stokes flow ($p = 0$, $n = 2$) past a circular cylinder is clear.

2. Statement of problem

The problem to be considered is the generalization of the classical hydrodynamical problem of determining the flow when a uniform stream is disturbed by an axially symmetric obstacle (or system of obstacles).

Weinstein [1] states the problem for $n = 1$ and shows how to solve it using his correspondence principle. Payne and Pell [5, 6, 7] discuss the corresponding problem for the case $n = 2$ in the context of Stokes flow and their accounts form the basis of the following statement of the general problem.

It is required to find a function $\Psi(x, y)$ which satisfies the equation

$$L^{n}_{-p}(\Psi) = 0 \tag{6}$$

in the region of the plane $y \geqq 0$ exterior to the boundary which can be one of two types (i) an arc C which joins two points on the x-axis but otherwise lies above the x-axis; or (ii) a closed contour C' which lies entirely above the x-axis. (Payne and Pell in fact consider the more general case in which there are several curves C or C'.)

The function Ψ must be such that on the curves C or C',

$$\Psi = \chi, \quad \frac{\partial^s \Psi}{\partial \nu^s} = 0 \quad \text{for } 1 \leqq s \leqq n-1, \tag{7}$$

where ν denotes the normal directed into the region defined above. The constant χ will be zero in case (i) but in case (ii) it will have a non-zero value which is prescribed. (In some problems the value of χ is fixed by the physical conditions: see Pell and Payne [7].)

Since Ψ is in any case constant on C or C', the tangential derivatives of Ψ are all zero there. Thus, to ensure that the normal derivatives vanish it is sufficient to require that the derivatives in some direction other than the tangential be zero. Thus, provided C and C' have their tangents parallel to the x-axis only at isolated points, the conditions (7) on Ψ can be replaced by conditions expressed in a form which facilitates the calculations: on C or C', Ψ must be such that

$$\Psi = \chi, \quad \frac{\partial^s \Psi}{\partial x^s} = 0 \quad \text{for } 1 \leqq s \leqq n-1. \tag{8}$$

Finally, to ensure uniform conditions at infinity, it is required that

$$\lim_{r\to\infty} \Psi(x, y) = Uy^{p+1}/(p+1) \tag{9}$$

where $r^2 = x^2+y^2$.

The two cases (i) and (ii) distinguished above will now be considered separately.

When the boundary profile is a curve C joining two points on the x-axis, $\Psi = 0$ on C and the function Ψ is conveniently expressed in the form

$$\Psi(x, y) = \frac{Uy^{p+1}}{p+1} - \psi(x, y).$$

https://doi.org/10.1017/S1446788700006327 Published online by Cambridge University Press

$\psi(x, y)$ must be such that $L^n_{-p}(\psi) = 0$, $\psi = o(y^{p+1})$ as $r \to \infty$ and, on C,

$$\psi = \frac{Uy^{p+1}}{p+1}, \quad \frac{\partial^s \psi}{\partial x^s} = 0 \quad \text{for } 1 \leqq s \leqq n-1. \tag{10}$$

It is here that the generalized correspondence principle is first invoked to transform the problem; this is made possible by the occurrence of the factor y^{p+1} both in the form assumed by Ψ at infinity and in the correspondence principle. The function ψ is replaced by $Uy^{p+1}\phi/(p+1)$ where ϕ is a solution of the equation

$$L^{(n)}_{p+2}(\phi) = 0 \tag{11}$$

and must be such that $\phi \to 0$ as $r \to \infty$ and ϕ satisfies the conditions on C:

$$\phi = 1, \quad \frac{\partial^s \phi}{\partial x^s} = 0 \quad \text{for } 1 \leqq s \leqq n-1. \tag{12}$$

The problem is thus reduced to finding this function ϕ, for which the boundary conditions on C and the conditions at infinity take a simple form. Once ϕ has been found, the original function Ψ is given by

$$\Psi(x, y) = \frac{Uy^{p+1}}{p+1}\{1-\phi(x, y)\}. \tag{13}$$

When the boundary profile consists of a closed curve C' in the upper half of the plane, $\Psi = \chi$ on C' and Ψ should be expressed [7] in the form

$$\Psi(x, y) = \frac{Uy^{p+1}}{p+1} - \psi(x, y) + \chi\bar{\psi}(x, y),$$

where $\psi(x, y)$ and $\bar{\psi}(x, y)$ are chosen so that each satisfies equation (6), each is $o(y^{p+1})$ as $r \to \infty$ and on C', ψ satisfies conditions (10) while $\bar{\psi}$ satisfies conditions (12) (with $\bar{\psi}$ written for ϕ). The correspondence principle is used again to transform ψ into ϕ exactly as before so that the problem is reduced to finding the two functions ϕ and $\bar{\psi}$, satisfying equations (11) and (6) respectively and each satisfying conditions on C' of type (12). The original function Ψ is then given by

$$\Psi(x, y) = \frac{Uy^{p+1}}{p+1}\{1-\phi(x, y)\} + \chi\bar{\psi}(x, y). \tag{14}$$

3. General solutions of $L^n_k(f) = 0$

In solving equations (11) and (6), both of type (1), it will be necessary to make use of general solutions of (1) which express the solution as linear combinations of terms involving arbitrary functions f_k which are solutions of the simple equation $L_k(f) = 0$.

https://doi.org/10.1017/S1446788700006327 Published online by Cambridge University Press

In particular, use will be made of the solutions

$$f_k + f_{k-2} + \cdots + f_{k-2(n-1)}, \tag{15}$$

$$f_{k-2(n-1)} + y^2 f_{k-2(n-3)} + \cdots + y^{2(n-1)} f_{k+2(n-1)}. \tag{16}$$

The solution (15) was given by Weinstein [8] and is valid for all k while (16), given in [9], is valid for $k \neq 1-2i$ where $i = 0, 1, 2, \cdots (n-2)$.

4. Use of curvilinear coordinates

It is clear that the application of the boundary conditions (12) will be much simplified if orthogonal coordinates ξ, η can be chosen in the $x-y$ plane so that the curve C or C' is given by, say, $\xi = \xi_0$. In these coordinates, the boundary conditions (12) are easily shown to become:

$$\text{when} \quad \xi = \xi_0, \quad \phi = 1 \quad \text{and} \quad \frac{\partial^s \phi}{\partial \xi^s} = 0 \quad \text{for } 1 \leqq s \leqq n-1. \tag{17}$$

Coordinate systems of this type will be used in all the examples considered.

When ξ and η are related to x and y by an equation of the form $x+iy = z(\xi+i\eta) = z(\zeta)$, it is known that if $h^2 = |dz/d\zeta|^{-2}$, then

$$L_k(f) = h^2 y^{-k} \left\{ \frac{\partial}{\partial \xi}\left(y^k \frac{\partial f}{\partial \xi} \right) + \frac{\partial}{\partial \eta}\left(y^k \frac{\partial f}{\partial \eta} \right) \right\}, \tag{18}$$

where $y = Y(\xi, \eta)$ will be a known function of ξ and η.

Furthermore, if $g = y^{k/2} f$, then

$$L_k(f) = h^2 y^{-k/2} \left\{ \frac{\partial^2 g}{\partial \xi^2} + \frac{\partial^2 g}{\partial \eta^2} - \frac{k(k-2)}{4h^2 y^2}\, g \right\}. \tag{19}$$

5. Application to circular boundary

The simplest class of problems to be considered is conveniently illustrated by the case when the profile in the $x-y$ plane is the circle $r = a$. The aim is to find a function Ψ to satisfy equation (6), boundary conditions (7) (with $\chi = 0$) on the circle $r = a$ and condition (9) at infinity.

From sections 2 and 4, it can be seen that the problem reduces to solving equation (11) subject to the conditions $\phi \to 0$ as $r \to \infty$ and, on $r = a$,

$$\phi = 1, \quad \frac{\partial^s \phi}{\partial r^s} = 0 \quad \text{for } 1 \leqq s \leqq n-1. \tag{20}$$

Weinstein's general solution of the iterated equation, quoted in (15),

https://doi.org/10.1017/S1446788700006327 Published online by Cambridge University Press

and the solution $f_k = r^{-k}$ of the equation $L_k(f) = 0$ are now used to produce a solution of (11) in the form

$$\phi = \left(\frac{r}{a}\right)^{-(p+2)} \sum_{t=0}^{n-1} \lambda_t^{(n)} \left(\frac{r^2}{a^2}\right)^t,$$

where the coefficients $\lambda_t^{(n)}$ are to be determined from the boundary conditions (20). Before this is done however, it can be observed that the condition $\phi \to 0$ as $r \to \infty$ is satisfied only when $p > 2(n-2)$. This means, in particular, that in the case of two-dimensional Stokes flow past a circular cylinder, given by $p = 0$, $n = 2$, there is no solution satisfying all the conditions — the Stokes paradox.

If, for any q, $(q)_0 = 1$, $(q)_s = q(q+1)\cdots(q+s-1)$, the system of linear equations E_s for the coefficients $\lambda_t^{(n)}$ resulting from boundary conditions (20) can be written in the form

$$\sum_{t=0}^{n-1} (p+2-2t)_s \lambda_t^{(n)} = \delta_{0s}, \qquad 0 \leqq s \leqq n-1, \tag{21}$$

where δ_{0s} is the Kronecker delta. This set of equations is equivalent to the set consisting of the equation E_0 and the $s-1$ equations obtained by the operation $E_{s+1}-(p+4-2n+s)E_s$ for $0 \leqq s \leqq n-2$. If new coefficients $\lambda_t^{(n-1)}$ are defined so that

$$2(n-1-t)\lambda_t^{(n)} = -(p+4-2n)\lambda_t^{(n-1)}, \qquad 0 \leqq t \leqq n-2,$$

the new system of equations is

$$\begin{aligned} \sum_{t=0}^{n-1} \lambda_t^{(n)} &= 1, \\ \sum_{t=0}^{n-2} (p+2-2t)_s \lambda_t^{(n-1)} &= \delta_{0s}, \qquad 0 \leqq s \leqq n-2. \end{aligned} \tag{22}$$

Comparison of (21) and (22) shows that the key step has been taken towards establishing the solution of (21) by mathematical induction. It is now readily conjectured and proved without difficulty that

$$\lambda_t^{(n)} = (-1)^{n-1-t} \binom{\frac{1}{2}p-t}{n-1-t} \binom{\frac{1}{2}p+1}{t}.$$

The solution for ϕ is

$$\phi(r) = \left(\frac{r}{a}\right)^{-(p+2)} \sum_{t=0}^{n-1} (-1)^{n-1-t} \binom{\frac{1}{2}p-t}{n-1-t} \binom{\frac{1}{2}p+1}{t} \left(\frac{r^2}{a^2}\right)^t$$

and the corresponding solution for Ψ is given by (13) as

$$\Psi = \frac{Ur^{p+1}\sin^{p+1}\theta}{p+1} \{1-\phi(r)\}.$$

https://doi.org/10.1017/S1446788700006327 Published online by Cambridge University Press

6. Application to problems in which $\phi = \phi(\xi)$

The method operated successfully in the last section because the profile on which the boundary conditions were to be applied was given by $r = a$ and it was possible to express the function ϕ in terms of r only. This in turn depended on the availability of the solution r^{-k} of the equation $L_k(f) = 0$.

In general, when a profile has equation $\xi = \xi_0$ say, solutions will be obtained in just the same way if a solution $f = f_k(\xi)$ can be found of the equation $L_k(f) = 0$. This can certainly be done when the coordinates ξ, η are such that $y = Y(\xi, \eta)$ can be factorized so that $Y(\xi, \eta) = Y_1(\xi)Y_2(\eta)$. In this case it can be seen from (18) that the equation $L_k(f)$ has a solution $f_k(\xi)$ given by

$$f_k(\xi) = \int_\alpha^\xi Y_1^{-k}(\xi)\,d\xi$$

where α is a constant to be chosen so that $f_k(\xi) \to 0$ as $r \to \infty$.

Weinstein's general solution (15) then leads to a formal solution of equation (11) for ϕ given by

$$\phi = \sum_{t=0}^{n-1} \lambda_t f_{p+2-2t}(\xi)$$

and the existence of a solution depends on the possibility of finding functions $f_{p+2-2t}(\xi)$, for all the relevant values of t, which satisfy the conditions at infinity. When the solution exists, the coefficients λ_t are found from the boundary conditions (17) in the same way as in the previous section. If the profile in the meridian plane is of the first type considered in section 2, the function Ψ is then given in terms of ϕ by (13). If the profile were of type (ii) however, a similar calculation for $\bar{\psi}$ in the form

$$\bar{\psi} = \sum_{t=0}^{n-1} \mu_t f_{p-2t}(\xi)$$

would be needed for Ψ to be given by (14).

The method can be illustrated in the case when the profile in the meridian plane is an ellipse. If the coordinates ξ, η are such that $x+iy = c\cosh(\xi+i\eta)$, the curve $\xi = \xi_0$ is an ellipse. Since $y = c\sinh\xi\sin\eta$, $Y(\xi) = \sinh\xi$ and the appropriate solution $f_k(\xi)$, satisfying the condition at infinity is

$$f_k(\xi) = \int_\xi^\infty \sinh^{-k}\xi\,d\xi$$

It is clear that the integral converges and the solution exists only for $k > 0$.

For Stokes flow, where $n = 2$,

$$\phi = \lambda_0 f_{p+2}(\xi) + \lambda_1 f_p(\xi).$$

Accordingly, there is no solution in the two-dimensional case $p = 0$. When $p = 1$, for axially symmetric flow, the boundary conditions (17) give

$$\lambda_0 f_3(\xi_0) + \lambda_1 f_1(\xi_0) = 1,$$
$$\lambda_0 f_3'(\xi_0) + \lambda_1 f_1'(\xi_0) = 0.$$

When these equations are solved for λ_0, λ_1 and the resulting function ϕ is put into (13), the stream function for Stokes flow past an ellipsoid of revolution in a uniform stream is obtained as

$$\tfrac{1}{2}Uc^2 \sinh^2 \xi \sin^2 \eta \left\{1 - \left[\tfrac{1}{2}(c_0^2+1) \log \frac{c+1}{c-1} - c\,\frac{c_0^2-1}{c^2-1}\right] \Big/ f(c_0)\right\},$$

where $c = \cosh \xi$, $c_0 = \cosh \xi_0$ and

$$f(c_0) = \tfrac{1}{2}(c_0^2+1) \log \frac{c_0+1}{c_0-1} - c_0.$$

This stream function was given by Payne and Pell [5] in their equation (8.7) which, however, contains a slight misprint.

7. Further applications

Solutions of the problem stated in section 2 which have been considered so far have depended on the ability to separate the variables in the equation $L_k(f) = 0$ when it is expressed, by means of (18), in terms of coordinates ξ, η.

If a new variable $g = y^{k/2} f$ is introduced, (19) shows that the equation $L_k(f) \doteq 0$ becomes

$$\frac{\partial^2 g}{\partial \xi^2} + \frac{\partial^2 g}{\partial \xi^2} - \frac{k(k-2)}{4h^2 y^2} g = 0. \tag{23}$$

Further cases which can be solved arise when the variables separate in this equation and this occurs when ξ and η are related to x and y so that $(h^2y^2)^{-1} = F(\xi) + G(\eta)$.

In particular, when $(h^2y^2)^{-1} = F(\xi)$, a family of functions g_k satisfying (23) can be found which are typically of the form

$$g_k = X_k(\alpha, \xi) \cos \alpha\eta \tag{24}$$

where α is an arbitrary separation constant. In (24), it is assumed that the functions $X_k(\alpha, \xi)$ can be chosen so that $y^{-k/2} g_k \to 0$ as $r \to \infty$ and that the variables ξ, η have been defined so that $\cos \alpha\eta$ is the appropriate form for the factor depending on η.

From (24), a solution $g_k(\xi, \eta)$ is obtained in the form

$$g_k(\xi, \eta) = \mathscr{S}_\alpha A(\alpha) X_k(\alpha, \xi) \cos \alpha\eta$$

where the summation process denoted by $\mathscr{S}_\alpha$ must be chosen to suit the range of values assumed by η in the region of the $x-y$ plane under consideration, a Fourier integral being required when η has an infinite range, a Fourier series when η has a finite range such as $(0, 2\pi)$. The result is a solution $f_k(\xi, \eta)$ of the equation $L_k(f) = 0$ in the form

$$f_k(\xi, \eta) = Y^{-k/2}(\xi, \eta) \mathscr{S}_\alpha A_k(\alpha) X_k(\alpha, \xi) \cos \alpha\eta \tag{25}$$

where $y = Y(\xi, \eta)$ as before.

This function can be used as the basis for solving the problem of section 2. The cases (i) and (ii) distinguished there are treated slightly differently and the method hinges on the choice of an appropriate general solution of the iterated equation of the type (1) which is involved.

When the boundary in the meridian plane is of type (i), an arc C joining two points on the x-axis, it is necessary to find a function ϕ satisfying equation (11). The general solution chosen is that given by (16), with $p+2$ in place of k. It will be observed that the powers of $y = Y(\xi, \eta)$ involved in the terms of (16) and in (25) are such that each term in the resulting general solution for ϕ has the same power of y as a factor. Accordingly, provided $p \neq -1-2i$ for $0 \leqq i \leqq n-2$, ϕ is given by

$$Y^\lambda(\xi, \eta)\phi(\xi, \eta) = \mathscr{S}_\alpha \left\{ \sum_{t=0}^{n-1} B_t(\alpha) X_{2\lambda+4t}(\alpha, \xi) \right\} \cos \alpha\eta \tag{26}$$

where $\lambda = \frac{1}{2}p+2-n$ and the functions $B_t(\alpha) = A_{2\lambda+4t}(\alpha)$ have to be determined from the boundary conditions (17). As before, the existence of the function $\phi(\xi, \eta)$ depends on the existence, for the relevant values of t, of the functions $X_{2\lambda+4t}(\alpha, \xi)$ which satisfy the required conditions at infinity.

If $Y^\lambda(\xi, \eta) \equiv F(\xi, \eta; \lambda)$, it is easily shown that the conditions (17) yield the following set of equations for the functions $A_t(\alpha)$:

$$F^{(s)}(\xi_0, \eta; \lambda) = \mathscr{S}_\alpha \left\{ \sum_{t=0}^{n-1} B_t(\alpha) X^{(s)}_{2\lambda+4t}(\alpha, \xi_0) \right\} \cos \alpha\eta \quad \text{for } 0 \leqq s \leqq n-1,$$

the superscript (s) denoting the s^{th} partial derivative with respect to ξ. If, in each of these equations the summation process $\mathscr{S}_\alpha$ is inverted, a set of n linear equations for the functions $B_t(\alpha)$ is obtained:

$$\mathscr{F}(\xi_0, \alpha; \lambda; s) = \sum_{t=0}^{n-1} B_t(\alpha) X^{(s)}_{2\lambda+4t}(\alpha, \xi_0) \quad \text{for } 0 \leqq s \leqq n-1,$$

where $\mathscr{F}(\xi_0, \alpha; \lambda; s)$ is the appropriate transform of $F^{(s)}(\xi_0, \eta; \lambda)$.

When these equations are solved for $B_t(\alpha)$, the function $\phi(\xi, \eta)$ is obtained and (13) then gives the required function Ψ.

When the profile in the $x-y$ plane is of type (ii), a closed curve C' in the upper half-plane, it has been shown in section 2 that, in addition to finding a function ϕ exactly as above, it is necessary also to find a function $\bar{\psi}$ satisfying equation (6) and boundary conditions (17) (with $\bar{\psi}$ replacing ϕ). In finding $\bar{\psi}$ to satisfy (6), it is convenient to use the generalized correspondence principle (2) to write $\bar{\psi}$ as $y^{p+1}\bar{\Phi}$ where $\bar{\Phi}$, like ϕ, satisfies equation (11). $\bar{\Phi}$ is then found in the same way as ϕ, the only change being to increase by $p+1$ the power of y which is a common factor of the terms making up the general solution. The result of this is to change the parameter λ which occurs in the solution for ϕ to $\mu = \lambda-p-1 = -\frac{1}{2}p+1-n$. The rest of the procedure remains the same and when ϕ and $\bar{\Phi}$ have both been obtained, (14) gives

$$\Psi = \frac{Uy^{p+1}}{p+1}\left[1-\phi+\frac{\chi(p+1)}{U}\bar{\Phi}\right] \tag{27}$$

The method is illustrated by showing how to find the stream function for axially symmetric Stokes flow of a viscous fluid in two cases previously considered by Pell and Payne [6, 7] who have solved the problems in detail. The account given here will be limited to demonstrating the systematic approach provided by the present method, the analysis being carried only far enough to link it to the original authors' presentation.

7.1. Stokes flow past a spindle

The profile in the meridian plane in this problem (see [6]) is the arc of a circle with centre on the y-axis joining the points $(\pm c, 0)$ on the x-axis. Coordinates ξ, η are chosen so that

$$x+iy = ic \cot \tfrac{1}{2}(\xi+i\eta)$$

and the profile can then be taken as the curve $\xi = \xi_0$. The region of flow is given by $0 < \xi \leqq \xi_0$, $-\infty < \eta < \infty$. Since $h^2y^2 = \sin^2 \xi$, the method given above can be applied and the range of values of η shows that the summation process $\mathscr{S}_\alpha$ should be an integration with respect to α. In the case when $q = \frac{1}{2}(k-1)$ is an integer, Payne [4] has obtained the basic solution $f_k(\xi, \eta)$ corresponding to (25) as

$$f_k(\xi, \eta) = (\cosh \eta-\cos \xi)^{k/2}\int_0^\infty A_k(\alpha)K_\alpha^{(q)}(\cos \xi) \cos \alpha\eta \, d\alpha$$

where (q) denotes the q^{th} derivative with respect to the argument and $K_\alpha(\cos \xi)$ is a Legendre function of complex degree known as a conal function. When this function is used to construct a general solution for $\phi(\xi, \eta)$

the result obtained, corresponding to (26), for axially symmetric Stokes flow ($p = 1$, $n = 2$), is

$$(28)\qquad (\cosh \eta - \cos \xi)^{-\frac{1}{2}} \phi(\xi, \eta) = \int_0^\infty \{B_1(\alpha) K_\alpha(\cos \xi) + \sin^2 \xi B_2(\alpha) K_\alpha^{(2)}(\cos \xi)\} \cos \alpha\eta \, d\alpha.$$

$B_1(\alpha)$ and $B_2(\alpha)$ are to be found so that $\phi = 1$, $\partial\phi/\partial\xi = 0$ when $\xi = \xi_0$, following the procedures outlined above.

At this stage, however, the analysis can be connected to Pell and Payne's [6] by observing that the function $K_\alpha(t)$ satisfies the equation

$$(1-t^2)K_\alpha^{(2)}(t) - 2tK_\alpha^{(1)}(t) - (\alpha^2 + \tfrac{1}{4})K_\alpha(t) = 0.$$

This can be used to transform (28) into

$$(29)\qquad (\cosh \eta - \cos \xi)^{-\frac{1}{2}} \phi(\xi, \eta) = \int_0^\infty \{A(\alpha) \cos \xi \, K_\alpha^{(1)}(\cos \xi) + B(\alpha) K_\alpha(\cos \xi)\} \cos \alpha\eta \, d\alpha$$

where the new functions $A(\alpha)$, $B(\alpha)$ are simply related to $B_1(\alpha)$, $B_2(\alpha)$. (29) with the conditions $\phi = 1$, $\partial\phi/\partial\xi = 0$ when $\xi = \xi_0$ to determine $A(\alpha)$ and $B(\alpha)$, may be compared with Pell and Payne's equation (2.6) and conditions (2.3), (2.4) to show that the two procedures are essentially equivalent.

Once $\phi(\xi, \eta)$ has been found, (13) gives the required function Ψ.

7.2. Stokes flow past a torus

In this problem (see [7]), the profile in the meridian plane is a circle, with its centre on the y-axis, lying entirely above the x-axis. If coordinates ξ, η are chosen so that

$$x + iy = -c \cot(\xi + i\eta),$$

such a circle is given by $\eta = \eta_0$ and the region exterior to the circle is given by $0 \leqq \eta < \eta_0$, $0 \leqq \xi < 2\pi$. It will be noted that the boundary is given by $\eta = \eta_0$ rather than by $\xi = \xi_0$ as in the rest of this paper. It will be necessary therefore to interchange the parts played by ξ and η but this slight inconvenience has been accepted as the use of a notation essentially the same as that used by Pell and Payne [7] facilitates comparison with their results.

Since $h^2 y^2 = \sinh^2 \eta$, the method of section 7 is applicable and the range of values of ξ shows that the summation process $\mathscr{S}_\alpha$ should be a Fourier sum over integral values of the parameter α. In the case when $q = \frac{1}{2}(k-1)$ is an integer, Weiss and Payne [10] show that the basic solution $f_k(\xi, \eta)$ corresponding to (25) is

$$f_k(\xi, \eta) = (\cosh \eta - \cos \xi)^{k/2} \sum_{n=0}^{\infty}{}' A_{kn} P_{n-\frac{1}{2}}^{(q)}(\cosh \eta) \cos n\xi$$

https://doi.org/10.1017/S1446788700006327 Published online by Cambridge University Press

where (q) denotes the q^{th} derivative with respect to the argument, $\sum'$ indicates that the term for $n = 0$ is to be multiplied by $\frac{1}{2}$ and $P_{n-\frac{1}{2}}(\cosh \eta)$ is a Legendre function.

The first problem to be solved here is that of finding the function ϕ. For this case of axially symmetric Stokes flow with $p = 1$, $n = 2$, $\phi(\xi, \eta)$ is given by a relation corresponding to (26):

$$(\cosh \eta - \cos \xi)^{-\frac{1}{2}} \phi(\xi, \eta) = \sum_{n=0}' \{B_{1n} P_{n-\frac{1}{2}}(\cosh \eta) + B_{2n} \sinh^2 \eta \, P^{(2)}_{n-\frac{1}{2}}(\cosh \eta)\} \cos n\xi. \tag{30}$$

The coefficients B_{1n}, B_{2n} are to be found so that $\phi = 1$ and $\partial\phi/\partial\eta = 0$ when $\eta = \eta_0$.

Again the analysis can be switched to Pell and Payne's [7] by noting that $P_{n-\frac{1}{2}}(t)$ satisfies the equation

$$(1-t^2) P^{(2)}_{n-\frac{1}{2}}(t) - 2t P^{(1)}_{n-\frac{1}{2}}(t) + (n^2 - \tfrac{1}{4}) P_{n-\frac{1}{2}}(t) = 0.$$

This is used to transform (30) into

$$(\cosh \eta - \cos \xi)^{-\frac{1}{2}} \phi(\xi, \eta) = \sum_{n=0}^{\infty}{}' \{A_n \cosh \eta \, P^{(1)}_{n-\frac{1}{2}}(\cosh \eta) + B_n P_{n-\frac{1}{2}}(\cosh \eta)\} \cos n\xi$$

which, with the boundary conditions on ϕ, may be compared with Pell and Payne's equation (3.13) and conditions (3.7).

It is also necessary to find the function $\bar{\Phi}$ which is given in terms of the basic solution $f_k(\xi, \eta)$ by an expression similar to (26) but with the modified parameter μ instead of λ. The expression obtained for $\bar{\Phi}$, is

$$(\cosh \eta - \cos \xi)^{\frac{3}{2}} \bar{\Phi}(\xi, \eta) = \sinh^2 \eta \sum_{n=0}^{\infty}{}' \{B_{1n} P_{n-\frac{1}{2}}(\cosh \eta) + B_{2n} \sinh^2 \eta \, P^{(2)}_{n-\frac{1}{2}}(\cosh \eta)\} \cos n\xi.$$

As before, this is transformed to

$$(\cosh \eta - \cos \xi)^{\frac{3}{2}} \bar{\Phi}(\xi, \eta) = \sinh^2 \eta \sum{}' \{C_n \cosh \eta \, P^{(1)}_{n-\frac{1}{2}}(\cosh \eta) + D_n P_{n-\frac{1}{2}}(\cosh \eta)\} \cos n\xi$$

which, with conditions $\bar{\Phi} = 1$, $\partial\bar{\Phi}/\partial\eta = 0$ on $\eta = \eta_0$ corresponds to Pell and Payne's relation (3.24) and conditions (3.8).

When ϕ and $\bar{\Phi}$ have been determined, Ψ is given by (27) which corresponds to Pell and Payne's (3.34) (in which there is a misplaced brace).

Acknowledgements

This paper was written while the author was a Visiting Member of the Institute for Fluid Dynamics and Applied Mathematics of the University of Maryland during a period of study leave from the Australian National University and he is grateful to both Universities for these privileges.

The work was supported in part by the Air Force Office of Scientific Research under Grant AF 1194-67.

References

[1] A. Weinstein, 'Generalized axially symmetric potential theory', *Bull. Amer. Math. Soc.* 59 (1953), 20—28.

[2] R. J. Weinacht, 'Fundamental solutions for a class of singular equations', *Contributions to Differential Equations III* (1964), 43—55.

[3] J. C. Burns, 'The iterated equation of generalized axially symmetric potential theory I, Particular solutions', *Journ. Austr. Math. Soc.* 7 (1967), 263—276.

[4] L. E. Payne, 'On axially symmetric flow and the method of generalized electrostatics', *Quart. Appl. Math.* 10 (1952), 197—204.

[5] L. E. Payne and W. H. Pell, 'The Stokes flow problem for a class of axially symmetric bodies', *Journ. Fluid Mech.* 7 (1960), 529—549.

[6] W. H. Pell and L. E. Payne, 'The Stokes flow about a spindle', *Quart. Appl. Math.* 18 (1960), 257—262.

[7] W. H. Pell and L. E. Payne, 'On Stokes flow about a torus', *Mathematika* 7 (1960), 78—92.

[8] A. Weinstein, 'On a class of partial differential equations of even order', *Ann. Mat. Pura Appl.* 39 (1955), 245—254.

[9] J. C. Burns, 'The iterated equation of generalized axially symmetric potential theory II, General solutions of Weinstein's type', *Journ. Austr. Math. Soc.* 7 (1967), 277—289.

[10] G. Weiss and L. E. Payne, 'Torsion of a shaft with a toroidal cavity', *Journ. Appl. Phys.* 25 (1954), 1321—1328.

The Australian National University, Canberra, A.C.T.

Institute for Fluid Dynamics and Applied Mathematics
University of Maryland, College Park, Maryland