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Small resolutions of minuscule Schubert varieties

Nicolas Perrin

Abstract

Let X be a minuscule Schubert variety. In this paper, we associate a quiver with X and use the combinatorics of this quiver to describe all relative minimal models $\widehat{\pi} : \widehat{X} \to X$. We prove that all the morphisms $\widehat{\pi}$ are small and give a combinatorial criterion for $\widehat{X}$ to be smooth and thus a small resolution of X. We describe in this way all small resolutions of X. As another application of this description of relative minimal models, we obtain the following more intrinsic statement of the main result of Perrin, J. Algebra **294** (2005), 431–462. Let $\alpha \in A_1(X)$ be an effective 1-cycle class. Then the irreducible components of the scheme $\mathbf{Hom}_\alpha(p^1, X)$ of morphisms from $\mathbb{P}^1$ to X and of class α are indexed by the set: $\mathfrak{ne}(\alpha) = \{\beta \in A_1(\widehat{X}) \mid \beta \text{ is effective and } \widehat{\pi}_*\beta = \alpha\}$ which is independent of the choice of a relative minimal model $\widehat{X}$ of X.

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Introduction

Schubert varieties have been intensively studied and are of great importance in representation theory. There are several ways to understand their geometry and singularities. One way is to describe their singular locus, the irreducible components of this locus and the singularity at a general point of any such component. This has been completely done in partial flag varieties only recently (see [Man01a], [Man01b], [BW03], [KLR03] and [Cor03]). For the general case, there are only partial results (for an account, see [BL00]). This description of singularities enables in particular the computation of some related Kazhdan–Lusztig polynomials. For this study, combinatoric tools are useful but it is also useful to construct special resolutions of Schubert varieties (see for example [Cor01] and [Cor03]). Another way to study Schubert varieties is to calculate the cohomology of line bundles and especially to prove some vanishing theorems. This has been done in many ways, one of them via resolutions of singularities (see [Kem76], [MR85], [RR85] or [Ram85]). In this paper, we want to study the geometry of Schubert varieties thanks to the study of some particular resolutions of these varieties.

A nice class of such resolutions are the Bott–Samelson resolutions (see [Dem74] or [Han73] for definitions). These are defined as towers of $\mathbb{P}^1$-fibrations and exist for any Schubert variety. We will recall their description as configuration varieties in § 2. For example in [Per05] and [Per06] we use Bott–Samelson resolutions to describe the schemes of rational and elliptic curves on a special class of Schubert varieties. However, these resolutions are big in the sense that the fibers have big dimensions and there are many contracted subvarieties.

Another class of resolutions is of particular importance, the small resolutions (see Definition 7.1). Here we use 'small' in the sense of intersection cohomology, and to avoid any confusion with small contractions in Mori theory (see for example [Mat02]) we will use of Totaro's convention in [Tot00] and call them *IH*-small resolutions. These resolutions are well suited for the calculation of Kazhdan–Lusztig polynomials. In particular Zelevinsky in [Zel83] constructed some *IH*-small resolutions for Grassmannian Schubert varieties and gave a geometric interpretation of the combinatorial computation of Kazhdan–Lusztig polynomials by Lascoux and Schützenberger in [LS81]. Later Sankaran and Vanchinathan in [SV94] and [SV95] described small resolutions for some Schubert varieties in Lagrangian Grassmannians and maximal isotropic Grassmannians. They calculated the corresponding Kazhdan–Lusztig polynomials. Furthermore in [SV94] they exhibited some Schubert varieties not admitting any small resolution. Any Schubert variety having singularities in codimension 2 will do because its normality prevents it from having a small resolution. Other examples of Schubert varieties without small resolutions are all singular but locally factorial Schubert varieties: indeed the purity theorem (see [Deb01, § 1.10] or [Gro67, Theorem 21.12.12]), says that there is no *IH*-small resolution.

In this paper we focus on minuscule Schubert varieties. They form a special class of Schubert varieties designed to generalize Grassmannian Schubert varieties (see § 3 for a precise definition). They also include maximal isotropic Grassmannian Schubert varieties. In particular all the Schubert varieties studied by Zelevinsky [Zel83] and most of the Schubert varieties studied by Sankaran and Vanchinathan [SV94] are minuscule. The Lagrangian Grassmannian Schubert varieties are 'cominuscule' Schubert varieties. Some of the results of this paper can be extended to this case. This will be done in [Per07]; we will briefly discuss this at the end of the introduction. Here, we generalize the constructions of Zelevinsky, Sankaran and Vanchinathan to any minuscule Schubert variety. We describe which ones admit an *IH*-small resolution and describe all such resolutions. We do not address the problem of calculating Kazhdan–Lusztig polynomials. This has been done in a combinatorial way for all minuscule Schubert varieties by Boe [Boe88] and we hope that our construction will lead to a geometric interpretation of these results.

Schubert varieties $X(w)$ are parametrized by elements w of a Weyl group W. In order to define our resolutions, we introduce a combinatorial object: to any reduced expression $\widetilde{w}$ of the element w, we associate a quiver $Q_{\widetilde{w}}$. We interpret the configuration variety $\widetilde{X}(\widetilde{w})$ isomorphic to the Bott–Samelson resolution defined by Magyar in [Mag98] in terms of the quiver. When w represents a minuscule Schubert variety, there exists a unique reduced expression $\widetilde{w}$ modulo commutation relations so that $Q_{\widetilde{w}}$ depends on w but not on $\widetilde{w}$. Furthermore, the quiver $Q_{\widetilde{w}}$ has a very special and rigid geometry. We define the peaks of such a quiver and the height of a peak (see Definition 4.6). As in the case of Zelevinsky's construction [Zel83], the choice of an ordering on the peaks leads to a partial resolution $\widehat{\pi}:\widehat{X}\to X(w)$. The variety $\widehat{X}$ is locally factorial but generally singular. Moreover we define special orderings respecting the order on the heights of the peaks and call them neat orderings by analogy with [Zel83] (see § 5.4, Construction 3) and recover all *IH*-small resolutions of [Zel83].

The varieties $\widehat{X}$ are interesting for the relative minimal model program, a relative version of the minimal model program initiated by Mori (see for example [KMM87] or [Mat02] for a presentation of the minimal model program). A relative minimal model of a variety X is a variety $\widetilde{X}$ together with a proper birational morphism $\pi:\widetilde{X}\to X$ having mild singularities (precisely $\widetilde{X}$ has terminal singularities; see [Mat02, p. 164]) and such that the canonical sheaf $K_{\widetilde{X}}$ exists, is $\mathbb{Q}$-Cartier and is numerically effective on the fibers of π. We study the relative minimal models of a minuscule Schubert variety $X(w)$ and prove the following theorem (see also Corollary 6.16).

Theorem 0.1. *The relative minimal models of $X(w)$ are the varieties $\widehat{X}$ obtained thanks to neat orderings.*

The following theorem shows that to generalize Zelevinsky's results to any minuscule Schubert variety, we need to allow some mild singularities: the relative minimal models play the role of *IH*-small resolutions.

Theorem 0.2. *The morphism $\widehat{\pi}:\widehat{X}\to X(w)$ from a relative minimal model to a minuscule Schubert variety is IH-small.*

Conversely, the following result of Totaro [Tot00] using a key result of Wisniewski [Wis91] tells us to look for *IH*-small resolutions in the class of relative minimal models.

Theorem 0.3 (Totaro–Wisniewski). *Any IH-small resolution of a normal variety X is a relative minimal model for X.*

In particular, in our situation, the *IH*-small resolutions of $X(w)$ are the smooth relative minimal models. We then give in Theorem 7.11 a combinatorial criterion on the quiver for the relative minimal model $\widehat{X}$ to be smooth. This criterion gives an effective way to tell which minuscule Schubert variety does admit an *IH*-small resolution and to describe all these *IH*-small resolutions.

At the end of the paper we sketch another way to describe the minuscule Schubert varieties that do not admit *IH*-small resolutions. If X has a relative minimal model $\widehat{\pi}:\widehat{X}\to X$ it has a unique relative canonical model $\pi:X_{\mathrm{can}}\to X$ with a factorization $\widehat{X}\to X_{\mathrm{can}}$ of $\widehat{\pi}$ through π (see [KMM87, Theorem 3-3-1]). For $X(w)$ a minuscule Schubert variety, we describe its relative canonical model $\pi:X_{\mathrm{can}}\to X(w)$. Any *IH*-small resolution $\widehat{\pi}:\widehat{X}\to X(w)$ factors through π and induces a crepant resolution $\widehat{X}\to X_{\mathrm{can}}$. In particular, the stringy Euler number e_{st} defined by Batyrev [Bat98] of X_{can} has to be an integer if $\widehat{\pi}$ exists. We give a formula for $e_{\mathrm{st}}(X_{\mathrm{can}})$. One can show that the minuscule Schubert varieties that do not admit *IH*-small resolutions are those such that $e_{\mathrm{st}}(X_{\mathrm{can}})\notin\mathbb{Z}$.

Another motivation for the study of (partial) resolutions of Schubert varieties (and in fact our motivation at the beginning of the study) is the following reformulation of our result in [Per05]. Let $X(w)$ be a minuscule Schubert variety and $\widehat{\pi}:\widehat{X}\to X(w)$ any relative minimal model. For an effective 1-cycle class $\alpha\in A_1(X(w))$ define the set

$$\mathfrak{ne}(\alpha)=\{\beta\in NE(\widehat{X})\mid \widehat{\pi}_*\beta=\alpha\},$$

where $NE(\widehat{X})$ is the cone of effective 1-cycles in $\widehat{X}$.

Theorem 0.4. *The irreducible components of* $\mathbf{Hom}_\alpha(\mathbb{P}^1,X(w))$ *of the scheme of morphisms from* $\mathbb{P}^1$ *to* $X(w)$ *of class* α *are indexed by* $\mathfrak{ne}(\alpha)$.

The same kind of results hold for cones over flag varieties (see [Per04]). Finally let us briefly discuss the case of cominuscule Schubert varieties (see § 3 for a definition). These share many properties with minuscule Schubert varieties. In particular one can associate a quiver with them and define the peaks of the quiver and the heights of the peaks. In many cases the same construction of $\widehat{\pi}:\widehat{X}\to X(w)$ as for minuscule Schubert varieties leads to relative minimal models of $X(w)$ with $\widehat{\pi}$ being *IH*-small. However, in some cases one needs to blow up a codimension 2 subvariety Z in $\widehat{X}$ to obtain a relative minimal model $\mathrm{Bl}_Z(\widehat{X})$. In this case the relative minimal model is only semismall over $X(w)$. These results on cominuscule Schubert varieties will be explained in [Per07].

Let us give an overview of the paper. In § 1 we recall some basic notation, definitions and results on Weyl groups. In § 2, we define the quiver associated to a reduced expression $\widetilde{w}$ and the corresponding configuration variety which is isomorphic to the Bott–Samelson resolution. We reformulate, after [LT04], the basic properties of this variety (Weil, Cartier and canonical divisors, 1-cycles and intersection formulae) in terms of the geometry of the quiver. In § 3 we recall the definitions and first properties of minuscule and cominuscule Schubert varieties. In § 4, we describe the particular geometry of a quiver associated to a minuscule Schubert variety and study its link with the geometry of the variety (Weil, Cartier and canonical divisors, 1-cycles and intersection formulae). In § 5, we construct and study a generalization of Bott–Samelson resolution which lies in between the Schubert variety and the Bott–Samelson resolution. In § 6, we describe the geometry of this generalization (Weil, Cartier and canonical divisors, 1-cycles and intersection formulae again) and describe the relative Mori theory for a minuscule Schubert variety. In the last section, we prove that all relative minimal models are *IH*-small and describe all *IH*-small resolutions of minuscule Schubert varieties. In this section, in contrast with the rest of the paper, we use a case-by-case analysis on the Dynkin diagrams. A general argument could be possible but we think it would be more complicated and would lead to much more combinatorics. In Appendix A, we describe the quivers of minuscule Schubert varieties. We use this description extensively in the last section.

In all the paper, we only consider projective varieties over the field $\mathbb{C}$ of complex numbers. We use the hypothesis on the base field only in Corollary 6.16 through the existence and termination of flops to prove that all minimal models are connected by flops.

1. Notation

Let G be a semisimple algebraic group. Fix T a maximal torus and B a Borel subgroup containing T. Let us denote by Δ the set of all roots, by Δ^+ (respectively Δ^-) the set of positive (respectively negative) roots, by Σ the set of simple roots associated to the data (G, T, B), by W the associated Weyl group and by l the length function on W. For a root $\alpha \in \Delta$, we denote by U_α the corresponding root subgroup and s_α the reflection in W. This reflection is said to be simple if α is a simple root. If P is a parabolic subgroup containing B we denote by W_P the Weyl group of (P, T). We will also denote by $\Sigma(P)$ the set of simple roots α such that $U_{-\alpha} \not\subset P$.

Definition 1.1. (i) Let $w \in W$. We define a parabolic subgroup P^w of G containing B by its set of simple roots $\Sigma(P^w) = \{\alpha \in \Sigma \mid l(ws_\alpha) = l(w) - 1\}$.

(ii) Let us denote by P_w the stabilizer of $X(w) = \overline{BwP^w}/P^w$ for the left action of G. It contains B and in terms of simple roots we have

$$\Sigma(P_w) = \{\alpha \in \Sigma \mid \overline{s_\alpha w} > \overline{w} \text{ for the Bruhat order in } W/W_{P^w}\}.$$

Remark 1.2. (i) The subgroup P^w is the largest parabolic subgroup of G such that the morphism $\overline{BwB}/B \to \overline{BwP^w}/P^w$ is birational.

(ii) The parabolic subgroup P_w is in general bigger than the stabilizer of the Schubert variety $\overline{BwB}/B$. Indeed, consider for example $G = SL_3$ and $w = s_{\alpha_1}s_{\alpha_2}$ with the notation of [Bou68]. The Schubert variety $\overline{BwB}/B$ is

$$\{(V_1, V_2) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} \mid V_1 \subset F_2 \text{ and } V_1 \subset V_2\},$$

where $F_2 \in \mathbb{P}^{2\vee}$ is fixed. Its stabilizer is the parabolic subgroup of G preserving F_2. On the other hand, the Schubert variety $\overline{BwP^w}/P^w$ is simply $\mathbb{P}^{2\vee}$ obtained from $\overline{BwB}/B$ by projection on the second factor. Its stabilizer is G so that $P_w = G$ in this case.

Definition 1.3. (i) Let $w \in W$. We define the support of w denoted by $\mathrm{Supp}(w)$ to be the set of simple roots α such that s_α appears in a reduced expression of w.

(ii) We denote by G_w the smallest semisimple subgroup of G containing all the groups U_α for $\alpha \in \mathrm{Supp}(w)$. It is easy to see that there is an isomorphism

$$X(w) = \overline{P_w w P^w}/P^w \simeq \overline{(P_w \cap G_w)w(P^w \cap G_w)}/(P^w \cap G_w).$$

(iii) We define the boundary of $\mathrm{Supp}(w)$ denoted by $\partial\mathrm{Supp}(w)$ to be the set of simple roots α not contained in $\mathrm{Supp}(w)$ and such that s_α does not commute with w.

Remark 1.4. The support of an element $w \in W$ is independent of the reduced expression. Indeed, commuting and braid relations do not change the support.

2. Quivers and configuration varieties

In this section, we associate to any reduced expression of an element $w \in W$ a quiver and a configuration variety. This construction works for any element in the Weyl group but to simplify some notation we restrict ourselves at least in the definition and the study of the configuration variety to the case of flag varieties G/P with P a maximal parabolic subgroup.

2.1 Quiver associated to a reduced expression

In this subsection, we take for P any parabolic subgroup of G containing B. Let us consider an element $\bar{w} \in W/W_P$ and let $w \in W$ be the shortest element in the class $\bar{w}$. To any reduced expression

$$\widetilde{w} = (\beta_1, \ldots, \beta_r) \quad \text{where } w = s_{\beta_1} \cdots s_{\beta_r} \tag{1}$$

of w, we associate a quiver with colored vertices. Let us first give the following definition.

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Definition 2.1. For a fixed reduced expression (1) of w, we define the successor $s(i)$ (respectively the predecessor $p(i)$) of an element $i \in [1, r]$ by $s(i) = \min\{j \in [1, r] \mid j > i \textit{ and } \beta_j = \beta_i\}$ respectively by $p(i) = \max\{j \in [1, r] \mid j < i \text{ and } \beta_j = \beta_i\}$.

Remark 2.2. Note that the successor and the predecessor of an element do not always exist.

We define the quiver $Q_{\widetilde{w}}$ associated to the reduced expression (1).

Definition 2.3. Let us denote by $Q_{\widetilde{w}}$ the quiver whose set of vertices is the set $[1, r]$ and whose arrows are given in the following way: there is an arrow from i to j if $\langle \beta_j^\vee, \beta_i \rangle \neq 0$ and $i < j < s(i)$ (or only $i < j$ if $s(i)$ does not exist).

This quiver comes with a coloration of its edges by simple roots via the map $\beta : [1, r] \to \Sigma$ such that $\beta(i) = \beta_i$.

Remark 2.4. (i) The datum of $Q_{\widetilde{w}}$ is equivalent to that of the reduced expression $\widetilde{w}$ modulo commutation relations.

(ii) This quiver seems to be the same as the one defined by Zelikson [Zel05] for ADE types.

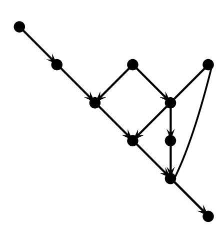
Example 2.5. Let $G = \mathrm{Spin}(16)$, and denote by $G^1_{\mathrm{iso}}(8, 16)$ one of the two connected components of maximal totally isotropic subspaces in $\mathbb{C}^{16}$ with respect to a non-degenerate quadratic form. Consider the following Schubert variety:

$$X(w) = \{V_8 \in \mathbb{G}_{\mathrm{iso}}(8, 16) \mid F_1 \subset V_8,\ \dim(V_8 \cap F_4) \geqslant 3 \text{ and } \dim(V_8 \cap F_6) \geqslant 4\},$$

where $F_1 \subset F_4 \subset F_6$ is a fixed partial flag of totally isotropic subspaces of respective dimensions one, four and six. A reduced expression for w is given by

$$w = s_{\alpha_7} s_{\alpha_5} s_{\alpha_6} s_{\alpha_8} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_7}$$

with the notation of [Bou68] and taking $G^1_{\mathrm{iso}}(8, 16) = G/P_{\alpha_7}$. The quiver Q_w has the following form.



2.2 Configuration varieties and Bott–Samelson resolution

From now on, we assume that $P = P_\varpi$, where P_ϖ, is the maximal parabolic subgroup of G associated to a fundamental weight ϖ. In this section, we describe – with a different formulation – the configuration variety $\widetilde{X}(\widetilde{w})$ associated to $\widetilde{w}$ a reduced expression (1) of w defined by Magyar in [Mag98]. This variety is isomorphic to the Bott–Samelson variety (cf. [Mag98]) and we will consider this as a definition of the Bott–Samelson variety.

Let β_i be a simple root and let us denote by P^{β_i} the minimal parabolic subgroup generated by B and $U_{-\beta_i}$. We have a projection morphism $\pi_{\beta_i} : G/B \to G/P^{\beta_i}$ whose fibers are isomorphic to $\mathbb{P}^1$. For any $x \in G/B$ we denote by $\mathbb{P}(x, \beta_i)$ the projective line $\pi_{\beta_i}^{-1}(\pi_{\beta_i}(x))$.

Definition 2.6. Let $\widetilde{w} = (\beta_1, \ldots, \beta_r)$ be a reduced expression of w. Then we set

$$\widetilde{X}(\widetilde{w}) = \left\{ (x_1, \ldots, x_r) \in \prod_{i=1}^{r} G/B \;\middle|\; x_0 = 1 \text{ and } x_i \in \mathbb{P}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r] \right\}.$$

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Remark 2.7. If we denote by P_{β_i} the maximal parabolic subgroup containing B and not containing $U_{-\beta_i}$, the restriction of the morphism $G/B \to G/P_{\beta_i}$ to $\mathbb{P}(x, \beta_i)$ is an isomorphism so that $\widetilde{X}(\widetilde{w})$ is isomorphic to

$$\left\{(x_1, \ldots, x_r) \in \prod_{i=1}^{r} G/P_{\beta_i} \;\middle|\; x_0 = 1 \text{ and } x_i \in \mathbb{P}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r]\right\}.$$

Because the element w is the shortest in the class $\bar{w}$ and because the expression $\widetilde{w}$ is reduced, the last root β_r has to be the unique simple root β such that $\langle \varpi^\vee, \beta\rangle = 1$, so that $P_{\beta_r} = P_\varpi$. The image of the projection morphism $\pi : \widetilde{X}(\widetilde{w}) \to G/P_{\beta_r}$ is the minuscule Schubert variety $X(\bar{w})$. The morphism $\pi : \widetilde{X}(\widetilde{w}) \to X(\bar{w})$ is birational; it is what is called the Bott–Samelson resolution.

2.3 Cycles on the configuration variety

2.3.1 *A basis of the Chow ring.* With the previous notation, we describe, in this paragraph, some particular elements in the Chow ring $A^*(\widetilde{X}(\widetilde{w}))$. We describe a basis of this ring and of the monoids of ample divisors and effective curves. We calculate the canonical divisor in terms of this basis.

For all $k \in [1, r]$, denote by X_k the image of $\widetilde{X}(\widetilde{w})$ in the product $\prod_{i=1}^{k} G/P_{\beta_i}$ and $X_0 = \{x_0\}$. We have natural projection morphisms $f_k : X_k \to X_{k-1}$ for all $k \in [1, r]$ which are $\mathbb{P}^1$-fibrations. The morphism $\sigma_k : X_{k-1} \to X_k$ defined by $\sigma_{k-1}(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, x_{p(k)})$ with $x_{p(k)} = 1$ if $p(k)$ does not exist is a section of f_k. We recover in this way the structure of $\widetilde{X}(\widetilde{w})$ as a tower of $\mathbb{P}^1$-fibrations with sections described in [Dem74].

Let us define the divisors $Z_i = f_r^{-1} \cdots f_{i+1}^{-1} \sigma_i(X_{i-1})$. We have

$$Z_i = \{(x_1, \ldots, x_r) \in \widetilde{X}(\widetilde{w}) \mid x_i = x_{p(i)}\}$$

in the configuration variety with $x_{p(i)} = 1$ if $p(i)$ does not exist. The divisors $(Z_i)_{i\in[1,r]}$ have normal crossings (cf. for example [Dem74]). Then for any subset K of $[1, r]$, one defines $Z_K = \bigcap_{i\in K} Z_i$. Let us recall the following (see for example [Dem74]) fact.

Fact 2.8. *The image by π of Z_K is the Schubert subvariety $X(y)$ where y is the longest element that can be written as a subword of $\widetilde{w}$ without any term s_i for $i \in K$.*

Denote by ξ_i the class of Z_i in $A^*(\widetilde{X}(\widetilde{w}))$. Following Demazure [Dem74], we define the sequence of roots $(\gamma_i)_{i\in[1,r]}$ associated to $\widetilde{w}$ by $\gamma_1 = \beta_1$, $\gamma_2 = s_{\beta_1}(\beta_2), \ldots, \gamma_r = s_{\beta_1} \cdots s_{\beta_{r-1}}(\beta_r)$. Demazure proves that the classes $(\xi_i)_{i\in[1,r]}$ generate the Chow ring.

Theorem 2.9 (Demazure [Dem74, Par. 4, Proposition 1]). *The Chow ring $A^*(\widetilde{X}(\widetilde{w}))$ of $\widetilde{X}(\widetilde{w})$ is isomorphic over $\mathbb{Z}$ to*

$$\frac{\mathbb{Z}[\xi_1, \ldots, \xi_r]}{(\xi_i \cdot \sum_{j=1}^{i} \langle \gamma_j^\vee, \gamma_i\rangle \xi_j \text{ for all } i \in [1, r])}.$$

Let us recall the following result due to Lauritzen and Thomsen [LT04].

Proposition 2.10. *The divisors $(\xi_i)_{1\leqslant i\leqslant r}$ form a basis of the monoid of effective divisors.*

Let us denote by T_i the pull-back on $\widetilde{X}(\widetilde{w})$ of the relative tangent sheaf of the fibration f_i. We denote by C_i the curve Z_K with $K = [1, r] \setminus \{i\}$ and recall some formulae in the ring $A^*(\widetilde{X}(\widetilde{w}))$ given in [Per05, Corollary 3.8 and Propositions 3.3 and 3.11]. If L is a line bundle, we will, by abuse of notation, still denote by L its first Chern class.

Proposition 2.11. *We have the following formulae:*

$$[C_i]\cdot\xi_j=\begin{cases}0 & \text{for } i>j,\\ 1 & \text{for } i=j,\\ \langle\beta_i^\vee,\beta_j\rangle & \text{for } i<j,\end{cases}\quad [C_i]\cdot T_j=\begin{cases}0 & \text{for } i>j,\\ \langle\beta_i^\vee,\beta_j\rangle & \text{for } i\leqslant j,\end{cases}\quad \text{and}\quad T_i=\sum_{k=1}^{i}\langle\gamma_k^\vee,\gamma_i\rangle\cdot\xi_k.$$

Because $\widetilde{X}(\widetilde{w})$ is a sequence of $\mathbb{P}^1$-fibrations with T_i as relative tangent bundle, we have the formula: $-K_{\widetilde{X}(\widetilde{w})}=\sum_{i=1}^r T_i$. Together with the preceding proposition we obtain the equality

$$-K_{\widetilde{X}(w)}=\sum_{k=1}^{r}\bigg(\sum_{i=k}^{r}\langle\alpha_k^\vee,\gamma_i\rangle\bigg)\xi_k.$$

2.3.2 *Ample divisors.* In this paragraph, we describe the ample divisors on $\widetilde{X}(\widetilde{w})$. This has already been done in [LT04] but we rephrase the results in terms of configuration varieties. We also get a description of the Mori cone (see [Mat02] for references on this cone).

We have natural morphisms $p_i:\widetilde{X}(\widetilde{w})\to G/P_{\beta_i}$ and, as P_{β_i} is maximal, the Picard group of, G/P_{β_i} is generated by a very ample invertible sheaf $\mathcal{O}_{G/P_{\beta_i}}(1)$ and we define on $\widetilde{X}(\widetilde{w})$ the invertible sheaf $\mathcal{L}_i=p_i^*(\mathcal{O}_{G/P_{\beta_i}}(1))$. These sheaves form a basis of the ample monoid (we will also call it the ample cone).

We also define curves Y_i for $i\in[1,r]$ by

$$Y_i=\left\{(x_1,\ldots,x_r)\in\prod_{i=1}^{r}G/P_{\beta_i}\;\middle|\; x_j=1 \text{ for } j\neq i \text{ and } x_i\in\mathbb{P}(1,\beta_i)\right\},$$

so that Y_i is isomorphic to $\mathbb{P}(1,\beta_i)$. We show that Y_i is contained in $\widetilde{X}(\widetilde{w})$.

Lemma 2.12. *For any $x_i\in\mathbb{P}(\bar{1},\beta_i)$, the element $(x_j)_{j\in[1,r]}$ of $\prod_{j=1}^r G/P_{\beta_j}$ such that $x_j=\bar{1}$ for all $j\neq i$ is in the configuration variety $\widetilde{X}(\widetilde{w})$.*

Proof. We only have to prove that, for any x_i in $\mathbb{P}(\bar{1},\beta_i)=P^{\beta_i}/B$, we have $\bar{1}\in\mathbb{P}(x_i,\beta_{i+1})$.

The element x_i can be lifted to some $b_i\in P^{\beta_i}$. The elements of $\mathbb{P}(x_i,\beta_{i+1})$ are the classes of elements of the form $b_ib_{i+1}\in P^{\beta_i}P^{\beta_{i+1}}$. If $\beta_{i+1}\neq\beta_i$, then $P^{\beta_i}\subset P_{\beta_{i+1}}$. In this case we set $b_{i+1}=1$ so that the class of $b_ib_{i+1}\in\mathbb{P}(x_i,\beta_{i+1})$ is $\bar{1}$ in $G/P_{\beta_{i+1}}$. If $\beta_{i+1}=\beta_i$, then we set $b_{i+1}=b_i^{-1}$ to get the result. □

We now describe the relations between the classes $[Y_i]$ (respectively $\mathcal{L}_i$) and $[C_i]$ (respectively ξ_i). The definitions of the curves Y_i and the line bundles $\mathcal{L}_i$ yield the following result.

Proposition 2.13. *We have the formula $[\mathcal{L}_i]\cdot[Y_j]=\delta_{i,j}$. In other words the families $(\mathcal{L}_i)_{i\in[1,r]}$ and $([Y_i])_{i\in[1,r]}$ are dual to each other.*

Let us prove that the family $([Y_i])_{i\in[1,r]}$ forms a basis of $A_1(\widetilde{X}(\widetilde{w}))$.

Proposition 2.14. *For all $i\in[1,r]$, we have $[Y_i]=[C_i]-[C_{s(i)}]$ (where $[C_{s(i)}]=0$ if $s(i)$ does not exist).*

As a consequence, the classes $([Y_i])_{i\in[1,r]}$ form a basis of $A_1(\widetilde{X}(\widetilde{w}))$ and the classes $(\mathcal{L}_i)_{i\in[1,r]}$ form a basis of $A^1(\widetilde{X}(\widetilde{w}))$.

Proof. On the one hand, the curve C_i is given by the equations $x_j=x_{p(j)}$ for $j\neq i$. This means that, for $j<i$, we have $x_j=1$ and, for all j with $\beta_j\neq\beta_i$, we also have $x_j=1$. The only indices k for which x_k may be different from 1 are such that $k=s^n(i)$ for some $n\in\mathbb{N}$. For such a k,

we have the equality $x_k = x_i$. Denote by $n(i)$ the biggest integer n such that $s^n(i)$ exists. The curve C_i (respectively $C_{s(i)}$) is the diagonal in the product

$$\prod_{k=0}^{n(i)} \mathbb{P}(1, \beta_{s^k(i)}) \quad \left(\text{respectively } \prod_{k=1}^{n(i)} \mathbb{P}(1, \beta_{s^k(i)})\right).$$

On the other hand, the curve Y_i corresponds to the first factor of the first product. In this product we thus have the required equality. □

Let us now give a description of the ample monoid and of the monoid of effective curves.

Corollary 2.15. *The closure in $A^1(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ of the cone of ample divisors is generated by the classes $[\mathcal{L}_i]$ and the cone of effective curves is generated by the classes $[Y_i]$. All ample divisors are very ample.*

Proof. Let D be ample on $\widetilde{X}(\widetilde{w})$; then $a_i = D \cdot [Y_i]$ is a positive integer. Because of Proposition 2.13, we have $D = \sum_{i=1}^r a_i \mathcal{L}_i$ and D lies in the cone generated by the $\mathcal{L}_i$.

Conversely, any divisor $\sum_{i=1}^r a_i \mathcal{L}_i$ with $a_i > 0$ gives the embedding of $\widetilde{X}(\widetilde{w})$ obtained by composing the inclusion in the product $\prod_{i=1}^r G/P_{\beta_i}$ with the Veronese morphism given by the very ample sheaf $\bigotimes_{i=1}^r \mathcal{O}_{G/P_{\beta_i}}(a_i)$.

In the same way we get the result on effective curves. □

Finally we calculate the divisors classes $[\mathcal{L}_i]$ in terms of the basis $(\xi_k)_{k \in [1,r]}$.

Proposition 2.16. *The kth coordinate of $\mathcal{L}_i$ in the basis $(\xi_i)_{i \in [1,r]}$ is 0 if $k > i$, 1 if $k = i$ and is given by the following formulae if $k < i$ and $\beta_k = \beta_i$ (respectively $\beta_k \neq \beta_i$):*

$$1 + \sum_{j=k+1, \beta_j = \beta_i}^{i} \langle \gamma_k^\vee, \gamma_j \rangle \quad \left(\textit{respectively} \sum_{j=k+1, \beta_j = \beta_i}^{i} \langle \gamma_k^\vee, \gamma_j \rangle\right).$$

In particular we have the following simple formula

$$\mathcal{L}_r = \sum_{k=1}^r \xi_k.$$

Proof. Let us recall from [Per05, Lemma 4.5] that the classes of curves

$$[\widehat{C}_i] = [C_i] + \sum_{k=i+1}^{n} \langle \gamma_i^\vee, \gamma_k \rangle [C_k]$$

form the dual basis to $(\xi_i)_{i \in [1,r]}$. The kth coordinate is thus given by the intersection $\mathcal{L}_i \cdot [\widehat{C}_k]$.

Applying Propositions 2.13 and 2.14 we get that

$$\mathcal{L}_i \cdot [C_j] = \begin{cases} 1 & \text{for } i > j \text{ and } \beta_i = \beta_j, \\ 0 & \text{otherwise.} \end{cases}$$

Applying this gives the first formula. For the case of $\mathcal{L}_r$, the formula is a consequence of the following formula from [Per05, Corollary 2.18]:

$$\sum_{j=k+1, \beta_j = \beta_r}^{r} \langle \gamma_k^\vee, \gamma_j \rangle = \begin{cases} 1 & \text{if } \beta_k \neq \beta_r, \\ 0 & \text{if } \beta_k = \beta_r. \end{cases}$$ □

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Table 1. Minuscule and cominuscule weights.

Type	Minuscule	Cominuscule
A_n	$\varpi_1 \cdots \varpi_n$	same weights
B_n	ϖ_n	ϖ_1
C_n	ϖ_1	ϖ_n
D_n	ϖ_1, ϖ_{n-1} and ϖ_n	same weights
E_6	ϖ_1 and ϖ_6	same weights
E_7	ϖ_7	same weight
E_8	none	none
F_4	none	none
G_2	none	none

3. Minuscule Schubert varieties

In this section we recall the notion of minuscule weight and study the related flag and Schubert varieties. Our basic reference will be [LMS79].

3.1 Definitions

Definition 3.1. Let ϖ be a fundamental weight:

(i) we say that ϖ is *minuscule* if we have $\langle \alpha^\vee, \varpi \rangle \leqslant 1$ for all positive roots $\alpha \in \Delta^+$;

(ii) we say that ϖ is *cominuscule* if $\varpi^\vee$ is a minuscule weight for the dual root system.

Remark 3.2. An equivalent definition of cominuscule weights is the following: ϖ is cominuscule if $\langle \alpha_0^\vee, \varpi \rangle = 1$ where α_0 is the longest root.

With the notation of Bourbaki [Bou68], the minuscule and cominuscule weights are as shown in Table 1.

Definition 3.3. Let ϖ be a minuscule (respectively cominuscule) weight and let P_ϖ be the associated parabolic subgroup. The flag variety G/P_ϖ is then said to be minuscule (respectively cominuscule). The Schubert varieties of a minuscule (respectively cominuscule) flag variety are called minuscule (respectively cominuscule) Schubert varieties.

Recall that we define in the notation a parabolic subgroup P^w. An element $w \in W$ is said to be minuscule (respectively cominuscule) if $P^w = P_\varpi$.

Remark 3.4. To study minuscule flag varieties and their Schubert varieties, it is sufficient to restrict ourselves to simply laced groups.

In fact the variety G/P_{ϖ_n} with $G = \mathrm{Spin}_{2n+1}$ is isomorphic to the variety $G'/P'_{\varpi_{n+1}}$ with $G' = \mathrm{Spin}_{2n+2}$ and Schubert varieties are identified via this isomorphism. The same situation occurs with G/P_{ϖ_1}, $G = \mathrm{Sp}_{2n}$ and $G'/P'_{\varpi_1}, G' = \mathrm{SL}_{2n}$.

3.2 First properties

Let us recall some properties of minuscule Schubert varieties and minuscule elements $w \in W$. For a general element $w \in W$ with P^w maximal, the quiver and the Bott–Samelson resolution are defined once a reduced expression, modulo commutation relations, $\widetilde{w}$ of w is fixed. For minuscule Schubert varieties there is a unique such choice, as follows.

Theorem 3.5 (Stembridge [Ste97]). *If $w \in W$ is a minuscule element then there exists a unique reduced expression $\widetilde{w}$ of w modulo commutation relations.*

In particular for a minuscule element $w \in W$ the quiver $Q_{\widetilde{w}}$ depends only on w and we denote it by Q_w. Another important result on minuscule Schubert varieties is the following (see [LMS79]).

Theorem 3.6. *Let P_ϖ be a parabolic subgroup associated to a minuscule weight. Then the Bruhat order in W/W_P is generated by simple reflections.*

Remark 3.7. This statement is equivalent to the fact that any divisor D on a minuscule Schubert variety $X(w)$ is a moving divisor, that is to say, D is not fixed by P_w.

Let us also recall a simple fact on reduced decompositions of minuscule elements. Let $\widetilde{w} = (s_{\beta_1}, \ldots, s_{\beta_r})$ be a reduced expression of a minuscule element $w \in W$. Set $w_i = s_{\beta_i} \cdots s_{\beta_r}$ for $i \in [1, r]$ and $w_{r+1} = 1$. Then we have the following fact (see for example the proof of Theorem 3.1 in [LMS79]).

Fact 3.8. *We have $\langle \beta_{i-1}^\vee, w_i(-\varpi)\rangle = -1$ for all $i \in [2, r+1]$. As a consequence we have for all $i \in [2, r]$:*

$$w_i(-\varpi) = -\varpi + \beta_r + \cdots + \beta_i.$$

We will see in the next section that this fact imposes strong conditions on the geometry of the quiver. In particular one can recover Theorems 3.5 and 3.6 from this fact and its consequences on the quiver (see [Per07]).

4. Geometry of minuscule quivers

In this section, we give an explicit description of the quiver $Q_{\widetilde{w}}$ given by a reduced expression $w = s_{\beta_1} \cdots s_{\beta_r}$ of the shortest element in the class $\bar{w} \in W/W_{P_\varpi}$. We also define invariants of the quiver and deduce some consequences on the geometry of the Schubert variety.

4.1 Minuscule conditions on the quivers

The following proposition describes all possible quivers for minuscule Schubert varieties.

Proposition 4.1. *Let $w \in W$ be a minuscule element and $\widetilde{w} = (s_{\beta_1}, \ldots, s_{\beta_r})$ a reduced expression of w.*

(i) *There is no arrow starting from the vertex r and β_r is the unique simple root with $\langle \beta_r^\vee, \varpi\rangle = 1$.*

(ii) *If a vertex $i < r$ of the quiver is such that $s(i)$ does not exist, then there is a unique arrow from i. If k is the end of the arrow we have $\langle \beta_i^\vee, \beta_k\rangle = -1$.*

(iii) *If a vertex i of the quiver is such that $s(i)$ exists, then there are exactly two arrows from i. If k_1 and k_2 are the ends of these arrows we have $\langle \beta_i^\vee, \beta_{k_1}\rangle = \langle \beta_i^\vee, \beta_{k_2}\rangle = -1$.*

Proof. (i) Fact 3.8 shows that we have $\langle \beta_r^\vee, \varpi\rangle = 1$.

(ii) Let i be such a vertex. In particular we have $\beta_i \neq \beta_r$ and $\langle \beta_i^\vee, \varpi\rangle = 0$. Fact 3.8 gives $\langle \beta_i^\vee, -\varpi + \beta_r + \cdots + \beta_{i+1}\rangle = \langle \beta_i^\vee, w_{i+1}(-\varpi)\rangle = -1$ and thus

$$\sum_{k=i+1}^{r} \langle \beta_i^\vee, \beta_k\rangle = -1.$$

We conclude because every term of this sum has to be either 0 or -1.

(iii) Let i be such a vertex. The same calculation as above shows that $\sum_{k=i+1}^{r} \langle \beta_i^\vee, \beta_k\rangle = \sum_{k=s(i)+1}^{r} \langle \beta_i^\vee, \beta_k\rangle = -1$ if $\beta_i \neq \beta_r$ and $\sum_{k=i+1}^{r} \langle \beta_i^\vee, \beta_k\rangle = \sum_{k=s(i)+1}^{r} \langle \beta_i^\vee, \beta_k\rangle = 0$ if $\beta_i = \beta_r$. In particular we always have

$$\sum_{k=i+1}^{s(i)} \langle \beta_i^\vee, \beta_k\rangle = 0.$$

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We conclude because the only positive term is $\langle \beta_i^\vee, \beta_{s(i)} \rangle = 2$ and every other term of this sum has to be either 0 or -1. $\square$

Remark 4.2. (i) The fact that there are always two vertices between i and $s(i)$ implies that the expression deduced from a quiver satisfying the conditions of the preceding proposition is always reduced and the quivers satisfying the conditions are always quivers associated to a minuscule Schubert variety.

(ii) A minuscule quiver is always connected: there is a path from any vertex i to the last vertex r.

Example 4.3. Set $G = SL_4$ and consider the reduced expressions $\widetilde{w}_1 = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2}$ and $\widetilde{w}_2 = s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$ with the notation of [Bou68]. The pictures of the associated quivers are shown below.

The first quiver is a minuscule quiver associated to the Grassmannian $\mathbb{G}(2,4)$ of lines in $\mathbb{P}^3$. The second one is not a minuscule quiver: condition (iii) of Proposition 4.1 is not satisfied for the vertex 1. The associated Schubert variety is isomorphic to the complete flag variety of SL_3.

4.2 Combinatorial description of the minuscule quivers

From Proposition 4.1, it is easy to describe the quivers Q_ϖ of a minuscule flag variety G/P_ϖ (see Appendix A for a list of these quivers). We now describe the quivers of minuscule Schubert varieties in G/P_ϖ as subquivers of Q_ϖ. Define a natural partial order on the quiver.

Definition 4.4. (i) We denote by $\preccurlyeq$ the partial order on the vertices of the quiver generated by the relations $i \preccurlyeq j$ if there exists an arrow from i to j.

(ii) Let A be a totally unordered set of vertices of the quiver Q_ϖ for the partial order $\preccurlyeq$. We denote by $Q_{\bar{A}}$ the full subquiver of Q_ϖ with vertices $i \in Q_\varpi$ such that there exists $a \in A$ with $i \preccurlyeq a$ and by Q_A the full subquiver of Q_ϖ whose vertices are not vertices of $Q_{\bar{A}}$.

Proposition 4.5. *The quivers of Schubert varieties in G/P_ϖ are in one-to-one correspondence with the subquivers Q_A of Q_ϖ for A any totally unordered set of vertices of Q_ϖ.*

Via this correspondence, the Bruhat order is given by the inclusion of quivers.

Proof. Let $X(w) \subset G/P_\varpi$ be a Schubert variety and denote by $w_\varpi \in W$ the element of minimal length such that $X(w_\varpi) = G/P_\varpi$. There exists a sequence $(\beta_1, \dots, \beta_i)$ of simple roots such that $w_\varpi = s_{\beta_1} \cdots s_{\beta_i} w$. Taking a reduced expression $w = s_{\beta_{i+1}} \cdots s_{\beta_r}$ of w we get a reduced expression of w_ϖ which is unique modulo commutation relations.

The vertices of the quiver Q_ϖ are indexed by $[1, r]$. Denote by $A = \{i_1, \dots, i_k\}$ the set of maximal elements for the partial order $\preccurlyeq$ of the set $[1, i]$. The set A is totally unordered and the quiver associated to $X(w)$ is Q_A.

The fact that this is a one-to-one correspondence comes from the unicity of the reduced expression. $\square$

Finally let us define some particular vertices of these quivers. In the following definition Q_w is the quiver of a minuscule Schubert variety $X(w)$.

Definition 4.6. (i) We call *peak* any vertex of Q_w minimal for the partial order $\preccurlyeq$. We denote by $\mathrm{Peaks}(Q_w)$ the set of peaks of Q_w.

(ii) We call *hole* of the quiver Q_w any vertex i of Q_ϖ satisfying one of the following properties:

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(a) the vertex i is in Q_w but $p(i) \notin Q_w$ and there are exactly two vertices $j_1 \preccurlyeq i$ and $j_2 \preccurlyeq i$ in Q_w with $\langle \beta_i^\vee, \beta_{j_k} \rangle \neq 0$ for $k = 1, 2$;

(b) the vertex i is not in Q_w, $s(i)$ does not exist in Q_ϖ and $\beta_i \in \partial\mathrm{Supp}(w)$.

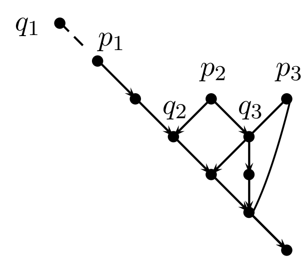
Because the vertex of the second type of holes is not a vertex in Q_w we call such a hole a virtual hole of Q_w. We denote by $\mathrm{Holes}(Q_w)$ the set of holes of Q_w.

(iii) The height $h(i)$ of a vertex i is the largest positive integer n such that there exists a sequence $(i_k)_{k\in[1,n]}$ of vertices with $i_1 = 1$, $i_n = r$ and such that there is an arrow from i_k to i_{k+1} for all $k \in [1, n-1]$.

Remark 4.7. (i) If $Q_w = Q_A$ as in Definition 4.4 then $\mathrm{Holes}(Q_w) = A$.

(ii) The height is well defined because there is at least one path from any vertex i to the last vertex r.

Example 4.8. Let $X(w)$ and Q_w be as in Example 2.5. Then the quiver Q_w has three peaks p_1, p_2 and p_3 and three holes q_1, q_2 and q_3 with q_1 a virtual hole as in the following picture.



The heights of p_1, p_2, p_3, q_1, q_2 and q_3 are respectively 6, 5, 5, 7, 4 and 4.

The following proposition gives a recursive way to calculate the height of a vertex.

Proposition 4.9. *Let Q be a quiver associated to a minuscule Schubert variety and i a vertex of this quiver. Then one of the following cases occurs.*

(i) *If $s(i)$ does not exist, then there exists a unique $k \succcurlyeq i$ with $\langle \beta_k^\vee, \beta_i \rangle = -1$ and we have*

$$h(i) = h(k) + 1.$$

(ii) *If $s(i)$ exists, then there exists a non-negative integer n and a sequence $(j_k, j'_k)_{k\in[0,n+1]}$ of vertices with $j_0 = i$, $j'_k = s(j_k)$ for $k \in [0, n]$, $\beta_{j_{n+1}} \neq \beta_{j'_{n+1}}$, $\langle \beta_{j_k}, \beta_{j_{k+1}} \rangle = -1$ and $\langle \beta'_{j_k}, \beta'_{j_{k+1}} \rangle = -1$ for $k \in [0, n]$ and $j_0 \preccurlyeq \cdots \preccurlyeq j_n \preccurlyeq j_{n+1}, j'_{n+1} \preccurlyeq j'_n \preccurlyeq \cdots \preccurlyeq j'_0$. In this case we have*

$$h(j_k) = 2n + 2 - k + h(s(i)) \quad \text{and} \quad h(j'_k) = k + h(s(i)) \quad \text{for all } k \in [0, n+1].$$

Proof. We prove these formulae by descending induction on i. If $i = r$ then $h(i) = 1$. Assume that the proposition is true for all $j > i$. In the first case, any sequence of arrows from i to r has to pass through the vertex k and we have $h(i) = h(k) + 1$.

In the second case, we first prove the existence of the sequence $(j_k, j'_k)_{k\in[0,n+1]}$. Let us denote by j_1 and j'_1 the two vertices j such that there is an arrow from i to j. If $\beta_{j_1} \neq \beta_{j'_1}$ then set $n = 0$ and we are done. Otherwise, assume (for example) that $j_1 < j'_1$ then $j'_1 = s(j_1)$. Indeed, otherwise there would exist $k \in [j_1, j'_1]$ and in particular $k < s(i)$ with $\beta_k = \beta_{j_1}$ thus $\langle \beta_i^\vee, \beta_k \rangle = -1$. By construction of the quiver, there must be an arrow from i to k and thus at least three arrows from i. This is impossible by Proposition 4.1. We can construct from (j_1, j'_1) a pair (j_2, j'_2) in the same way and by induction a sequence $(j_k, j'_k)_k$. As long as $\beta_{j_k} = \beta_{j'_k}$ we can go on. This has to stop because the quiver is finite.

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The formula on the heights follows by induction. Any sequence from i to r has to go through j_1 or j_1'. As the height of j_1 is bigger than the one of $j_1' = s(j_1)$, by induction we must have $h(i) = h(j_1) + 1$ and we apply the induction hypothesis on j_1 to conclude. $\square$

Remark 4.10. By changing the order of commuting factors, we may assume in the preceding proposition that $k = i+1$ in the first case and that $j_k = i + k$ and $j_k' = i + 2n + 3 - k$ for all $k \in [0, n+1]$ in the second one.

We can now describe the stabilizer P_w of a Schubert variety $X(w)$ in terms of its quiver Q_w.

Proposition 4.11. *We have the equality* $\Sigma(P_w) = \beta(\mathrm{Holes}(Q_w))$.

Proof. A simple root β is in $\Sigma(P_w)$ if and only if $s_\beta \bar{w} > \bar{w}$ (for the Bruhat order in W/W_{P_w}). But from unicity of reduced expressions and our characterization (Proposition 4.1) of quivers associated to a reduced expression, we see that this is equivalent to $\beta \in \beta(\mathrm{Holes}(Q_w))$. $\square$

Corollary 4.12. *Let* $Q_{w'}$ *be the quiver of a Schubert subvariety* $X(w')$ *of* $X(w)$ *stable under* P_w. *Then* $\beta(\mathrm{Holes}(Q_{w'})) \subset \beta(\mathrm{Holes}(Q_w))$.

Example 4.13. Let $X(w)$ and Q_w as in Example 2.5 then we have

$$\Sigma(P_w) = \beta(\mathrm{Holes}(Q_w)) = \{\alpha_1, \alpha_4, \alpha_6\}.$$

The parabolic P_w is the stabilizer of the partial flag $F_1 \subset F_4 \subset F_6$.

4.3 Weil and Cartier divisors

In this section, we describe some well-known results on Weil and Cartier divisors of a minuscule Schubert variety $X(w)$ in terms of the quiver. In particular we recover the fact that all Schubert divisors are of multiplicity one in the hyperplane class (see for example [Ses78]).

Proposition 4.14. *The divisor class group* $\mathrm{Weil}(X(w))$ *is the free* $\mathbb{Z}$*-module generated by the classes* $D_i := \pi_* \xi_i$ *for* $i \in \mathrm{Peaks}(Q_w)$.

The Picard group $\mathrm{Pic}(X(w)) \subset \mathrm{Weil}(X(w))$ *is isomorphic to* $\mathbb{Z}$ *and is generated by the element* $\mathcal{L}(w) := \pi_* \mathcal{L}_r = \mathcal{O}_{G/P_{\beta_r}}(1)|_{X(w)}$. *We have the formula*

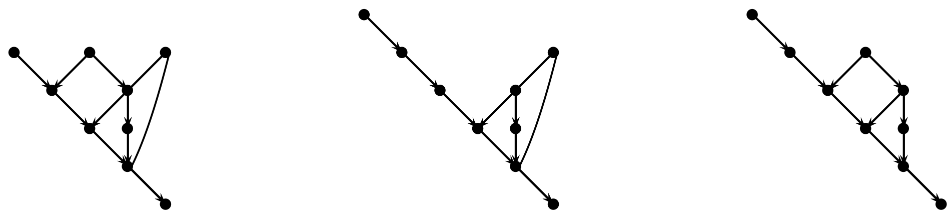
$$\mathcal{L}(w) = \sum_{i \in \mathrm{Peaks}(Q_w)} D_i.$$

Proof. It is well known (see for example [Bri05]) that the Picard group is isomorphic to $\mathbb{Z}$ and generated by $\mathcal{L}(w)$ and that the group of Weil divisors is freely generated by the divisorial Schubert varieties. These varieties are the images by $\pi : \widetilde{X}(\widetilde{w}) \to X(w)$ of the non-contracted divisors Z_i. Now let $\widetilde{w}^i$ be the expression obtained from the reduced expression $\widetilde{w}$ by removing the simple root β_i. According to Fact 2.8, the image of Z_i is not contracted if and only if $\widetilde{w}^i$ is reduced or equivalently the quiver $Q_{\widetilde{w}(i)}$ corresponds to a reduced expression. It is clear that this is the case if and only if $i \in \mathrm{Peaks}(Q_w)$.

The last formula is an application of Proposition 2.16 and expresses the well-known fact that all Schubert divisors are of multiplicity one in the minuscule case. $\square$

Corollary 4.15. *A minuscule Schubert variety is locally factorial if and only if its quiver has a unique peak.*

Example 4.16. Let $X(w)$ and Q_w be as in Example 2.5 and keep the notation of Example 4.8. Then the quivers of the divisors D_{p_1}, D_{p_2} and D_{p_3} are as shown in the diagrams.



D_{p_1} D_{p_2} D_{p_3}

These divisors are described as follows:

$$\begin{aligned}
D_{p_1} &= \{V_8 \in \mathbb{G}_{\mathrm{iso}}(8,16) \mid F_2 \subset V_8, \dim(V_8 \cap F_4) \geqslant 2 \text{ and } \dim(V_8 \cap F_6) \geqslant 4\},\\
D_{p_2} &= \{V_8 \in \mathbb{G}_{\mathrm{iso}}(8,16) \mid F_1 \subset V_8 \text{ and } \dim(V_8 \cap F_5) \geqslant 4\},\\
D_{p_3} &= \{V_8 \in \mathbb{G}_{\mathrm{iso}}(8,16) \mid F_1 \subset V_8, \dim(V_8 \cap F_4) \geqslant 2 \text{ and } \dim(V_8 \cap F_8) \geqslant 6\},
\end{aligned}$$

where $F_1 \subset F_2 \subset F_4 \subset F_5 \subset F_6 \subset F_8$ is a partial flag of totally isotropic subspaces of respective dimensions one, two, four, five, six and eight and such that $F_8 \in \mathbb{G}^1_{\mathrm{iso}}(8,16)$. The Schubert variety $X(w)$ is not locally factorial.

4.4 Canonical divisor

In this section we compute the canonical divisor of $X(w)$. The Schubert varieties are generally singular and not Gorenstein (see [WY04] for a characterization of Gorenstein Schubert varieties for SL_n). We cannot therefore define the canonical divisor as a Cartier divisor.

The canonical divisor $K_{X(w)}$ of a Schubert variety $X(w)$ is well defined as a Weil divisor class thanks to the divisor of a top degree form on the smooth locus of $X(w)$. The properties of Schubert varieties (they are normal, Cohen–Macaulay with rational singularities) and the Bott–Samelson resolution $\pi : \widetilde{X}(\widetilde{w}) \to X(w)$ enable one however to calculate $K_{X(w)}$ by $K_{X(w)} = \pi_*(K_{\widetilde{X}(\widetilde{w})})$ (see for example [BK05, § 3.4]).

Let us denote by $h(w)$ the lowest height of a peak in Q_w (the quiver associated to $X(w)$). We have the following result.

Proposition 4.17. *We have the formula*

$$-K_{X(w)} = \sum_{i \in \mathrm{Peaks}(Q_w)} (h(i)+1)D_i = (h(w)+1)\mathcal{L}(w) + \sum_{i \in \mathrm{Peaks}(Q_w)} (h(i)-h(w))D_i.$$

Proof. The second part of the formula comes from the first one and Proposition 4.14.

To prove the first part, we use the fact that $K_{X(w)} = \pi_*(K_{\widetilde{X}(\widetilde{w})})$ and the formula of § 2.3.1. We are left to prove the following lemma.

Lemma 4.18. *We have the formula*

$$\sum_{k=i}^{r} \langle \gamma_i^\vee, \gamma_k \rangle = \sum_{k \succcurlyeq i} \langle \gamma_i^\vee, \gamma_k \rangle = h(i)+1.$$

Proof. We proceed by descending induction and use Proposition 4.9 and Remark 4.10. We have the following two cases.

(i) If $s(i)$ does not exist, then there exists a unique $k \succcurlyeq i$ with $\langle \beta_k^\vee, \beta_i \rangle = -1$ and we have

$$h(i) = h(k)+1.$$

(ii) If $s(i)$ exists, then there exists a non-negative integer n and a sequence $(j_k, j'_k)_{k\in[0,n+1]}$ of vertices with $j_0 = i$, $j'_k = s(j_k)$ for $k \in [0,n]$, $\beta_{j_{n+1}} \neq \beta_{j'_{n+1}}$, $\langle \beta_{j_k}, \beta_{j_{k+1}}\rangle = -1$ and $\langle \beta'_{j_k}, \beta'_{j_{k+1}}\rangle = -1$ for $k \in [0,n]$ and $j_0 \preccurlyeq \cdots \preccurlyeq j_n \preccurlyeq j_{n+1}, j'_{n+1} \preccurlyeq j'_n \preccurlyeq \cdots \preccurlyeq j'_0$. Furthermore we may assume that $j_k = i+k$ and $j'_k = i+2n+3-k$ for all $k \in [0, n+1]$.

We proceed by descending induction on i. If $i = r$, there is a unique term and the sum is $\langle \gamma_r^\vee, \gamma_r\rangle = 2 = h(r)+1$. We assume that the formula is true for all $j \geqslant i+1$. Let us use the following sequence $\widetilde{\alpha}_j = s_{\beta_r}\cdots s_{\beta_{r-j+2}}(\beta_{r-j+1})$ satisfying the equality $\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r+1-i}\rangle = \langle \gamma_k^\vee, \gamma_i\rangle$ (see for example [Per05]).

Calculating $\widetilde{\alpha}_{r+1-i} = s_{\beta_r}\cdots s_{\beta_{i+1}}(\beta_i)$ we find

$$\widetilde{\alpha}_{r+1-i} = \begin{cases} \widetilde{\alpha}_{r-i} + s_{\beta_r}\cdots s_{\beta_{i+2}}(\beta_i) & \text{in the first case;}\\ \widetilde{\alpha}_{r-i-n} + \widetilde{\alpha}_{r-i-n-1} - \widetilde{\alpha}_{r-i-2n-2} & \text{in the second case.}\end{cases}$$

Let us now calculate the sum

$$\sum_{k\succcurlyeq i}\langle \gamma_k^\vee, \gamma_i\rangle = \sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r+1-i}\rangle.$$

In the first case, set $\alpha = s_{\beta_r}\cdots s_{\beta_{i+2}}(\beta_i)$. If $j \geqslant i+2$, then we have $\langle \beta_i^\vee, \beta_j\rangle = 0$ so that $\alpha = \beta_i$. Furthermore, the root $\widetilde{\alpha}_{r+1-j}$ is a sum of simple roots contained in the set $\{\beta_j, \dots, \beta_r\}$ so, for $j \geqslant i+2$, we have $\langle \widetilde{\alpha}^\vee_{r+1-j}, \alpha\rangle = 0$. In this case the sum equals

$$\begin{aligned}\sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r+1-i}\rangle &= \sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i}\rangle + \sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \alpha\rangle\\ &= \sum_{k\succcurlyeq i+1}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i}\rangle + \langle \widetilde{\alpha}^\vee_{r+1-i}, \widetilde{\alpha}_{r-i}\rangle + \langle \widetilde{\alpha}^\vee_{r+1-i}, \alpha\rangle + \langle \widetilde{\alpha}_{r-i}, \alpha\rangle\\ &= h(i+1) + 1 - \langle \beta_i^\vee, \beta_{i+1}\rangle + \langle (\beta_i + \beta_{i+1})^\vee, \beta_i\rangle + \langle \beta_{i+1}^\vee, \beta_i\rangle\\ &= h(i+1) + 1 + 1 + 1 - 1 = h(i+1) + 2.\end{aligned}$$

In the second case, it is an easy exercise to see that the simple roots $\{\beta_k\}_{k\in[i,i+n+2]}$ form a diagram of type D_{n+2} (with the notation of [Bou68] the root β_{i+k} is the $(k+1)$th root of the diagram). We can then calculate

$$\widetilde{\alpha}_{r+1-i-k} = \begin{cases} s_{\beta_r}\cdots s_{\beta_{i+2n+4}}\left(\displaystyle\sum_{j=0}^{2n+2-k}\beta_{i+j}\right) & \text{for all } k \in [0, n+1],\\[2ex] s_{\beta_r}\cdots s_{\beta_{i+2n+4}}\left(\displaystyle\sum_{j=0}^{2n+3-k}\beta_{i+j}\right) & \text{for all } k \in [n+2, 2n+3].\end{cases}$$

The sum is in this case

$$\begin{aligned}&\sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r+1-i}\rangle\\ &\quad= \sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n}\rangle + \sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n-1}\rangle - \sum_{k\succcurlyeq i}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-2n-2}\rangle\\ &\quad= \sum_{k\succcurlyeq i+n+1}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n}\rangle + \sum_{k\succcurlyeq i+n+2}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n-1}\rangle - \sum_{k\succcurlyeq i+2n+3}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-2n-2}\rangle\\ &\qquad+ \sum_{k=0}^{n}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n}\rangle + \sum_{k=0}^{n+1}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n-1}\rangle - \sum_{k=0}^{2n+2}\langle \widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-2n+2}\rangle.\end{aligned}$$

The description of $\widetilde{\alpha}_{r+1-i-k}$ shows that for all $k \in [0, n]$ (respectively $k \in [1, 2n+2]$) the linear forms $\langle\widetilde{\alpha}_{r+1-i-k}, \cdot\rangle$ have value 1 at roots $\widetilde{\alpha}_{r-i-n}$ and $\widetilde{\alpha}_{r-i-n-1}$ (respectively $\widetilde{\alpha}_{r-i-2n-2}$). By the same argument, we have $\langle\widetilde{\alpha}^\vee_{r-i-n}, \widetilde{\alpha}_{r-i-n-1}\rangle = 0$ and $\langle\widetilde{\alpha}^\vee_{r+1-i}, \widetilde{\alpha}_{r-i-2n-2}\rangle = 0$. This gives us the following formulae:

$$\sum_{k=0}^{n}\langle\widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n}\rangle = n+1, \quad \sum_{k=0}^{n+1}\langle\widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-n-1}\rangle = n+1, \quad \sum_{k=0}^{2n+2}\langle\widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r-i-2n+2}\rangle = 2n+2.$$

Using the induction hypothesis we get

$$\begin{aligned}
&\sum_{k \succcurlyeq i}\langle\widetilde{\alpha}^\vee_{r+1-k}, \widetilde{\alpha}_{r+1-i}\rangle \\
&\quad = h(i+n+1) + 1 + h(i+n+2) + 1 - h(i+2n+3) - 1 + n + 1 + n + 1 - 2n + 2 \\
&\quad = h(i+n+1) + h(i+n+2) + 1 - h(i+2n+3).
\end{aligned}$$

We conclude in both cases thanks to Proposition 4.9. Lemma 4.18 is proved. □

This lemma completes the proof of Proposition 4.17. □

Corollary 4.19. *The Schubert variety $X(w)$ is Gorenstein if and only if all the peaks of its quiver have the same height. In this case we have $-K_{X(w)} = (h(w)+1)\mathcal{L}(w)$.*

Remark 4.20. For $G = SL_n$, we recover a particular case of the result of Woo and Yong [WY04] on Gorenstein Schubert varieties.

Example 4.21. Let $X(w)$ and Q_w be as in Example 2.5 and keep the notation of Examples 4.8 and 4.16. The variety $X(w)$ is not Gorenstein; we have

$$-K_{X(w)} = 7D_{p_1} + 6D_{p_2} + 6D_{p_3}.$$

5. Generalization of Bott–Samelson's construction

Let us now construct some projective varieties $\widehat{X}(\widehat{w})$ with at most terminal singularities together with birational morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$. These constructions generalize the *IH*-small resolutions of Zelevinsky [Zel83] and Sankaran and Vanchinathan [SV94]. In this section, we do not need to assume $X(w)$ minuscule and we will perform the construction in a more general setting.

Recall that the Bott–Samelson varieties can be seen as towers of $\mathbb{P}^1$-fibrations coming from reduced expressions $\widetilde{w}$ of an element w. Many generalizations of this construction (for example in [Zel83] and [SV94] but also in [Per00] and even in the general construction of Contou-Carrère [Con88]) are constructed as towers of locally trivial fibrations with fibers isomorphic to a fixed flag variety thanks to a more general decomposition $\widehat{w}$ of w as a product of elements in the Weyl group. For the varieties $\widehat{X}(\widehat{w})$ we make the same construction but we allow locally trivial fibrations with fiber a Schubert variety with at most locally factorial singularities (respectively at most Gorenstein in the case of the relative canonical model).

5.1 Elementary construction

Let us explain the following elementary construction. As in [Dem74], the variety $\widehat{X}(\widehat{w})$ will be constructed by successive applications of this elementary construction. Let $u \in W$ and Y be a variety with action of a parabolic subgroup P_Y of G containing B and assume that $P^u \cap G_u \subset P_Y$ (with the notation of § 1). We define

$$\widehat{Y}(u) = \overline{(P_u \cap G_u)u(P^u \cap G_u)} \times^{(P^u \cap G_u)} Y.$$

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Lemma 5.1. (i) *The variety $\widehat{Y}(u)$ is a locally trivial fibration over $X(u)$ with fibers isomorphic to Y.*

(ii) *Define the parabolic subgroup $P_{\widehat{Y}(u)}$ by*

$$\Sigma(P_{\widehat{Y}(u)}) = (\Sigma(P_Y) \cap \mathrm{Supp}(u)^c) \cup \partial\mathrm{Supp}(u) \cup (\Sigma(P_u) \cap \mathrm{Supp}(u)).$$

Then $P_u \cap G_u \subset P_{\widehat{Y}(u)}$ and the action of $P_u \cap G_u$ on $\widehat{Y}(u)$ extends to an action of $P_{\widehat{Y}(u)}$ on $\widehat{Y}(u)$.

Proof. (i) The first part of the proposition comes from the isomorphism between $X(u)$ and $\overline{(P_u \cap G_u)u(P^u \cap G_u)}/(P^u \cap G_u)$.

(ii) Remark that $\Sigma(P_u \cap G_u) = \Sigma(P_u) \cup \mathrm{Supp}(u)^c$ so we have the inclusion $\Sigma(P_{\widehat{Y}(u)}) \subset \Sigma(P_u \cap G_u)$ and the inclusion $P_u \cap G_u \subset P_{\widehat{Y}(u)}$.

For the second part, let us remark that one can replace the groups $P_u \cap G_u$ and $P^u \cap G_u$ by any bigger groups A and B such that the natural map $\overline{(P_u \cap G_u)u(P^u \cap G_u)}/(P^u \cap G_u) \to \overline{AuB}/B$ is an isomorphism and $B \subset P_Y$. For example, we take A and B such that

$$\Sigma(A) = (\Sigma(P_Y) \cap \mathrm{Supp}(u)^c) \cup \partial\mathrm{Supp}(u) \cup (\Sigma(P_u) \cap \mathrm{Supp}(u))$$

and

$$\Sigma(B) = (\Sigma(P_Y) \cap \mathrm{Supp}(u)^c) \cup (\Sigma(P^u) \cap \mathrm{Supp}(u)).$$

We have the required isomorphism and $B \subset P_Y$ (simply because $\Sigma(P_Y) \subset \Sigma(B)$). Then $\widehat{Y}(u)$ is isomorphic to $\overline{AuB} \times^B Y$ and A acts on $\widehat{Y}(u)$. $\square$

5.2 Construction of the resolution

Definition 5.2. (i) Let $w \in W$. A sequence $(w_1, \ldots, w_n)$ of elements of W such that $w = w_1 \cdots w_n$ is called a *generalized decomposition* and denoted by $\widehat{w}$. If moreover we have the equality $l(w) = \sum_{i=1}^n l(w_i)$ then we will say that the generalized decomposition is *reduced*.

(ii) Let us associate to any generalized decomposition a sequence of parabolic subgroups $(P_i)_{i \in [1,n]}$ defined by $P_n = P_{w_n}$ and

$$\Sigma(P_i) = (\Sigma(P_{i+1}) \cap \mathrm{Supp}(w_i)^c) \cup \partial\mathrm{Supp}(w_i) \cup (\Sigma(P_{w_i}) \cap \mathrm{Supp}(w_i)).$$

(iii) We will say that a generalized reduced decomposition is *admissible* if for all $i \in [1, n-1]$ we have $P^{w_i} \cap G_{w_i} \subset P_{i+1}$.

(iv) We will say that a generalized reduced decomposition is *good* if for all $i \in [1, n-1]$ we have $P^{w_i} \cap G_{w_i} \subset P_{w_{i+1} \cdots w_n}$ and $\partial\mathrm{Supp}(w_i) \subset \Sigma(P_{w_i \cdots w_n})$.

Proposition 5.3. *Let $\widehat{w}$ be an admissible reduced generalized decomposition of a minuscule element w of W. One can define by descending induction on n the varieties $\widehat{X}_i(\widehat{w})$ by $\widehat{X}_n(\widehat{w}) = X(w_n)$ and for $i < n$ by $\widehat{X}_i(\widehat{w}) = \widehat{Y}(w_i)$ where $Y = \widehat{X}_{i+1}(\widehat{w})$. The group $P_{\widehat{X}_i(\widehat{w})}$ is the group P_i.*

Furthermore, if the decomposition is good, then the group $P_{\widehat{X}_i(\widehat{w})}$ is the group $P_{w_i \cdots w_n}$ and in particular any good generalized reduced decomposition is admissible.

Proof. We proceed by induction. The variety $\widehat{X}_n(\widehat{w})$ is well defined and we have $P_{\widehat{X}_n(\widehat{w})} = P_{w_n}$. Assume that $\widehat{X}_{i+1}(\widehat{w})$ is well defined and that $P_{\widehat{X}_{i+1}(\widehat{w})} = P_{i+1}$. To prove that $\widehat{X}_i(\widehat{w})$ exists, we have to prove that $P^{w_i} \cap G_{w_i} \subset P_{\widehat{X}_{i+1}(\widehat{w})}$ but this holds by hypothesis. The fact that $P_{\widehat{X}_i(\widehat{w})} = P_i$ comes from Lemma 5.1.

Now in the case of a good generalized reduced decomposition, we have to prove that $P_i = P_{\widehat{X}_i(\widehat{w})} = P_{w_i \cdots w_n}$. We know from Lemma 5.1 that

$$\Sigma(P_{\widehat{X}_i(\widehat{w})}) = (\Sigma(\widehat{X}_{i+1}(\widehat{w})) \cap \mathrm{Supp}(w_i)^c) \cup \partial\mathrm{Supp}(w_i) \cup (\Sigma(P_{w_i}) \cap \mathrm{Supp}(w_i)).$$

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Let $\beta \in \Sigma(P_{w_i \cdots w_n})$. If $\beta \in \mathrm{Supp}(w_i)$ then β has to be a hole of the quiver of w_i so that $\beta \in \Sigma(P_{w_i})$ and $\beta \in \Sigma(P_{\widehat{X}_i(\widehat{w})})$. If $\beta \notin \mathrm{Supp}(w_i)$ and $\beta \notin \partial\mathrm{Supp}(w_i)$ then s_β commutes with w_i and we have $\beta \in \Sigma(P_{w_{i+1} \cdots w_n})$ and $\beta \in \Sigma(P_{\widehat{X}_i(\widehat{w})})$. Finally if $\beta \in \partial\mathrm{Supp}(w_i)$ we also have $\beta \in \Sigma(P_{\widehat{X}_i(\widehat{w})})$.

Conversely, let $\beta \in \Sigma(P_{\widehat{X}_i(\widehat{w})})$. If $\beta \in \mathrm{Supp}(w_i)$ then $\beta \in \Sigma(P_{w_i})$ so β corresponds to a hole of the quiver of w_i and thus has to be a hole of the quiver of $w_i \cdots w_n$. If $\beta \in \partial\mathrm{Supp}(w_i)$ we are done by hypothesis, and finally if β is neither in $\mathrm{Supp}(w_i)$ nor in $\partial\mathrm{Supp}(w_i)$ then $\beta \in \Sigma(P_{w_{i+1} \cdots w_n})$ by induction hypothesis. Thus β corresponds to a hole of the quiver of $w_{i+1} \cdots w_n$ and does not appear in the quiver of w_i. It is thus still a hole of the quiver of $w_i \cdots w_n$. ☐

Definition 5.4. With the notation of Proposition 5.3, we denote by $\widehat{X}(\widehat{w})$ the variety $\widehat{X}_1(\widehat{w})$.

Corollary 5.5. *The variety $\widehat{X}(\widehat{w})$ is a tower of locally trivial fibrations f_i with fibers isomorphic to $X(w_i)$.*

Lemma 5.6. *Let $\widehat{w} = (w_1, \dots, w_n)$ be an admissible reduced generalized decomposition of a minuscule element w of W. Let $i \in [1, n-1]$ and assume that for any pair $(\beta, \beta') \in \mathrm{Supp}(w_i) \times \mathrm{Supp}(w_{i+1})$ we have $\langle \beta^\vee, \beta' \rangle = 0$.*

Then $w_i w_{i+1} = w_{i+1} w_i$, the generalized decomposition $\widehat{w}'$ given by $w = w'_1 \cdots w'_n$ where $w'_k = w_k$ for $k \notin \{i; i+1\}$, $w'_i = w_{i+1}$ and $w'_{i+1} = w_i$ is admissible and reduced, and the morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ and $\widehat{\pi}' : \widehat{X}(\widehat{w}') \to X(w)$ are the same.

Proof. We simply have to look at the following situation. Let A and B be parabolic subgroups of a semisimple group G, and C and D be parabolic subgroups of a semisimple group G'. Assume that B and D act on a variety X and commute pairwise. Consider the variety $\overline{AuB} \times^B \overline{CvD} \times^D X$ (B acts on $\overline{CvD} \times^D X$ thanks to its action on X). It is isomorphic to $(\overline{AuB} \times \overline{CvD}) \times^{B \times D} X$ and the construction is completely symmetric.

Let us remark that the variety $\widehat{X}(\widehat{w})$ is also isomorphic to the variety $\widehat{X}(\widehat{w}'')$ where $\widehat{w}''$ is such that $w''_k = w_k$ for $k < i$, $w''_i = w_i w_{i+1}$ and $w''_k = w_{k+1}$ for $k > i+1$. ☐

Thanks to this lemma, we may assume that the support of any element w_i is connected (otherwise replace w_i by a product of elements having a connected support).

5.3 Link with the Bott–Samelson resolution

In this section, we show that the Bott–Samelson resolution $\widetilde{X}(\widetilde{w})$ of a minuscule element factors through any 'pseudo-resolution' $\widehat{X}(\widehat{w})$ constructed above. Thus, we may view $\widehat{X}$ as a projection from the Bott–Samelson resolution $\widetilde{X}(\widetilde{w})$.

Let $\widehat{w}$ be an admissible generalized reduced decomposition of w and let us fix for any $i \in [1, n]$ a (unique) reduced expression $w_i = s_{1,i} \cdots s_{r_i,i}$ denoted $\widetilde{w}_i$.

Lemma 5.7. *For any $i \in [1, n]$, the expression*

$$w = \left(\prod_{k=1}^{i} \prod_{j=1}^{r_k} s_{j,k} \right) \cdot \prod_{k=i+1}^{n} w_k$$

denoted $\widehat{w}'_i$ is an admissible generalized reduced decomposition of w.

Proof. Let us denote by w'_k for $k \in [1, N]$ the terms of the generalized decomposition. It is clear that it is reduced. Let us prove that it is admissible. Because the decomposition $\widehat{w}$ is admissible, it is clear that the inclusion $P^{w'_{N-k}} \cap G_{w'_{N-k}} \subset P_{N-k+1}$ holds for $k \leqslant i+2$. But if $k \geqslant i+1$ then w'_{N-k} is a simple reflection s_β and $G_{w'_{N-k}}$ is the semisimple subgroup of rank 1 in G containing U_β. The group $P^{w'_{N-k}} \cap G_{w'_{N-k}}$ is contained in the Borel subgroup B and *a fortiori* in P_{N-k+1}. ☐

Remark 5.8. Let us remark that the classical Bott–Samelson resolution $\widetilde{X}(\widetilde{w})$ is given by $\widehat{X}(\widehat{w}'_1)$.

Proposition 5.9. *There is a morphism $\pi_i : \widehat{X}(\widehat{w}'_i) \to \widehat{X}(\widehat{w}'_{i+1})$ for all $i \in [1,n]$ (for $i = n$, let us set $\widehat{X}(\widehat{w}'_{n+1}) := \widehat{X}(\widehat{w})$) and the morphism $\pi : \widetilde{X}(\widetilde{w}) \to X(w)$ from the Bott–Samelson resolution to the Schubert variety factors through π_i. In particular we will denote by $\widetilde{\pi}$ the morphism from $\widetilde{X}(\widetilde{w})$ to $\widehat{X}(\widehat{w})$.*

Proof. The variety $\widehat{X}(\widehat{w}'_i)$ is the quotient of the product

$$\left(\prod_{k=1}^{i}\prod_{j=1}^{r_k}\overline{(P_{s_{j,k}}\cap G_{s_{j,k}})s_{j,k}(P^{s_{j,k}}\cap G_{s_{j,k}})}\right)\times\prod_{k=i+1}^{n}\overline{(P_{w_k}\cap G_{w_k})w_k(P^{w_k}\cap G_{w_k})}$$

by the product

$$\left(\prod_{k=1}^{i}\prod_{j=1}^{r_k}P^{s_{j,k}}\cap G_{s_{j,k}}\right)\times\prod_{k=i+1}^{n}P^{w_k}\cap G_{w_k}.$$

The action respects multiplication and in particular the multiplication map on the ith factor

$$\prod_{j=1}^{r_i}\overline{(P_{s_{j,i}}\cap G_{s_{j,i}})s_{j,i}(P^{s_{j,i}}\cap G_{s_{j,i}})}\to\overline{(P_{w_i}\cap G_{w_i})w_i(P^{w_i}\cap G_{w_i})}$$

and the identity map on all the other factors is still defined modulo the action giving a map $\widehat{X}(\widehat{w}'_i) \to \widehat{X}(\widehat{w}'_{i+1})$. This map is simply the identity on all but one fibrations (the one with fiber $X(w_i)$) and on this fibration it is given by the map $\widetilde{X}(\widetilde{w}_i) \to X(w_i)$ from the Bott–Samelson resolution to the Schubert variety.

The last claim is a simple consequence of the associativity of the product: the morphism from $\widetilde{X}(\widetilde{w}) = \widehat{X}(\widehat{w}'_1)$ is given by the product of all the terms and the factorizations are given by making the product in a certain order. □

Remark 5.10. (i) The morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ is $P_{\widehat{X}(\widehat{w})}$-equivariant, and in particular if the generalized reduced decomposition $\widehat{w}$ is good, it is P_w-equivariant.

(ii) Let us explain the construction of $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X$ in terms of the quiver Q_w of w and the configuration variety. Denote by $m(Q_w)$ the set of maximal elements of Q_w for the partial order $\preccurlyeq$.

The Bott–Samelson morphism $\pi : \widetilde{X}(\widetilde{w}) \to X(w)$ is given by the projection from the configuration variety $\widetilde{X}(\widetilde{w}) \subset \prod_{i\in Q_w} G/P_{\beta_i}$ on the product $\prod_{i\in m(Q_w)} G/P_{\beta_i}$.

To give a generalized decomposition $\widetilde{w}$ of w is equivalent to giving a partition of the vertices of the quiver by subquivers $(Q_{w_i})_{i\in[1,n]}$. Then the morphism from $\widetilde{X}(\widetilde{w})$ to $\widehat{X}(\widehat{w}'_i)$ is given by the projection on the product $\prod_{k=1}^{i}\prod_{j\in Q_{w_k}} G/P_{\beta_j} \times \prod_{k=i+1}^{n}\prod_{j\in m(Q_{w_k})} G/P_{\beta_j}$.

With the notation of Remark 5.10, we generalize Fact 2.8 to the morphism $\widetilde{\pi}$. Let $\widehat{w} = (w_1, \ldots, w_n)$ be a generalized reduced decomposition of w and let $\widetilde{w}_i = (s_{\beta_{i,j}})$ be reduced expressions $w_i = s_{\beta_{i,1}} \cdots s_{\beta_{i,r_i}}$. Then the product of these expression gives a reduced expression $s_{\beta_{1,1}} \cdots s_{\beta_{n,r_n}} = s_{\beta_1} \cdots s_{\beta_r}$ of w. We denote by $p_{\widehat{w}}(Q_w)$ the set of vertices of Q_w which are peaks for the quiver Q_{w_i} to which they belong.

Corollary 5.11. *The variety Z_K is not contracted by $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to \widehat{X}(\widehat{w})$ if and only if for any $i \in [1,n]$ the part of the subword $\prod_{k\in[1,r]\setminus K} s_{\beta_k}$ corresponding to a subword of $w_i = s_{\beta_{i,1}} \cdots s_{\beta_{i,r_i}}$ is reduced.*

In particular, assume that all the w_i are minuscule elements. The divisor class group of $\widehat{X}(\widehat{w})$ has a basis given by $\widetilde{\pi}_*[Z_j]$ for $j \in p_{\widehat{w}}(Q_w)$, and $A_1(\widehat{X}(\widehat{w}))$ has a basis indexed by $[1,n]$ given by $\widetilde{\pi}_*[C_{j_i}]$ for j_i the maximal vertex of some quiver Q_{w_i} with $i \in [1,n]$.

Proof. This comes from the fiberwise description of the morphism from $\widetilde{X}(\widetilde{w})$ to $\widehat{X}(\widehat{w})$ and Fact 2.8. Recall that $\widehat{X}(\widehat{w})$ is a tower of locally trivial fibrations with fibers $X(w_i)$ for $i \in [1,n]$. Thus the result follows directly from the case of minuscule Schubert varieties. $\square$

5.4 Constructing generalized reduced decompositions

In this section, we give a way of constructing good generalized reduced decompositions of an element w into a product of minuscule elements $(w_i)_{i\in[1,n]}$.

Definition 5.12. Let $A \subset \mathrm{Peaks}(Q_w)$ be a subset of the set of peaks of Q_w. We denote by $Q_w(A)$ the full subquiver of Q_w containing the vertices i of Q_w such that $i \not\succcurlyeq j$ for all $j \in \mathrm{Peaks}(Q_w) \setminus A$.

The quiver $Q_w(A)$ is different from Q_w as soon as A is different from $\mathrm{Peaks}(Q_w)$.

Proposition 5.13. (i) *Each connected component C of the quiver $Q_w(A)$ is isomorphic to the quiver of a minuscule Schubert variety and in particular has a unique maximal element $m(C)$ for the partial order $\preccurlyeq$.*

(ii) *When A has a unique element then $Q_w(A)$ is connected.*

(iii) *The quiver $\widehat{Q}_w(A)$ obtained from Q_w by removing the vertices of $Q_w(A)$ is also the quiver of a minuscule Schubert variety.*

(iv) *The set* $\mathrm{Peaks}(Q_w(A))$ *is A and the set $p(\widehat{Q}_w(A))$ is* $\mathrm{Peaks}(Q_w) \setminus A$.

Proof. (i) (a) Let us prove that in any connected component C there is a unique maximal element for the partial order $\preccurlyeq$. Let j_1 and j_2 be two such maximal elements. By connectedness, there exists a sequence of vertices $i_0 = j_1, i_1, \ldots, i_n = j_2$ such that there is an arrow linking i_k and i_{k+1}. Let us take a minimal such sequence (that is to say, n is minimal) and let x be the smallest integer in $[0,n]$ such that $i_x \preccurlyeq i_{x-1}$ and $i_x \preccurlyeq i_{x+1}$. Such an element exists because j_1 and j_2 are maximal. By minimality of n we have $i_{x-1} \neq i_{x+1}$ and thanks to Proposition 4.1 the vertex $s(i_x)$ exists. The arrows arriving at $s(i_x)$ come from $i_{x-1} \in C$, $i_{x+1} \in C$ and maybe from a third vertex $k \preccurlyeq i_x$ (and thus $k \in C$). The vertex $s(i_x)$ has to be in C. If we replace i_x by $s(i_x)$, we get a new sequence of length n but with i_{x-1} being the first term such that $i_{x-1} \preccurlyeq i_{x-2}$ and $i_{x-1} \preccurlyeq i_x$. By induction we get a sequence of length n such that the smallest $x \in [0,n]$ with $i_x \preccurlyeq i_{x-1}$ and $i_x \preccurlyeq i_{x+1}$ is $x = 1$. This tells us that $s(i_1) \in C$ and $j_1 = i_0 \preccurlyeq s(i_1)$ in C, which is a contradiction to the maximality of j_1.

(b) Let us now prove that any connected component C of $Q(A)$ satisfies the conditions of § 4.1. Let k be a vertex of C such that $s(k)$ does not exist or is not in C.

In the first case, this means that there is at most one arrow from k, and denote by j the end vertex of this arrow. If j is not in C (or does not exist) then k is the maximal element of C. Otherwise j is in C and there is exactly one arrow from k in C.

In the second case, we have two vertices k_1 and k_2 such that the arrows arriving at $s(k)$ come from k_1, k_2 and possibly a third one $k_3 \preccurlyeq k$ which has to be in C. As $s(k) \notin C$, at least one of the two vertices k_1 and k_2 has to be out of C. If both are out of C then k is the unique maximal element of C. Otherwise exactly one vertex from $\{k_1, k_2\}$ is in C.

If k is a vertex of C such that $s(k) \in C$, then we have two vertices k_1 and k_2 such that the arrows arriving at $s(k)$ come from k_1, k_2 and eventually a third one $k_3 \preccurlyeq k$ which has to be in C. These two elements have to be in C otherwise $s(k)$ would not be in C.

(c) We are left to prove that if $m(C)$ is the maximal element of C then $\beta(m(C))$ is a simple minuscule root for some semisimple subgroup of G.

If the Dynkin diagram of G is of type A_n this is always true because any simple root is minuscule. Likewise, if the set $\beta(C)$ of simple roots is of type A_n we are done. So let us assume that $\beta(C)$

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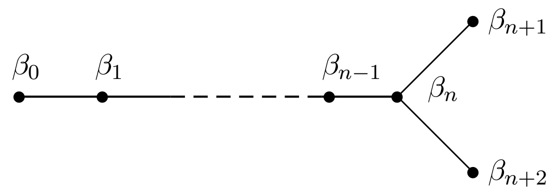
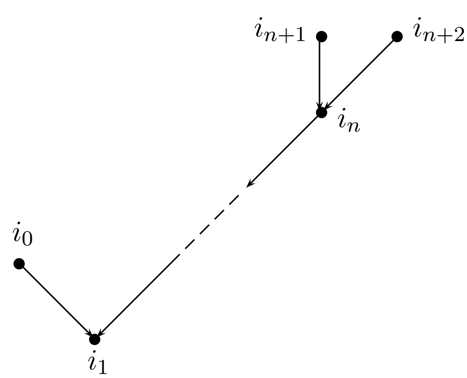
contains a trivalent root γ and a root on each branch of the Dynkin diagram (remark that because C is connected, so is $\beta(C)$).

Denote by $m(C)$ the maximal element of C and by β the simple root $\beta(m(C))$. If this root were not a minuscule root of the Dynkin diagram $\beta(C)$ (a sub-Dynkin diagram of that of G) then we would have the situation

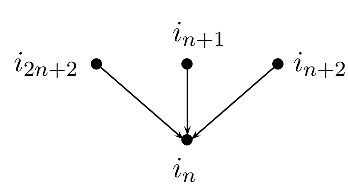


such that $\beta_1 = \beta$, $\beta_n = \gamma$ and all the simple roots β_i are in $\beta(C)$. There are two distinct simple roots β_{n+1} and β_{n+2} in $\beta(C)$ and not in $[\beta, \gamma]$ such that $\langle \gamma^\vee, \beta_{n+i} \rangle \neq 0$ for $i = 1, 2$. There is a simple root β_0 in $\beta(C)$ different from all the β_i for $i \in [1, n+2]$ such that $\langle \beta^\vee, \beta_0 \rangle \neq 0$.

Denote by i_k the biggest (for $\preccurlyeq$) vertex in C such that $\beta(i_k) = \beta_k$ for all $k \in [0, n+2]$. Then because C satisfies the properties of Proposition 4.1, we see that, in C, for all $k \in [2, n+1]$ there exists a unique arrow from i_k and it goes to i_{k-1}. In the same way there exists a unique arrow from i_{n+2} and it goes to i_n and from i_0 to $i_1 = m(C)$. This means that in C we have the following subquiver.



Now let us consider the subquiver Q' of Q corresponding to the vertices i such that $i \succcurlyeq i_0$ or $i \succcurlyeq i_0$ or $i \succcurlyeq i_0$. This is a quiver corresponding to a minuscule Schubert variety. Each time there is a hole i in the quiver, we can add a new vertex j such that $\beta(j) = \beta(i)$ to obtain a quiver which still corresponds to a minuscule Schubert variety. We can thus add a vertex i_{n+3} with $\beta(i_{n+3}) = \beta$ and, by induction, vertices i_{n+2+k} with $\beta(i_{n+2+k}) = \beta(i_k)$ for all $k \in [1, n-1]$. In this new quiver we have the following subquiver.



But we also could have chosen $\beta(i_{2n+2}) = \beta(i_n)$ proving that in the quiver Q_ϖ of the minuscule flag variety there exists a vertex j with $s(j) = i$. Then between j and i_n there would be three vertices (namely i_{n+1}, i_{n+2} and i_{2n+2}) having an arrow to i_n. This contradicts Proposition 4.1 for Q_ϖ.

(ii) If A has a unique element then all vertices of $Q_w(A)$ are bigger than this element and $Q_w(A)$ is connected.

(iii) The quiver $\widehat{Q}_w(A)$ is obtained from Q_w by removing all the vertices smaller than $m(C)$ for any connected component C of $Q_w(A)$. It is thus (see Proposition 4.5) the quiver of a minuscule Schubert variety.

(iv) This is clear from the definitions. □

To construct a partition of the quiver Q_w of a minuscule element w into quivers $(Q_{w_i})_{i\in[1,n]}$ with w_i minuscule elements, it suffices to give a partition $(A_i)_{i\in[1,n]}$ of the set $\mathrm{Peaks}(Q_w)$ of the peaks of the quiver. Indeed, given such a partition $(A_i)_{i\in[1,n]}$, we define by induction a sequence $(Q_i)_{i\in[0,n]}$ of quivers with $Q_0 = Q$ and $Q_{i+1} = \widehat{Q}_i(A_{i+1})$ $Q_{w_1} = Q_w(A_1)$. We then denote by Q_{w_i} the quiver $Q_{i-1}(A_i)$. The quivers $(Q_{w_i})_{i\in[1,n]}$ form a partition of Q_w. Each quiver Q_{w_i} is associated to a minuscule element w_i.

Remark 5.14. (i) For such partitions (giving a reduced generalized decomposition $\widehat{w}$), we have $p_{\widehat{w}}(Q_w) = \mathrm{Peaks}(Q_w)$.

(ii) The vertices of Q_{w_i} are the vertices x of Q_w such that there exists a peak $p \in A_i$ with $p \preccurlyeq x$ and $p' \not\preccurlyeq x$ for any peak p' in A_j with $j > i$.

For such partitions $(Q_{w_i})_{i\in[1,n]}$ of the quiver Q_w coming from partitions $(A_i)_{i\in[1,n]}$ of $\mathrm{Peaks}(Q_w)$ we have a reduced generalized decomposition $w = w_1 \cdots w_n$ denoted $\widehat{w}$.

Proposition 5.15. *The reduced generalized decomposition $\widehat{w}$ is good.*

Proof. We have to prove that the inclusions $\Sigma(P_{w_{i+1}\cdots w_n}) \subset \Sigma(P^{w_i} \cap G_{w_i})$ and $\partial\mathrm{Supp}(w_i) \subset \Sigma(P_{w_i\cdots w_n})$ hold. But the set $\Sigma(P_{w_i\cdots w_n})$ is the set $\beta(\{i \in Q_w \mid i \text{ is a hole of } Q_{w_i\cdots w_n}\})$ where $Q_{w_i\cdots w_n}$ is the subquiver of Q_w whose vertices are in $\bigcup_{k\geqslant i} Q_{w_k}$ (see Proposition 4.11).

The set $\Sigma(P^{w_i} \cap G_{w_i})$ is the set $\beta(m_i) \cup \mathrm{Supp}(w_i)^c$ where m_i is the maximal vertex of Q_{w_i}. So for the first inclusion we only have to prove that for any simple root $\beta \in \mathrm{Supp}(w_i)\cap\Sigma(P_{w_{i+1}\cdots w_n})$ we have $\beta = \beta(m_i)$. But as $\beta \in \mathrm{Supp}(w_i)$, there exists $j \in Q_{w_i}$ such that $\beta(j) = \beta$. Let j be the biggest such vertex. If j were not the biggest element m_i in Q_{w_i} then there would exist in Q_{w_i} an element k with an arrow from j to k. But then we distinguish between the two cases: $s(j)$ exists or not. If $s(j)$ exists, it is a hole of $Q_{w_{i+1}\cdots w_n}$. There are two vertices k_1 and k_2 having an arrow to $s(j)$. Between j and $s(j)$ there are three vertices k, k_1 and k_2. This is impossible thanks to Proposition 4.1. If $s(j)$ does not exist, then j is a virtual hole of $Q_{w_{i+1}\cdots w_n}$ and there is a vertex k' such that $\langle\beta(k'), \beta(j)\rangle \neq 0$. So $s(j)$ does not exist but there are two vertices k and k' having an arrow coming from j. This is impossible thanks to Proposition 4.1.

For the second inclusion, let β be a simple root in $\partial\mathrm{Supp}(w_i)$; then there exists a vertex $j \in Q_{w_i}$ with $\langle\beta(j), \beta\rangle \neq 0$. If β is not in the support of $w_i \cdots w_n$ then β is the simple root of a virtual hole and $\beta \in \Sigma(P_{w_i\cdots w_n})$. If β is in this support then there exists a vertex k such that $\beta(k) = \beta$. Let k be the smallest such vertex. We have an arrow from j to k, thus k is not a peak of $Q_{w_i\cdots w_n}$ and thus not a peak of $Q_{w_{i+1}\cdots w_n}$ (see the previous proposition). In particular there exists a vertex $x \in Q_{w_i\cdots w_n}$ with an arrow from x to k. But then k is the smallest vertex with $\beta(k) = \beta$ in $Q_{w_i\cdots w_n}$ and there are two arrows arriving at k. Thus k is a hole of $Q_{w_i\cdots w_n}$ and we are done. □

We now give here three types of partitions of Q_w constructed in this way.

Construction 1. Choose any order $\{i_1, \ldots, i_n\}$ on the set $\mathrm{Peaks}(Q_w)$ of the peaks of Q_w and set $A_k = \{i_k\}$.

Construction 2. Define a partition $(A_i)_{i\in[1,n]}$ by induction: A_1 is the set of peaks with minimal height and A_{i+1} is the set of peaks in $\mathrm{Peaks}(Q_w) \setminus \bigcup_{k=1}^{i} A_k$ with minimal height.

Before giving the last construction let us fix some notation and prove the following proposition. Recall that $p_{\widehat{w}}(Q_w) = \mathrm{Peaks}(Q_w)$ (with these constructions) is the set of all vertices j of Q_w such that there exists an integer $i \in [1, n]$ with $j \in \mathrm{Peaks}(Q_{w_i})$. Let us denote by $m_{\widehat{w}}(Q_w)$ the set of vertices j of Q_w such that j is a maximal element of Q_{w_i} for some $i \in [1, n]$.

The partial order $\preccurlyeq$ induces a partial order on $m_{\widetilde{w}}(Q_w)$. Let us finally prove the following result.

Proposition 5.16. *Let $i \in m_{\widehat{w}}(Q_w)$. Then there exists a unique minimal element $f(i)$ in $m_{\widehat{w}}(Q_w)$ for $\preccurlyeq$ such that $i \prec f(i)$ (i.e. $i \preccurlyeq f(i)$ and $i \neq f(i)$).*

Proof. Let us first prove the following lemma.

Lemma 5.17. *Let j and k be in $m_{\widehat{w}}(Q_w)$ such that there exists $x \in Q_w$ with $x \preccurlyeq j$ and $x \preccurlyeq k$. Then we have either $j \preccurlyeq k$ or $k \preccurlyeq j$.*

Proof. We proceed by induction on $a+b$ where a and b are the indices in $[1, n]$ such that $j \in Q_{w_a}$ and $k \in Q_{w_b}$. Let x be a maximal element (for $\preccurlyeq$) such that $x \preccurlyeq j$ and $x \preccurlyeq k$ and suppose that x is different from j and k.

If there is a unique arrow from x say going to a vertex y, then we must have $y \preccurlyeq j$ and $y \preccurlyeq k$ contradicting the maximality. Let y_1 and y_2 be the two target vertices of the two arrows from x. If $\beta(y_1) = \beta(y_2)$ then $y_1 \preccurlyeq y_2$ (or the converse) and we have $y_1 \preccurlyeq j$ and $y_1 \preccurlyeq k$ contradicting the maximality. We thus have $\beta(y_1) \neq \beta(y_2)$ and $y_1 \preccurlyeq j$ but $y_1 \not\preccurlyeq k$ and $y_2 \not\preccurlyeq j$ but $y_2 \preccurlyeq k$. This also implies that $s(x)$ exists because y_1 and y_2 are connected to the biggest element r of the quiver and so the segments $[\beta(y_1), \beta(r)]$ and $[\beta(y_1), \beta(r)]$ are contained in the set $\beta(\{z \in Q_w \mid z \succcurlyeq x, z \neq x\})$ and thus $\beta(x)$ is in this set. So $s(x)$ exists and we have $s(x) \not\preccurlyeq j$ and $s(x) \not\preccurlyeq k$.

Now let c, d and e be in $[1, n]$ such that $s(x) \in Q_{w_c}$, $j_1 \in Q_{w_d}$ and $j_2 \in Q_{w_e}$. We must have $c \geqslant d$ and $c \geqslant e$ because $s(x) \succcurlyeq j_1, j_2$. We must also have $a \geqslant d$ and $b \geqslant e$. But if p is a peak in A_c such that $s(x) \succcurlyeq p$, we must have $p \preccurlyeq j_1$ or $p \preccurlyeq j_2$ which implies (see Remark 5.14) that $d \geqslant c$ or $e \geqslant c$. We thus have $c = d$ or $c = e$. Assume for example that $c = d$ and denote by m the maximal element of Q_{w_c}. If $c = d = a$ then $j_2 \preccurlyeq s(x) \preccurlyeq m = j$ and $j_2 \preccurlyeq k$, a contradiction to the maximality of x. So $c = d < a$, but we have $x \preccurlyeq m$ and $x \preccurlyeq j$ and by induction we must have $m \preccurlyeq j$. Then we have $j_2 \preccurlyeq s(x) \preccurlyeq m \preccurlyeq j$ and $j_2 \preccurlyeq k$, again a contradiction to the maximality of x. This completes the proof of Lemma 5.17. ◻

The preceding lemma proves that for $i \in m_{\widehat{w}}(Q_w)$ (and even for any $i \in Q_w$) the set $\{j \in m_{\widehat{w}}(Q_w) \mid j \succcurlyeq i\}$ is totally ordered and thus there exists a minimal element $f(i)$. This completes the proof of Proposition 5.16. ◻

We can now give the last construction which is a particular case of Construction 1. Because in Construction 1 there is a bijection between $\mathrm{Peaks}(Q_w)$ and $m_{\widehat{w}}(Q_w)$ we define thanks to the preceding proposition the function f on the set $\mathrm{Peaks}(Q_w)$ simply by the following: if $p \in \mathrm{Peaks}(Q_w)$ is in Q_{w_i} and if $m \in m_{\widehat{w}}(Q_w)$ is the maximal element of Q_{w_i} then $f(m)$ is the maximal element of some Q_{w_j}. There is a unique peak q in Q_{w_j} and we define $f(p) = q$.

Construction 3. An ordering $\{i_1, \ldots, i_n\}$ on the set $\mathrm{Peaks}(Q_w)$ of peaks of Q_w will be called neat (by analogy with [Zel83]) if for all $k \in [1, n-1]$ we have $h(i_k) \leqslant h(f(i_k))$. In this case we set $A_k = \{i_k\}$.

Choosing a neat ordering is equivalent to choosing an ordering $\{i_1, \ldots, i_n\}$ on the set $\mathrm{Peaks}(Q_w)$ of the peaks of Q_w such that if i_k and i_{k+1} are adjacent in the quiver then $h(i_k) \leqslant h(i_{k+1})$.

These constructions may produce non-connected subquivers Q_{w_i} but thanks to Lemma 5.6 we may assume (replacing these quivers by their connected components) that all the quivers Q_{w_i} are quivers of minuscule Schubert varieties.

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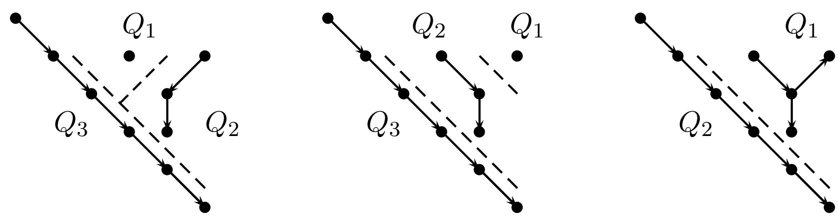
Example 5.18. Let $X(w)$ and Q_w be as in Example 2.5 and keep the notation of Example 4.8. There are six different orders on the set of peaks $\{p_1, p_2, p_3\}$ of Q_w. Only two of them are neat: (p_2, p_3, p_1) and (p_3, p_2, p_1). The construction of Sankaran and Vanchinathan in [SV94] is equivalent to choosing p_3 as last peak. In this example, this choice never gives a neat ordering and in particular (we will see why in the next section) they do not obtain a small resolution for $X(w)$.

Let us describe the varieties $\widehat{X}(\widehat{w}_1)$ and $\widehat{X}(\widehat{w}_2)$ obtained from the neat orders (p_2, p_3, p_1) and (p_3, p_2, p_1) and the variety $\widehat{X}_{\mathrm{can}}(w)$ obtained from Construction 2. The partitions of the quivers are the following.



We have:

$$\widehat{X}(\widehat{w}_1) = \left\{ \begin{array}{l} (V_5, V_8', V_8) \in \mathbb{G}_{\mathrm{iso}}(5,16) \times \mathbb{G}^2_{\mathrm{iso}}(8,16) \times \mathbb{G}^1_{\mathrm{iso}}(8,16) \mid \\ F_4 \subset V_5 \subset F_6, V_5 \subset V_8', F_1 \subset V_8 \text{ and } \dim(V_8' \cap V_8) = 7 \end{array} \right\},$$

$$\widehat{X}(\widehat{w}_2) = \left\{ \begin{array}{l} (W_8, V_8', V_8) \in \mathbb{G}^1_{\mathrm{iso}}(8,16) \times \mathbb{G}^2_{\mathrm{iso}}(8,16) \times \mathbb{G}^1_{\mathrm{iso}}(8,16) \mid \\ F_6 \subset W_8,\ F_4 \subset V_8', \dim(V_8' \cap W_8) = 7, F_1 \subset V_8 \text{ and} \dim(V_8' \cap V_8) = 7 \end{array} \right\},$$

$$\widehat{X}_{\mathrm{can}}(\widehat{w}) = \left\{ \begin{array}{l} (V_8', V_8) \in \mathbb{G}_{\mathrm{iso}}(5,16) \times \mathbb{G}^2_{\mathrm{iso}}(8,16) \times \mathbb{G}^1_{\mathrm{iso}}(8,16) \mid \\ F_4 \subset V_8', \dim(V_8' \cap F_6) \geqslant 5, F_1 \subset V_8 \text{ and} \dim(V_8' \cap V_8) = 7 \end{array} \right\},$$

where $G^i_{\mathrm{iso}}(8,16)$ for $i \in \{1,2\}$ are the two connected components of the Grassmannian of maximal isotropic subspaces. The projection of any of these three varieties to $X(w)$ is given by projection on the last factor V_8.

6. Relative Mori theory of minuscule Schubert varieties

In this section we describe all relative canonical and minimal models of a minuscule Schubert variety $X(w)$. We only consider generalized reduced decompositions $\widehat{w}$ of w obtained via one of the three constructions of the preceding section. Construction 3 will give all relative minimal models of $X(w)$ and Construction 2 will give the relative canonical model of $X(w)$.

6.1 Ample divisors and effective curves

Recall that we described in Corollary 5.11 a basis of divisors and 1-cycles on $\widehat{X}(\widehat{w})$ in the following way. Let $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to \widehat{X}(\widehat{w})$ be the morphism from the Bott–Samelson resolution to the partial resolution $\widehat{X}(\widehat{w})$. The group $A^1(\widehat{X}(\widehat{w}))$ has a basis given by $D_i = \widetilde{\pi}_*[Z_i]$ for $i \in p_{\widehat{w}}(Q_w) = \mathrm{Peaks}(Q_w)$ and the group $A_1(\widehat{X}(\widehat{w}))$ has a basis given by $\widetilde{\pi}_*[C_i]$ for $i \in m_{\widehat{w}}(Q_w)$. More generally, the cellular decomposition of $\widetilde{X}(\widetilde{w})$ (cf. for example [Wil04]) will induce a cellular decomposition on $\widehat{X}(\widehat{w})$ so that the Chow groups of $\widehat{X}(\widehat{w})$ are generated by classes of B-stable subvarieties and are free over $\mathbb{Z}$. Recall also that $\widehat{X}(\widehat{w})$ is a tower of locally trivial fibrations with fibers Schubert varieties so that the Picard group is free and dual to the group of 1-cycles.

We have seen in Remark 5.10 that the morphism $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to \widehat{X}(\widehat{w})$ is the projection from $\widetilde{X}(\widetilde{w})$ to the product $\prod_{i \in m_{\widehat{w}}(Q_w)} G/P_i$. We have a projection $p_i : \widehat{X}(\widehat{w}) \to G/P_i$ for all $i \in m_{\widehat{w}}(Q_w)$. Let us define the invertible sheaf $\mathcal{M}_i = p_i^*(\mathcal{O}_{G/P_i}(1))$ for all $i \in m_{\widehat{w}}(Q_w)$. We have $\mathcal{L}_i = \widetilde{\pi}^* \mathcal{M}_i$

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for all $i \in m_{\widehat{w}}(Q_w)$. Because of the description of $\widehat{X}(\widehat{w})$ as a tower of locally trivial fibrations with fibers isomorphic to minuscule Schubert varieties $X(w_i)$, we have the following fact.

Fact 6.1. *The family $(\mathcal{M}_i)_{i\in m_{\widehat{w}}(Q_w)}$ is a basis of* $\mathrm{Pic}(\widehat{X}(\widehat{w}))$.

Recall that we gave a basis $([Y_i])_{i\in[1,r]}$ of the monoid of effective 1-cycles of $\widetilde{X}(\widetilde{w})$ in § 2.3.2. Because $[Y_i] = [C_i] - [C_{s(i)}]$ we see that we have the following fact.

Fact 6.2. *The family $(\widetilde{\pi}_*[Y_i])_{i\in m_{\widehat{w}}(Q_w)}$ is a basis of $A_1(\widehat{X}(\widehat{w}))$ dual to the basis $(\mathcal{M}_i)_{i\in m_{\widehat{w}}(Q_w)}$.*

Proof. The pull-back of $\mathcal{M}_i$ by $\widetilde{\pi}$ is $\mathcal{L}_i$ and we have $\mathcal{L}_i \cdot [Y_j] = \delta_{i,j}$ so by the projection formula we get the result. □

Proposition 6.3. *The family $([\widetilde{\pi}_*Y_i])_{i\in m_{\widehat{w}}(Q_w)}$ is a basis of the cone of classes of effective 1-cycles and the family $(\mathcal{M}_i)_{i\in m_{\widehat{w}}(Q_w)}$ is a basis of the closure of the ample cone.*

Proof. The embedding of $\widehat{X}(\widehat{w})$ in $\prod_{i\in m_{\widehat{w}}(Q_w)} G/P_i$ is given by $\bigotimes_{i\in m_{\widehat{w}}(Q_w)} \mathcal{M}_i$. The cone generated by the $\mathcal{M}_i$ is thus contained in the closure of the ample cone.

Conversely, let A be an ample divisor and let $a_i = A \cdot [\widetilde{\pi}_*Y_i]$ for $i \in m_{\widehat{w}}(Q_w)$. We must have $a_i > 0$ and the divisor $A - \sum_i a_i\mathcal{M}_i$ is numerically trivial and we get the result.

By duality we have the result on curves. □

Proposition 4.14 has the following relative version.

Proposition 6.4. *We have the formula*

$$\mathcal{M}_i = \sum_{k\in\mathrm{Peaks}(Q_w),k\preccurlyeq i} D_k.$$

Proof. Because the variety $\widehat{X}(\widehat{w})$ is normal, we have $\mathcal{M}_i = \widetilde{\pi}_*\mathcal{L}_i$ and we obtain the formula in the same way as in Proposition 4.14 thanks to the fact that all quivers Q_{w_i} are associated to minuscule Schubert varieties. □

Likewise, we may generalize Corollary 4.15 as follows.

Corollary 6.5. *The variety $\widehat{X}(\widehat{w})$ is locally factorial if and only if for all $i \in [1, n]$ the quiver Q_{w_i} has a unique peak.*

6.2 Canonical divisor of $\widehat{X}(\widehat{w})$

As in Proposition 4.17, we have $K_{\widehat{X}(\widehat{w})} = \widetilde{\pi}_*K_{\widetilde{X}(\widetilde{w})}$. The same argument gives the following fact.

Fact 6.6. *We have*

$$-K_{\widehat{X}(\widehat{w})} = \sum_{k\in\mathrm{Peaks}(Q_w)} (h(k) + 1)D_k.$$

For the three constructions, the peaks of a fixed quiver Q_{w_i} all have the same height so we can define $h(w_i)$ to be the height of any peak of Q_{w_i}. Set $h(w_{n+1}) = -1$. Proposition 6.4 gives us the following (by induction on the ordering of $m_{\widehat{w}}(Q_w)$).

Corollary 6.7. *We have the formula*

$$-K_{\widehat{X}(\widehat{w})} = \sum_{i\in m_{\widehat{w}}(Q_w)} (h(w_i) - h(w_{f(i)}))\mathcal{M}_i$$

and in particular $\widehat{X}(\widehat{w})$ is Gorenstein.

6.3 Types of singularities

In this section we are going to prove that the variety $\widehat{X}(\widehat{w})$ has terminal singularities in the case of Constructions 1, 2 and 3. For the definition of terminal and canonical singularities, see [Mat02].

For this we use the resolution $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to \widehat{X}(\widehat{w})$ and compare the canonical divisor $K_{\widetilde{X}(\widetilde{w})}$ to the pull-back of the canonical divisor $K_{\widehat{X}(\widehat{w})}$. We need the following fact coming directly from the formula of § 2.3.1 and Lemma 4.18.

Fact 6.8. *We have the formula*

$$-K_{\widetilde{X}(\widetilde{w})} = \sum_{i=1}^{r} (h(i)+1)\xi_i.$$

We can now prove the following proposition.

Proposition 6.9. *The variety $\widehat{X}(\widehat{w})$ has terminal (and hence canonical) singularities.*

Proof. By Corollary 6.7, we have

$$-\widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i \in m_{\widehat{w}}(Q_w)} (h(w_i) - h(w_{f(i)}))\mathcal{L}_i.$$

But thanks to Proposition 2.16 and the fact that the quivers Q_{w_i} are the quivers of minuscule Schubert varieties we have

$$\mathcal{L}_i = \sum_{k \preccurlyeq i} \xi_k,$$

giving

$$-\widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i=1}^{n} \biggl((h(w_i)+1) \sum_{k \in Q_{w_i}} \xi_k \biggr).$$

We get for the difference

$$K_{\widetilde{X}(\widetilde{w})} - \widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i=1}^{n} \sum_{k \in Q_{w_i}} (h(w_i) - h(k))\xi_k.$$

But $h(w_i)$ is the biggest height of an element in Q_{w_i} so $h(w_i) - h(k) \geqslant 0$ with equality if and only if k is a peak of the quiver, that is to say, if and only if ξ_i is not contracted by $\widetilde{\pi}$. □

6.4 Description of the relative minimal and canonical models

We can now prove our results on relative minimal and canonical models of minuscule Schubert varieties. Recall from [Mat02] that a variety $\pi : Y \to X$ is a relative minimal (respectively *the* relative canonical) model of X if Y has terminal singularities and K_Y is numerically effective (strictly numerically effective) on the fibers of π, that is to say, if for any curve C contracted by π one has $[K_Y] \cdot [C] \geqslant 0$ (respectively $[K_Y] \cdot [C] > 0$).

Theorem 6.10. (i) *The varieties $\widehat{X}(\widehat{w})$ obtained from a neat ordering (Construction 3) are relative minimal models of $X(w)$.*

(ii) *The variety $\widehat{X}(\widehat{w})$ obtained from Construction 2 is the relative canonical model of $X(w)$.*

Proof. (i) We have to prove that any curve C contracted by $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ satisfies $[C] \cdot K_{\widehat{X}(\widehat{w})} \geqslant 0$. The class $[C]$ can be written as $\sum_i a_i[\widetilde{\pi}_* Y_i]$ with $a_i \geqslant 0$ and $a_n = 0$ (because the curve is contracted). We just have to prove the non-negativity of the intersections $K_{\widehat{X}(\widehat{w})} \cdot [\widetilde{\pi}_* Y_j]$ for $j \in [1, n-1]$. We have $K_{\widehat{X}(\widehat{w})} \cdot [\widetilde{\pi}_* Y_j] = h(w_{f(j)}) - h(w_j)$ and by Construction 3 this intersection is non-negative.

(ii) It suffices to prove that the contracted curves have a positive intersection with the canonical divisor and this comes from the argument of part (i) and Construction 2. □

To study relations between relative minimal models, we need to study Mori contractions. They arise in the following setting: let $\pi : Y \to X$ be a projective morphism from a variety Y with terminal singularities. The subcone $\mathfrak{C}(Y)$ of the cone $NE(Y) \otimes_{\mathbb{Z}} \mathbb{R}$ of effective 1-cycles with real coefficients modulo numerical equivalence consisting of classes $[C]$ such that $[K_Y] \cdot [C] < 0$ is locally polyhedral (see [Mat02]). An edge of this subcone is called an extremal ray and defines a morphism $Y \to Z$ (a Mori contraction) factorizing π if any class in that ray is contracted by π. If the morphism $Y \to Z$ is small in the sense of Mori theory (i.e. has an exceptional set in codimension at least 2) then Z has non-terminal singularities but one conjectures that a relative flip or a relative flop of this morphism exists (see [Mat02] for a definition) giving rise to a new variety $\widetilde{Y} \to X$ with terminal singularities. Iterating this process should converge to relative minimal models.

In our situation the whole relative Mori program works perfectly. Let $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ be a variety obtained from one of Constructions 1, 2 or 3. The cone $NE(\widehat{X}(\widehat{w}))$ is simplicial and we will explicitly describe the extremal rays contracted by $\widehat{\pi}$. Moreover, we will describe all relative flips and relative flops of the morphism $\widehat{\pi}$ and prove the existence and termination of flips and flops (see [Mat02] for a definition of flips, flops and termination of flips and flops).

Fact 6.11. (i) *The extremal rays of $\widehat{X}(\widehat{w})$ are given by the classes $[\widetilde{\pi}_* Y_j]$ such that $h(w_{f(j)}) < h(w_j)$.*

(ii) *If $\widehat{X}(\widehat{w})$ is obtained from Construction 3, then there is no extremal ray. However, if D is any effective divisor, then the $(K_{\widehat{X}(\widehat{w})} + D)$-extremal rays are given by the classes $[\widetilde{\pi}_* Y_j]$ such that $(K_{\widehat{X}(\widehat{w})} + D) \cdot [\widetilde{\pi}_* Y_j] < 0$.*

Proof. (i) Let $[C]$ be the class of an effective curve. Then there exist non-negative integers a_i such that $[C] = \sum_i a_i [\widetilde{\pi}_* Y_i]$. Denote by μ_j (respectively ν_k and ω_l) the classes $[\widetilde{\pi}_* Y_i]$ such that $K_{\widehat{X}(\widehat{w})} \cdot [\widetilde{\pi}_* Y_i] < 0$ (respectively > 0 and $= 0$). For each j and k there is a linear combination with positive coefficients $x_{j,k}\mu_j + y_{j,k}\nu_k$ such that $K_{\widehat{X}(\widehat{w})} \cdot (x_{j,k}\mu_j + y_{j,k}\nu_k) = 0$. It is easy to check that if $K_{\widehat{X}(\widehat{w})} \cdot [C] < 0$ then $[C]$ has to be a linear combination with non-negative coefficients of classes (μ_j), $(x_{j,k}\mu_j + y_{j,k}\nu_k)$ and (ω_l), proving the result.

(ii) The same proof with $K_{\widehat{X}(\widehat{w})} + D$ instead of $K_{\widehat{X}(\widehat{w})}$ works. □

Let us consider a fixed ordering $(p_1, \dots, p_n)$ on the peaks of the quiver Q_w and let $\widehat{w}$ be the good reduced generalized decomposition it induces (cf. Constructions 1, 2 and 3). Let us denote by $\widehat{w}'$ the good reduced generalized decomposition induced by the ordering $(q_1, \dots, q_n)$ on the peaks where $q_k = p_k$ for $k \notin \{i, i+1\}$, $q_i = p_{i+1}$ and $q_{i+1} = p_i$. Denote by $\widehat{w}''$ the good reduced generalized decomposition given by $w''_k = w_k$ for $k < i$, $w''_i = w_i w_{i+1}$ and $w_k = w_{k+1}$ for $k > i$, that is to say, obtained by the partition $(A_k)_{k \in [1, n-1]}$ of $\mathrm{Peaks}(Q_w)$ given by $A_k = \{p_k\}$ for $k < i$, $A_i = \{p_i, p_{i+1}\}$ and $A_k = \{p_{k+1}\}$ for $k > i$. Denote by k_i (respectively k_{i+1}) the maximal vertex of Q_{w_i} (respectively $Q_{w_{i+1}}$).

We have morphisms (see for example Remark 5.10)

$$f : \widehat{X}(\widehat{w}) \to \widehat{X}(\widehat{w}'') \quad \text{and} \quad f' : \widehat{X}(\widehat{w}') \to \widehat{X}(\widehat{w}'').$$

Proposition 6.12. *Let $\widehat{X}(\widehat{w})$ be as obtained from Construction 1, 2 or 3.*

(i) *If $k_i \not\preccurlyeq k_{i+1}$ (i.e. if $i+1 \neq f(i)$) then the morphisms f and f' are isomorphisms.*

(ii) *If $f(i) = i+1$ and $\widetilde{\pi}_*[Y_{k_i}] \cdot K_X > 0$, then f (respectively f') is the small contraction corresponding to the extremal ray $\mathbb{R}^{\geqslant 0}\widetilde{\pi}_*[Y_{k_i}]$ (respectively $\mathbb{R}^{\geqslant 0}\widetilde{\pi}_*[Y_{k'_i}]$) and f is the flip of f'.*

(iii) *If $f(i) = i+1$ and $\widetilde{\pi}_*[Y_{k_i}] \cdot K_X = 0$, denote $D = D_{i+1}$ (respectively $D' = D'_i$). Then for $\varepsilon > 0$, the class $\widetilde{\pi}_*[Y_{k_i}]$ (respectively $\widetilde{\pi}_*[Y_{k'_i}]$) is extremal for $K_{\widehat{X}(\widehat{w})} + \varepsilon D$ (respectively $K_{\widehat{X}(\widehat{w}')} + \varepsilon D'$) and f*

(respectively f') is the small contraction corresponding to the extremal ray $\mathbb{R}^{\geqslant 0}\widetilde{\pi}_*[Y_{k_i}]$ (respectively $\mathbb{R}^{\geqslant 0}\widetilde{\pi}_*[Y_{k'_i}]$). The morphism (f', D') is the flop of (f, D).

Proof. (i) Because k_i and k_{i+1} are not comparable for $\preccurlyeq$, we have thanks to Lemma 5.17 that w_i and w_{i+1} satisfy the hypothesis of Lemma 5.6 and we have the result.

In the cases where $f(i) = i+1$, we already know that $\widehat{X}(\widehat{w})$ and $\widehat{X}(\widehat{w}')$ are locally factorial with terminal singularities. We also know that $\widehat{X}(\widehat{w}'')$ is normal and the morphisms are birational and Mori-small (the group of Weil divisors has a basis given by the peaks). Because of our description of Picard groups we also have $\rho(\widehat{X}(\widehat{w})/\widehat{X}(\widehat{w}'')) = \rho(\widehat{X}(\widehat{w}')/\widehat{X}(\widehat{w}'')) = 1$ where if $X \to Y$ is a morphism, the integer $\rho(X/Y)$ is the relative Picard number.

(ii) We are left to study the divisors $K_{\widehat{X}(\widehat{w})}$ and $K_{\widehat{X}(\widehat{w}')}$ on the fibers of f and f'. We will not describe these fibers in detail in this proposition, but more details will be given in the next section. Because all the sheaves $\mathcal{M}_{k_j}$ for $j \neq i$ are already defined on $\widehat{X}(\widehat{w}'')$ they are trivial on the fibers of f and the sheaf $\mathcal{M}_{k_i}$ on $\widehat{X}(\widehat{w})$ is relatively ample with respect to f. The restriction of $K_{\widehat{X}(\widehat{w})}$ to the fibers of f is given by $(K_{\widehat{X}(\widehat{w})} \cdot \widetilde{\pi}_*[Y_{k_i}])\mathcal{M}_{k_i}$ so that $K_{\widehat{X}(\widehat{w})}$ is ample, anti-ample or trivial according to the positivity of the intersection $K_{\widehat{X}(\widehat{w})} \cdot \widetilde{\pi}_*[Y_{k_i}]$ and thus according to the height of the peaks. This proves in case (ii) that $K_{\widehat{X}(\widehat{w})}$ is f-ample.

Furthermore, the fibers of f are contained in $G/P_{\beta(k_i)}$ and thus the classes of contracted curves are proportional to $\widetilde{\pi}_*[Y_{k_i}]$.

In the same way we get that $-K_{\widehat{X}(\widehat{w}')}$ is f'-ample and the result.

(iii) Proposition 6.4 tells us that $D = D_{i+1}$ satisfies $\widetilde{\pi}_*[Y_{k_i}] \cdot D < 0$ so the class $\widetilde{\pi}_*[Y_{k_i}]$ is extremal for $K_{\widehat{X}(\widehat{w})} + \varepsilon D$.

The Bott–Samelson variety $\widetilde{X}(\widetilde{w})$ is a resolution of the birational morphism between $\widehat{X}(\widehat{w})$ and $\widehat{X}(\widehat{w}')$ and D is the image of $\xi_{p_{i+1}}$ whose image in $\widehat{X}(\widehat{w}')$ is D' so that D' is the strict transform of D.

We are left to study the divisors $K_{\widehat{X}(\widehat{w})}$, $K_{\widehat{X}(\widehat{w}')}$, D and D' on the fibers of the morphisms f and f'. The computation in case (i) proves the triviality of $K_{\widehat{X}(\widehat{w})}$ and $K_{\widehat{X}(\widehat{w}')}$ on the fibers. For D and D', the same argument as for the canonical sheaves proves that their restriction to the fibers of f (respectively f') is a positive multiple of $-\mathcal{M}_{k_i}$ (respectively $\mathcal{M}_{k'_i}$), concluding the proof. □

Remark 6.13. If $f(i) = i+1$ and $[Y_{k_i}] \cdot K_{\widehat{X}(\widehat{w})} < 0$ then by symmetry f' is the flip of f.

COROLLARY 6.14. (i) *The varieties $\widehat{X}(\widehat{w})$ obtained from Construction 1 are linked by flips and flops and any variety obtained from $\widehat{X}(\widehat{w})$ by flips and flops comes from this construction.*

(ii) *The varieties $\widehat{X}(\widehat{w})$ obtained from Construction 3 are linked by flops and any variety obtained from $\widehat{X}(\widehat{w})$ by flops comes from this construction.*

Proof. (i) We know that the extremal rays (or more generally the $(K_{\widehat{X}(\widehat{w})} + D)$-extremal rays) of $\widehat{X}(\widehat{w})$ are generated by the classes $\widetilde{\pi}_*[Y_{k_i}]$ with $K_{\widehat{X}(\widehat{w})} \cdot \widetilde{\pi}_*[Y_{k_i}] < 0$ (respectively $(K_{\widehat{X}(\widehat{w})} + D) \cdot \widetilde{\pi}_*[Y_{k_i}] < 0$). But the associated flip or flop gives a variety obtained by Construction 1.

(ii) The same argument works in this case because $h(w_{f(i)}) = h(w_i)$ and we stay in the class of varieties obtained from Construction 3. □

Remark 6.15. For any variety obtained from Construction 1 we have proved the existence and termination of flips and flops (in the sense of Matsuki [Mat02]).

COROLLARY 6.16. *The relative minimal models of $X(w)$ are exactly the varieties obtained from Construction 3.*

Proof. We use Theorem 12-1-8 of [Mat02], the fact that varieties obtained from Construction 3 are relative minimal models, and the existence and termination of flops for these varieties. ☐

Example 6.17. Let $X(w)$ and Q_w be as in Example 2.5 and keep the notation of Examples 4.8 and 5.18. The minimal models of $X(w)$ are $\widehat{X}(\widehat{w}_1)$ and $\widehat{X}(\widehat{w}_2)$. They are linked by a flop. The canonical model of $X(w)$ is $\widehat{X}_{\mathrm{can}}(w)$.

7. Small *IH*-resolutions of minuscule Schubert varieties

In this section we prove that the morphisms $\widehat{X}(\widehat{w}) \to X(w)$ obtained from Construction 3 are *IH*-small. We then discuss the smoothness of $\widehat{X}(\widehat{w})$ and describe all *IH*-small resolutions of minuscule Schubert varieties. Let us first recall the definition of an *IH*-small morphism.

Definition 7.1. A proper birational morphism $\pi : Y \to X$ is said to be *IH-small* if for all $k > 0$, we have the inequality

$$\mathrm{codim}_X\{x \in X \mid \dim(\pi^{-1}(x)) = k\} > 2k.$$

An *IH*-small morphism $\pi : Y \to X$ is an *IH*-small resolution of X if Y is smooth.

In this section we will use a case-by-case analysis.

7.1 Necessary condition

Let us first prove the following proposition showing that among the morphisms $\pi : \widehat{X}(\widehat{w}) \to X(w)$ obtained from Construction 1 (or from an ordering on the peaks) only the ones coming from Construction 3 (or from neat orderings) can be small.

Proposition 7.2. *Let $\widehat{X}(\widehat{w})$ be as obtained from Construction 1 but not from Construction 3. Then the morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ is not IH-small.*

Proof. If $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ were small then, because $\widehat{X}(\widehat{w})$ has terminal singularities, it would be a relative minimal model (this is explained in the proof of [Tot00, Proposition 8.3]). This is not the case by Corollary 6.16.

One can give an explicit subvariety in $X(w)$ not satisfying the *IH*-small condition for $\widehat{\pi}$. Let i be a peak of Q_w such that $h(f(i)) < h(i)$ (such a peak exists because the resolution is not obtained from Construction 3). Let us consider the smallest (for $\preccurlyeq$) vertex $j \in Q_w$ such that $j \succcurlyeq i$ and $j \succcurlyeq f(i)$. Then one can prove that the image $\widetilde{\pi}(Z_k)$ of the divisor $Z_j \subset \widetilde{X}(\widetilde{w})$ in $\widehat{X}(\widehat{w})$ is of codimension $h(f(i)) - h(j) + 1$ and that its image $\pi(Z_j)$ in $X(w)$ is of codimension $h(i) - h(j) + h(f(i)) - h(j) + 1$. The fiber above $\pi(Z_j)$ contains $\widetilde{\pi}(Z_j)$ and is of dimension at least $h(i) - h(j)$. But we have $\mathrm{codim}_{X(w)}(\pi(Z_j)) = h(i) - h(j) + h(f(i)) - h(j) + 1 \leqslant 2(h(i) - h(j))$. ☐

On the contrary, when we choose a neat ordering on the peaks then we obtain the following result.

Theorem 7.3. *The morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ obtained from Construction 3 are IH-small.*

We prove this theorem in §§ 7.2, 7.3 and 7.4. Let us first give an easy corollary.

Corollary 7.4. *The morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ obtained from Construction 2 is IH-small.*

Proof. Indeed, any morphism $\widehat{\pi}'$ obtained from Construction 3 factors through the morphism $\widehat{\pi}$ and it is easy to verify that this implies, as $\widehat{\pi}'$ is *IH*-small, that $\widehat{\pi}$ is *IH*-small. ☐

7.2 Fibers

We will adapt the approach of [SV94] in our setting. The idea that choosing a neat ordering in the peaks will produce *IH*-small morphisms comes from Zelevinsky's paper [Zel83].

Let us recall that the varieties $\widehat{X}(\widehat{w})$ were constructed by induction, the first step being given by the morphism $p : \overline{PuH} \times^H X(v) \to X(w)$ where P is the stabilizer of $X(w)$, v is obtained from w by removing the subquiver of the partition containing the first peak and u corresponds to the removed vertices. The group H is the intersection of P with the stabilizer of $X(v)$. By induction there exists a resolution $\pi' : \widehat{X}(\widehat{v}) \to X(v)$ equivariant under the stabilizer of $X(v)$. The resolution $\widehat{\pi}$ is given by the fiber product $\widehat{\pi} : \widehat{X}(\widehat{w}) = \overline{PuH} \times^H \widehat{X}(\widehat{v}) \to X(w)$.

Let us first prove a lemma giving a description of the fibers of $\widehat{\pi}$ and a formula on their dimension. This lemma is directly inspired by Lemma 2.1 of [SV94]. If $w' \leqslant w$ in the Bruhat order, let us denote by $U(w')$ the P-orbit (P is the stabilizer of $X(w)$) of $e_{w'}$ (the fixed point of the torus corresponding to the Schubert cell of w') in $X(w)$. Because $\widehat{\pi}$ is P-equivariant, all the fibers of points in $U(w')$ are isomorphic and to calculate

$$f_{\widehat{\pi},w'} := \dim(\widehat{\pi}^{-1}(w'))$$

it is enough to calculate $\dim(\widehat{\pi}^{-1}(U(w'))) - \dim(U(w'))$.

Lemma 7.5. *Define the set*

$$S(w',w) = \left\{ (u',v') \in W \;\middle|\; \begin{array}{c} u' \leqslant u \text{ and } v' \leqslant v, \text{ in the Bruhat order and} \\ PX(u'v') = X(w'), PX(u') = X(u') \text{ and } HX(v') = X(v') \end{array} \right\}.$$

(i) *We have*

$$p^{-1}(U(w')) = \bigcup_{(u',v') \in S(w',w)} Pu'H \times^H He_{v'}.$$

(ii) *We have*

$$\widehat{\pi}^{-1}(U(w')) = \bigcup_{(u',v') \in S(w',w)} Pu'H \times^H \pi'^{-1}(He_{v'}).$$

(iii) *This gives the formula*

$$\begin{aligned} f_{\widehat{\pi},w'} &= \mathrm{Card}(Q_{u'}) + f_{\pi',v'} + \mathrm{Card}(Q_{v'}) - \mathrm{Card}(Q_{w'}) \\ &= \mathrm{Card}(Q_{u'}) + f_{\pi',v'} - \mathrm{codim}_{X(w')}(X(v')) \end{aligned}$$

for some $(u',v') \in S(w',w)$.

Proof. (i) Let $(u',v') \in S(w',w)$. We have the inclusions:

$$p(Pu'H \times^H He_{v'}) \subset Pe_{u'}He_{v'} \subset Pe_{u'}X(v') \subset PX(u'v') = X(w').$$

Furthermore $p(Pu'H \times^H He_{v'})$ is a P-orbit and contains $Pe_{u'}e_{v'} = Pe_{w'}$, so that $Pu'H \times^H He_{v'}$ is contained in $p^{-1}(U(w'))$.

Conversely, if $(x,y) \in \overline{PuH} \times^H X(v)$ is such that $p(x,y) = xy \in U(w')$, then there are elements u' and v' in the Weyl group such that we have $PX(u') = X(u')$, $HX(v') = X(v')$ and $\overline{PxH} \times^H \overline{HyP_\varpi}/P_\varpi = \overline{Pu'H} \times^H X(v')$. But then there exists $(p,q) \in P \times H$ such that $pe_{u'} = x$ and $qe_{v'} = y$ so that we have $(x,y) \in Pu'H \times^H He_{v'}$. Furthermore, there exists $p' \in P$ such that $p'xy = e_{w'}$ and thus $p'pe_{u'}qe_{v'} = e_{w'}$. This implies, because $HX(v') = X(v')$, that $PX(u'v') = X(w')$.

(ii) Follows directly from part (i).

(iii) Because $\dim U(w') = \mathrm{Card}(Q_{w'})$, we only need to prove that

$$\dim(\widehat{\pi}^{-1}(U(w'))) = \mathrm{Card}(Q_{u'}) + f_{\pi',v'} + \mathrm{Card}(Q_{v'})$$

for some $(u',v') \in S(w',w)$. But this follows from the equality $\dim(Pu'H/H) = \mathrm{Card}(Q_{u'})$ together with part (ii). ∎

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Remark 7.6. (i) In the case where $\overline{PuH}/H$ is a flag variety, we recover Lemma 2.1 of [SV94]. In this case we must have $u' = u$ because $X(u)$ is the unique P-stable Schubert subvariety of $X(u)$. We then have $\mathrm{Card}(Q_{u'}) = \mathrm{codim}_{X(w)}(X(v)) = \mathrm{Card}(Q_w) - \mathrm{Card}(Q_v)$.

(ii) More generally, the Schubert variety $X(u')$ is a Schubert subvariety of $X(u)$ with the same stabilizer so that if i is a hole of its quiver then $\beta(i) \in \beta(\mathrm{Holes}(Q_u))$. The Schubert variety $X(v')$ is a Schubert subvariety of $X(v)$ with stabilizer $P_u \cap P_v$. If i is a hole of $Q_{v'}$ then $\beta(i)$ has to be in the union $\beta(\mathrm{Holes}(Q_v)) \cup \beta(\mathrm{Holes}(Q_w))$.

Let us now analyze the condition $PX(u'v') = X(w')$. The quiver of the Schubert variety $X(u'v')$ is obtained by gluing the quiver of $X(u')$ above the quiver of $X(v')$. Furthermore, if $X(a)$ is a Schubert subvariety of $X(w)$, the quiver of the Schubert variety $PX(a)$ is the smallest subquiver Q of Q_w containing the quiver Q_a and such that $\beta(\mathrm{Holes}(Q)) \subset \beta(\mathrm{Holes}(Q_w))$. In particular, if we denote by A the set of vertices $i \in Q_a$ such that i is not the successor of an element of Q_a and $\beta(i) \in \beta(\mathrm{Holes}(Q_w))$ then the set of non-virtual holes of $PX(a)$ is A and the virtual holes are associated to simple roots in $\beta(\mathrm{Holes}(Q_w)) \setminus \beta(A)$.

If i is a hole of $Q_{w'}$ such that $\beta(i)$ is not in the support of u, let j be the smallest vertex in $Q_{v'}$ such that $\beta(j) = \beta(i)$. Then j will be a vertex of $Q_{v'u'}$ with no predecessor and has to be a hole of $PX(u'v') = X(w')$. We must thus have $j = i$. In particular all the holes of $Q_{w'}$ associated to simple roots not in the support of u have to be holes of v'.

The proof of Theorem 7.3 goes as follows. Because the morphism $\widehat{\pi}$ is P-equivariant where P is the stabilizer of $X(w)$, we need to prove that, for any $w' \in W$ such that $X(w')$ is stable under P in $X(w)$, we have $\mathrm{codim}_{X(w)}(X(w')) > 2f_{\widehat{\pi},w'}$.

For the classical cases (A_n and D_n) we introduce two functions Γ and q such that

$$\mathrm{codim}_{X(w)}(X(w')) = \Gamma(w', w) + q(w', w).$$

The function q takes only non-negative values and is positive if $w' \neq w$.

We then proceed by induction on the number of peaks of w (or on the number of fibrations in $\widehat{X}(\widehat{w})$) and prove the stronger result:

$$\Gamma(w', w) \geqslant 2f_{\widehat{\pi},w'}.$$

Because of the previous lemma, it is enough to prove that for all $(u', v') \in S(w', w)$ we have

$$\Gamma(w', w) \geqslant 2(\mathrm{Card}(Q_{u'}) + f_{\pi',v'} - \mathrm{codim}_{X(w')}(X(v'))).$$

Let $\theta \in W$ such that $X(\theta)$ is the closure of the orbit of $X(v')$ in $X(v)$ under P_v. We have $f_{\pi',v'} = f_{\pi',\theta}$ and by the induction hypothesis we have $2f_{\pi',\theta} \leqslant \Gamma(\theta, v)$. We are thus reduced to proving that

$$2(\mathrm{Card}(Q_{u'}) - \mathrm{codim}_{X(w')}(X(v'))) \leqslant \Gamma(w', w) - \Gamma(\theta, v).$$

We prove this formula in the following section for G a group with Dynkin diagram of type A_n or D_n.

7.3 The A_n and D_n cases

To prove Theorem 7.3, we will need to describe the elements of $S(w', w)$ and compute the terms in the formula of Lemma 7.5. The only two difficult cases of minuscule Schubert varieties – for A_n or D_n types – will be the cases of Grassmannians (the varieties constructed are those of Zelevinsky [Zel83]) and of maximal isotropic subspaces in an even-dimensional vector space endowed with a non-degenerate quadratic form (some of these cases have been treated in [SV94] and we complete their study). Indeed, the other minuscule Schubert varieties for A_n and D_n are contained in quadrics and one can use a direct study (see § 7.4).

In the following sections we will prove Theorem 7.3 for both cases A_n and D_n at the same time (for the quadric case, see § 7.4). The proofs of both cases are very similar and we will indicate where they differ.

7.3.1 *The functions q and Γ.* Let us consider the quiver Q_w of a minuscule Schubert variety $X(w)$ in the homogeneous variety X with $X = \mathbb{G}(p,q)$ the Grassmannian of p-dimensional subvector spaces of a q-dimensional vector space or $X = \mathbb{G}_{\mathrm{iso}}(p,2p)$ the Grassmannian of p-dimensional isotropic subvector spaces of a $2p$-dimensional vector space endowed with a non-degenerate quadratic form.

We may assume that all the simple roots are in the support of w (otherwise we consider the action of a subgroup of G) so that there is no virtual hole in Q_w. The set of simple roots $\beta(\mathrm{Holes}(Q_w))$ can be written as $\{\alpha_{k_1},\dots,\alpha_{k_s}\}$. Let us denote by $t_1,\dots,t_s$ the holes such that $\beta(t_i)=\alpha_{k_i}$. Because of Proposition 4.1, for all $i\in[2,s]$ there exists exactly one peak p_i between the holes t_{i-1} and t_i.

Furthermore, in the A_n *case*, there must be a peak p_1 (respectively p_{s+1}) with $\beta(p_1)=\alpha_k$ (respectively $\beta(p_{s+1})=\alpha_k$) with $k<k_1$ (respectively $k>k_s$). In particular we see that the number of peaks equals $s+1$.

In the D_n *case*, there must be a peak p_1 with $\beta(p_1)=\alpha_k$ with $k<k_1$. If $\alpha_{k_s}\notin\{\alpha_{p-1},\alpha_p\}$ then there must be a peak p_{s+1} with $\beta(p_{s+1})=\alpha_{p-1}$ or α_p and in this case there are $s+1$ peaks, otherwise there are s peaks. If $\alpha_{k_s}\in\{\alpha_{p-1},\alpha_p\}$, we have $\alpha_{k_s}=\alpha_{p-1+i}$ with $i=0$ or $i=1$. We define p_{s+1} to be the smallest vertex (for $\preccurlyeq$) of Q_w with $\beta(p_{s+1})=\alpha_{p-i}$.

Let us now define the following sequences $(a_i(w))_{i\in[1,s+1]}$ and $(b_i(w))_{i\in[0,s]}$ of integers (we will sometimes simply denote them by a_i and b_i omitting w).

A_n case:

$$a_i(w)=h(p_i)-h(t_i)\quad\text{and}\quad b_i(w)=h(p_{i+1})-h(t_i)\quad\text{for } i\in[1,s],$$
$$a_{s+1}(w)=p-\sum_{i=1}^{s}a_i(w),$$
$$b_0(w)=q-p-\sum_{i=1}^{s}b_i(w).$$

D_n case and $\alpha_{k_s}\notin\{\alpha_{p-1},\alpha_p\}$:

$$\begin{aligned}
a_i(w)&=h(p_i)-h(t_i)\quad\text{for } i\in[1,s],\\
b_i(w)&=h(p_{i+1})-h(t_i)\quad\text{for } i\in[1,s-1],\\
b_s(w)&=h(p_{s+1})-h(t_s)+\tfrac{1}{2},\\
a_{s+1}(w)&=p-\sum_{i=1}^{s}a_i(w),\\
b_0(w)&=\frac{1}{2}h(p_{s+1})-\sum_{i=1}^{s}b_i(w).
\end{aligned}$$

D_n case and $\alpha_{k_s}\in\{\alpha_{p-1},\alpha_p\}$:

$$a_i(w)=h(p_i)-h(t_i)\quad\text{and}\quad b_i(w)=h(p_{i+1})-h(t_i)\quad\text{for } i\in[1,s-1],$$
$$a_s(w)=h(p_s)-h(p_{s+1})\quad\text{and}\quad b_s(w)=h(p_{s+1})-h(p_{s+1})-\tfrac{1}{2},$$
$$a_{s+1}(w)=p-\sum_{i=1}^{s}a_i(w),$$
$$b_0(w)=\frac{1}{2}h(p_{s+1})-\sum_{i=1}^{s}b_i(w).$$

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It is an easy game on the quiver and the description of configuration varieties to verify that in both cases

$$X(w) = \{V \in X \mid \dim(V \cap \mathbb{C}^{n_i}) \geqslant m_i \text{ for all } i \in [1, s]\},$$

where the $\mathbb{C}^k$ form a complete flag of subspaces (isotropic in the D_n *case*). In the A_n *case*, we have $n_i = \sum_{k=1}^{i}(a_k + b_{k-1})$ and $m_i = \sum_{k=1}^{i} a_k$. In the D_n *case*, we have, in the first case, $n_i = \sum_{k=1}^{i}(a_k + b_{k-1})$ and $m_i = \sum_{k=1}^{i} a_k$ for $i \in [1, s]$, and, in the second case, $n_i = \sum_{k=1}^{i}(a_k + b_{k-1})$ and $m_i = \sum_{k=1}^{i} a_k$ for $i \in [1, s-1]$, $n_s = p$, $m_s = 1 + \sum_{k=1}^{s} a_k$ and $\mathbb{C}^{n_s} \in G/P_{\alpha_{k_s}}$, where $P_{\alpha_{k_s}}$ is the maximal parabolic subgroup associated to the simple root α_{k_s}.

Let $X(w')$ be a Schubert subvariety of $X(w)$ with the same stabilizer. Then we must have $\beta(\mathrm{Holes}(Q_{w'})) \subset \beta(\mathrm{Holes}(Q_w))$. For any hole t_i of Q_w let us define the depth of w' in t_i to be the integer given by:

in the A_n *case*, $\quad c_i = \mathrm{Card}\{j \in Q_w \backslash Q_{w'} \mid \beta(j) = \beta(t_i)\}$;

in the D_n *case*, $\quad c_i = \begin{cases} \mathrm{Card}\{j \in Q_w \backslash Q_{w'} \mid \beta(j) = \beta(t_i)\} & \text{for } \beta(t_i) \notin \{\alpha_{p-1}, \alpha_p\}, \\ 2\,\mathrm{Card}\{j \in Q_w \backslash Q_{w'} \mid \beta(j) = \beta(t_i)\} & \text{for } \beta(t_i) \in \{\alpha_{p-1}, \alpha_p\}. \end{cases}$

The same game on the quiver and the description of configuration varieties shows that the associated sequences are given by:

in the A_n *case*, $\quad \begin{cases} a_i(w') = a_i(w) + c_i - c_{i-1} & \text{for all } i \in [1, s+1], \\ b_i(w') = b_i(w) + c_i - c_{i+1} & \text{for all } i \in [0, s]; \end{cases}$

in the D_n *case*, $\quad \begin{cases} a_i(w') = a_i(w) + c_i - c_{i-1} & \text{for all } i \in [1, s+1], \\ b_i(w') = b_i(w) + c_i - c_{i+1} & \text{for all } i \in [0, s]; \end{cases}$

with $c_0 = c_{s+1} = 0$ in the A_n *case* and with $c_0 = 0$ and $c_{s+1} = c_s$ in the D_n *case*. We have in both cases:

$$X(w') = \{V \in X \mid \dim(V \cap \mathbb{C}^{n_i}) \geqslant m_i + l_i \text{ for all } i \in [1, s]\},$$

with $l_i = \sum_{k=1}^{i} c_k$. One can calculate the codimension of $X(w')$ in $X(w)$ as follows.

Fact 7.7. *We have the formula*

$$\mathrm{codim}_{X(w)}(X(w')) = \mathrm{Card}(Q_w) - \mathrm{Card}(Q_{w'}) = \Gamma(w', w) + q(w', w),$$

where in the A_n case we have

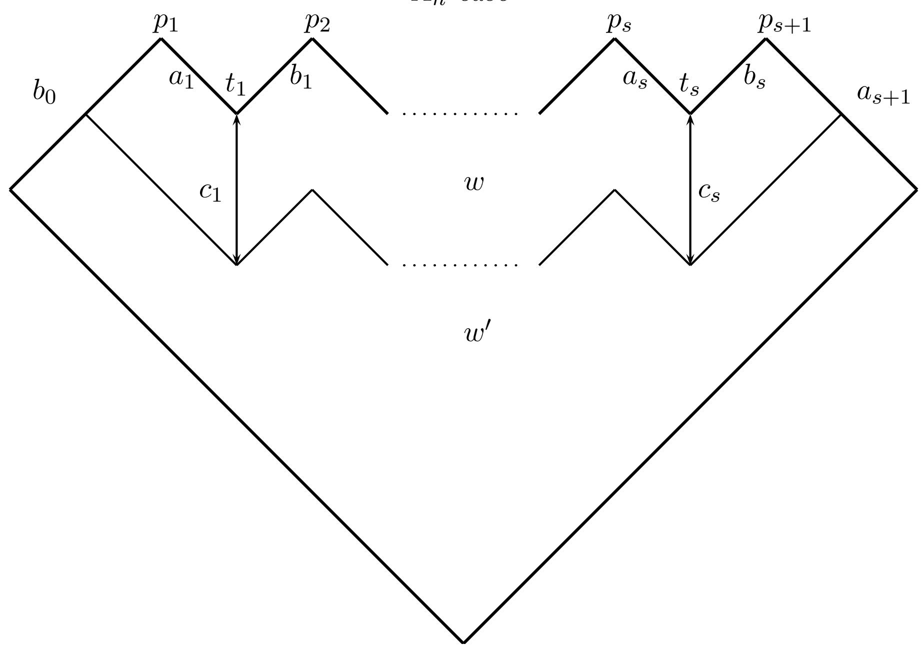
$$\Gamma(w', w) = \sum_{i=1}^{s} c_i(a_i + b_i) \quad \text{and} \quad q(w', w) = \frac{1}{2}\sum_{i=1}^{s+1}(c_i - c_{i-1})^2$$

and in the D_n case we have

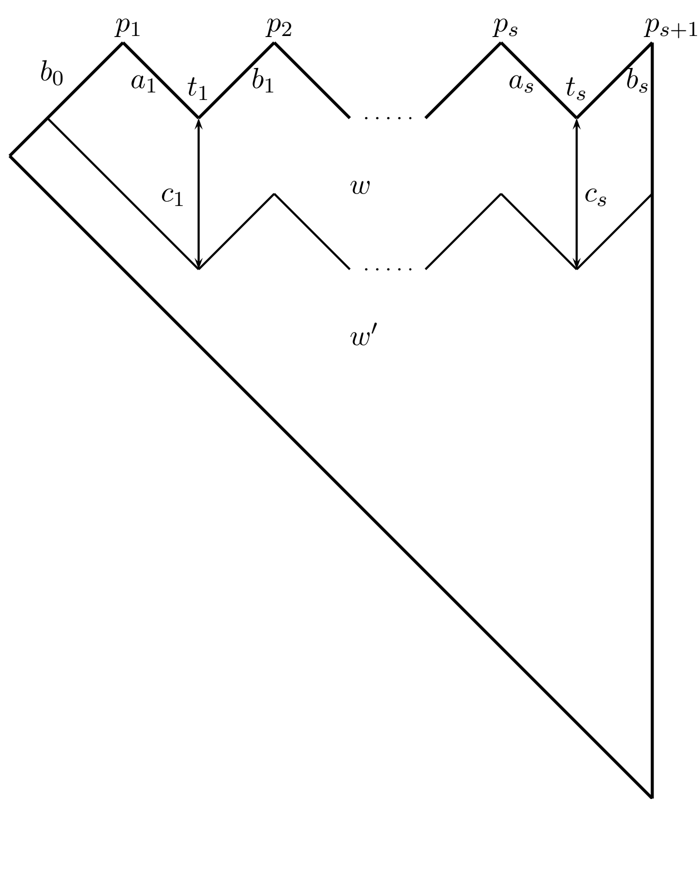
$$\Gamma(w', w) = \sum_{i=1}^{s} c_i(a_i + b_i) \quad \text{and} \quad q(w', w) = \frac{1}{2}\sum_{i=1}^{s}(c_i - c_{i-1})^2.$$

Proof. These formulae can be checked directly. We will rather obtain them geometrically from the quivers depicted below, where we only draw the boundary. The numbers a_i and b_i indicate the number of vertices and all arrows are going down (see Appendix A for a full description of the quivers).

A_n *case*



D_n *case* and $\alpha_{k_s} \notin \{\alpha_{p-1}, \alpha_p\}$



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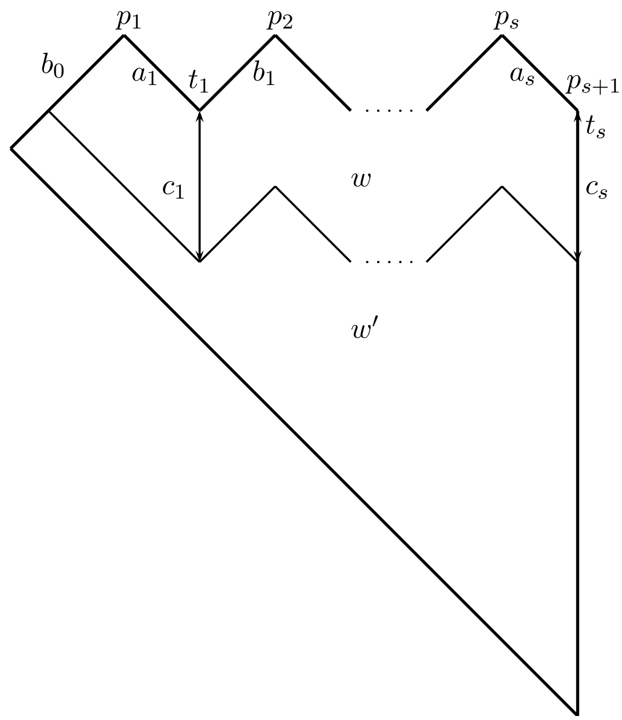
D_n *case* and $\alpha_{k_s} \in \{\alpha_{p-1}, \alpha_p\}$



The dimension of a Schubert variety is the number of vertices of its quiver. The codimension $\operatorname{codim}_{X(w)}(X(w'))$ is thus given by the difference of the numbers of vertices. We find respectively in the A_n *case*, in the D_n *case* for $\alpha_{k_s} \notin \{\alpha_{p-1}, \alpha_p\}$ and in the D_n *case* for $\alpha_{k_s} \in \{\alpha_{p-1}, \alpha_p\}$:

$$\operatorname{codim}_{X(w)}(X(w')) = \begin{cases} \displaystyle\sum_{i=1}^{s} c_i(a_i+b_i) + \sum_{i=1}^{s} c_i^2 - \sum_{i=1}^{s-1} c_ic_{i+1}, \\ \displaystyle\sum_{i=1}^{s-1} c_i(a_i+b_i) + c_s(a_s+b_s-\tfrac{1}{2}) - \tfrac{1}{2}c_s(c_s-1) - \sum_{i=1}^{s-1} c_ic_{i+1} + \sum_{i=1}^{s} c_i^2, \\ \displaystyle\sum_{i=1}^{s-1} c_i(a_i+b_i) + a_sc_s - \sum_{i=1}^{s-1} c_ic_{i+1} + \sum_{i=1}^{s-1} c_i^2 + \tfrac{1}{2}c_s(c_s+1). \end{cases}$$

Simple calculations give the desired formulae. We remark that $q(w', w) > 0$ for $w' \neq w$. □

7.3.2 *Proof of Theorem 7.3.* Let u and v be as in § 7.2. Let us assume that v is obtained from w by removing the kth peak of Q_w. In the A_n *case*, we obtain the following situation.

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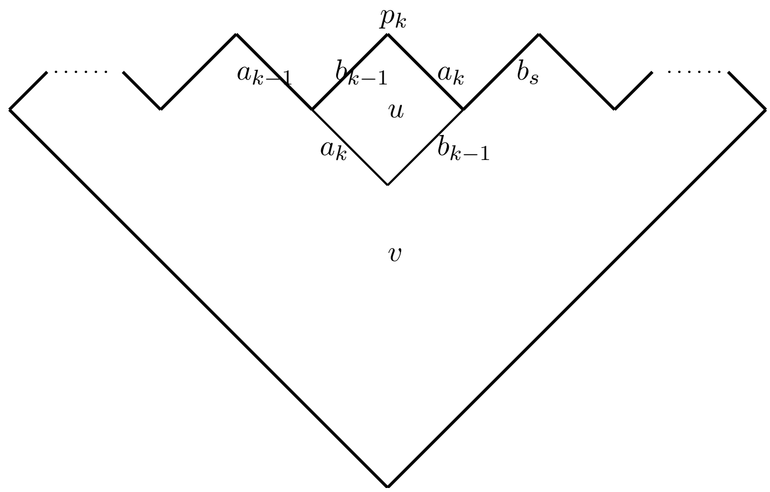
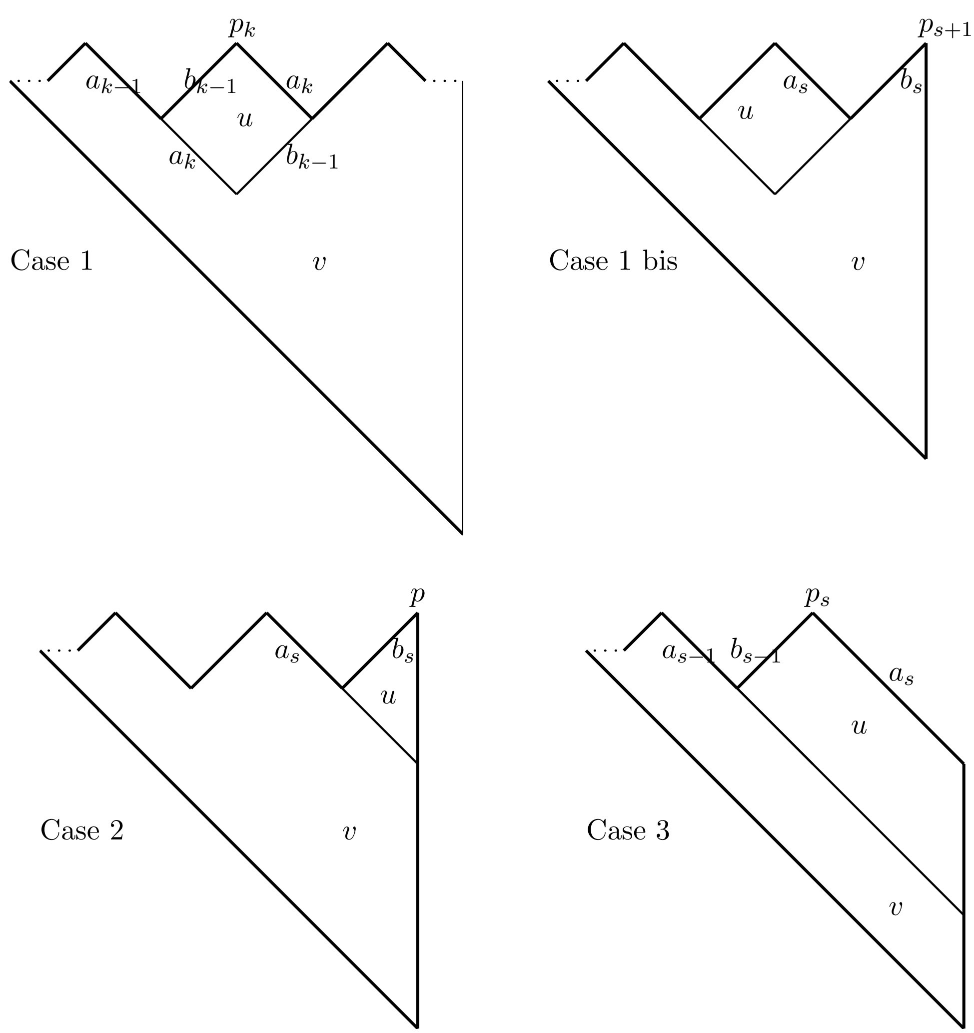


In the D_n *case*, we obtain the following four situations.



In the D_n case, Case 1 leads to the same calculations as in the A_n *case* and we will do the calculation only for the A_n *case*. Case 1 bis has been done with these techniques in [SV94], and so we will not perform the calculation again.

In the A_n *case*, the sequences of integers $(a_i(v))_{i\in[1,s]}$ and $(b_i(v))_{i\in[0,s-1]}$ are given by:

$$a_i(v)=\begin{cases} a_i(w) & \text{for } i<k-1,\\ a_k(w)+a_{k-1}(w) & \text{for } i=k-1,\\ a_{i+1}(w) & \text{for } i\geqslant k,\end{cases} \quad\text{and}\quad b_i(v)=\begin{cases} b_i(w) & \text{for } i<k-1,\\ b_k(w)+b_{k-1}(w) & \text{for } i=k-1,\\ b_{i+1}(w) & \text{for } i\geqslant k,\end{cases}$$

where $a_0(w)=b_0(w)$.

In the D_n *case*, for Case 2, the sequences of integers $(a_i(v))_{i\in[1,s+1]}$ and $(b_i(v))_{i\in[0,s]}$ are given by:

$$a_i(v)=\begin{cases} a_i(w) & \text{for } i<s,\\ a_s(w)+b_s(w)-\frac{1}{2} & \text{for } i=s,\\ a_{s+1}(w)-(b_s(w)-\frac{1}{2}) & \text{for } i=s+1,\end{cases} \quad\text{and}\quad b_i(v)=\begin{cases} b_0(w)+b_s(w)-\frac{1}{2} & \text{for } i=0,\\ b_i(w) & \text{for } 0<i<s,\\ \frac{1}{2} & \text{for } i=s.\end{cases}$$

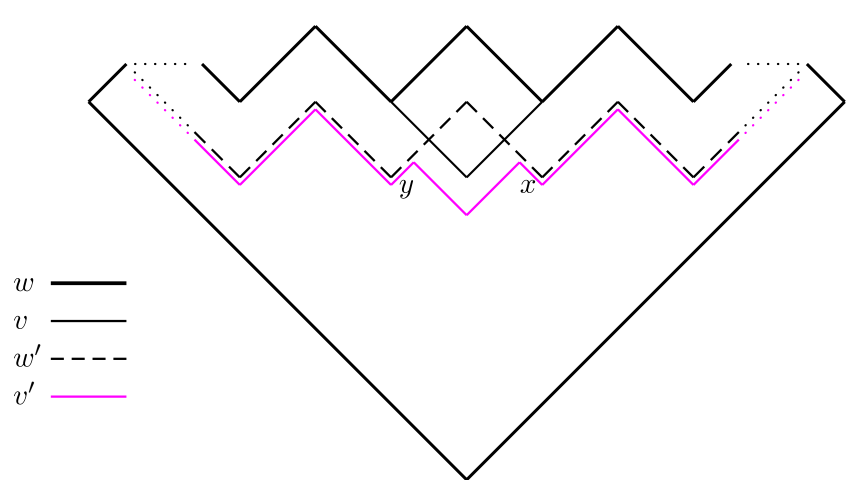
For Case 3, the sequences of integers $(a_i(v))_{i\in[1,s]}$ and $(b_i(v))_{i\in[0,s-1]}$ are given by:

$$a_i(v)=\begin{cases} a_i(w) & \text{for } i<s-1,\\ a_s(w)+a_{s-1}(w)+b_{s-1}(w) & \text{for } i=s-1,\\ a_{s+1}(w)-b_{s-1}(w) & \text{for } i=s,\end{cases} \quad\text{and}\quad b_i(v)=\begin{cases} b_i(w) & \text{for } i<s-1,\\ b_s(w) & \text{for } i=s-1.\end{cases}$$

Furthermore, in the A_n *case* and Case 2 of the D_n *case*, the quiver Q_u of u has no hole, meaning that the variety $\overline{PuH}/H$ is smooth and u' has to be equal to u. In these cases we only need to determine v'.

Let us now consider the quiver Q obtained by intersecting in Q_w the quivers Q_v and $Q_{w'}$. The quiver of v' has to be a subquiver of Q such that (see Remark 7.6) all the holes of $Q_{w'}$ are holes of $Q_{v'}$, and $Q_{v'}$ may have one more hole corresponding to the hole of v which is not a hole of w.

A_n *case*



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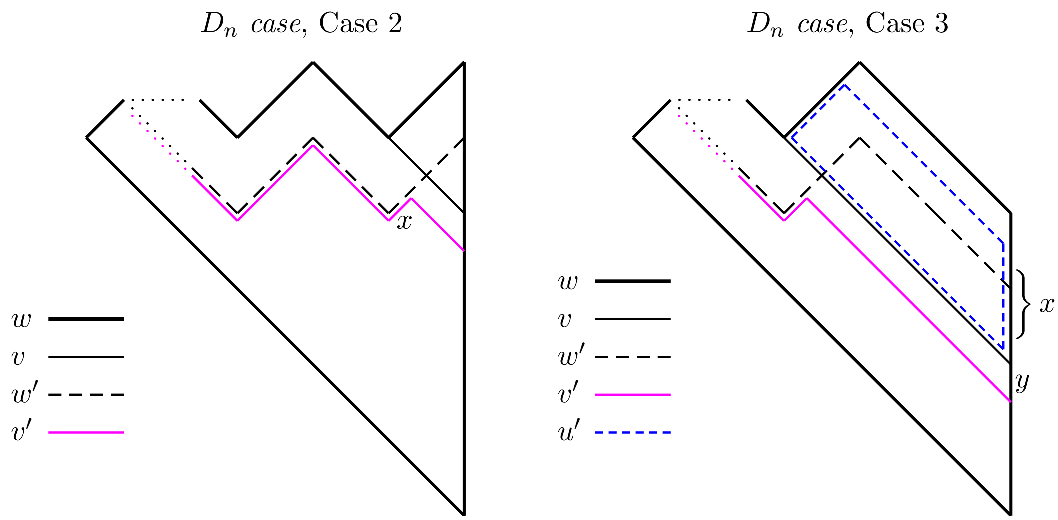

In the A_n *case*, the quiver $Q_{v'}$ has $s+1$ holes and the sequences of integers $(a_i(v'))_{i\in[1,s+2]}$ and $(b_i(v'))_{i\in[0,s+1]}$ are given by:

$$a_i(v') = \begin{cases} a_i(w') & \text{for } i \leqslant k-1, \\ a_k(w') - x & \text{for } i = k, \\ x & \text{for } i = k+1, \\ a_{i-1}(w') & \text{for } i > k+1, \end{cases} \quad \text{and} \quad b_i(v') = \begin{cases} b_i(w') & \text{for } i < k-1, \\ y & \text{for } i = k-1, \\ b_{k-1}(w') - y & \text{for } i = k, \\ b_{i-1}(w) & \text{for } i \geqslant k+1, \end{cases}$$

where $x \in [0, c_k]$, $y \in [0, c_{k-1}]$ and $c_{k-1} - y = c_k - x$. Indeed, the last formula is given by the fact that the unique hole different from those of $Q_{w'}$ has to be associated to the same root as the hole of v which is not a hole of w. This gives the equality

$$\sum_{i=1}^{k-1}(a_i(v) + b_{i-1}(v)) = \sum_{i=1}^{k}(a_i(v') + b_{i-1}(v'))$$

and the equality $c_{k-1} - y = c_k - x$. The fact that $x \in [0, c_k]$ and $y \in [0, c_{k-1}]$ is equivalent to the fact that $X(v')$ is a Schubert subvariety of $X(v)$.

In the D_n *case*, for Case 2, the quiver $Q_{v'}$ has $s+1$ holes and the sequences $(a_i(v'))_{i\in[1,s+2]}$ and $(b_i(v'))_{i\in[0,s+1]}$ are given by:

$$a_i(v') = \begin{cases} a_i(w') & \text{for } i \leqslant s, \\ b_s(w') - \frac{1}{2} - x & \text{for } i = s+1, \end{cases} \quad \text{and} \quad b_i(v') = \begin{cases} b_i(w') & \text{for } i < s, \\ x & \text{for } i = s, \\ \frac{1}{2} & \text{for } i = s+1, \end{cases}$$

where $x \in [0, c_s]$. For Case 3, the quiver $Q_{v'}$ has s holes and the sequences $(a_i(v'))_{i\in[1,s+1]}$ and $(b_i(v'))_{i\in[0,s]}$ are given by:

$$a_i(v') = \begin{cases} a_i(w') & \text{for } i \leqslant s-1, \\ a_s(w) + b_{s-1}(w) - c_{s-1} + y & \text{for } i = s, \end{cases} \quad \text{and} \quad b_i(v') = \begin{cases} b_i(w') & \text{for } i < s-1, \\ c_{s-1} - y & \text{for } i = s-1, \\ b_s(w') & \text{for } i = s. \end{cases}$$

In Case 3, we also get that u' has a unique hole,

$$\begin{cases} a_0(u') = x \quad \text{and} \quad a_1(u') = a_s(w) + b_{s-1}(w) - x = a_1(u) + b_0(u) - x, \\ b_0(u') = x \quad \text{and} \quad b_1(u') = b_1(u) = b_s(w). \end{cases}$$

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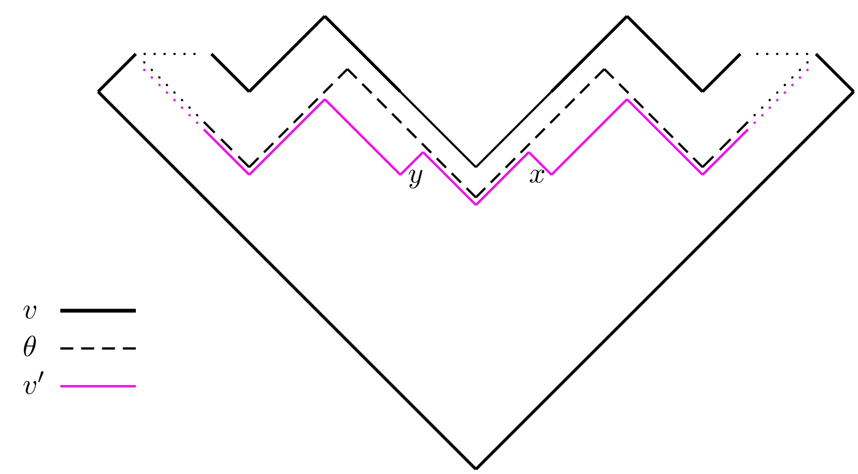
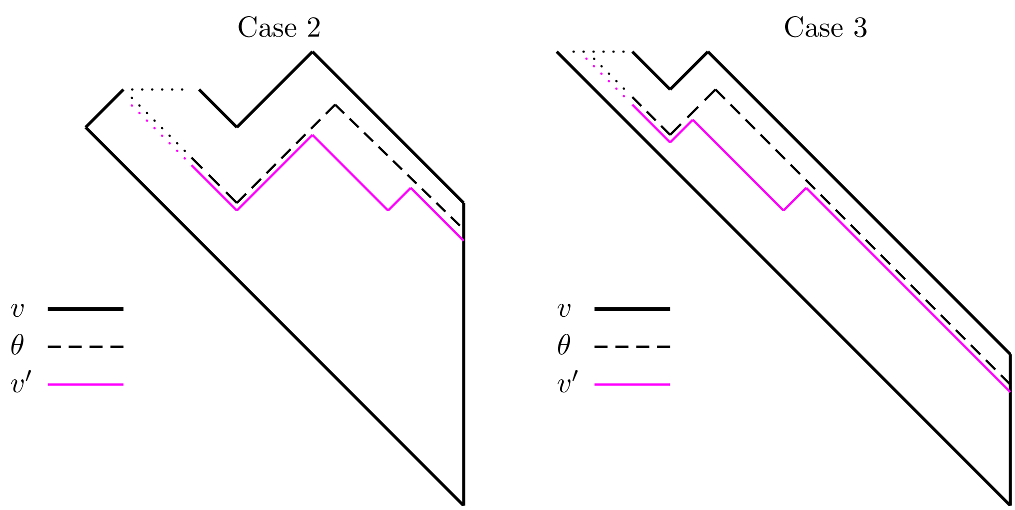
In Case 3, we have $y \in [c_{s-1} - c_s, c_{s-1}]$, $x \in [b_{s-1}(w') - c_k, b_{s-1}(w')]$ and $b_{s-1} - x = c_s - y$. Indeed, the last formula follows from the fact that the last hole of $Q_{u'v'}$ has to be the same hole as the last hole of $Q_{w'}$. This implies that $b_{s-1}(v') + b_0(u') = b_{s-1}(w')$ and we have the equality $b_{s-1} - x = c_s - y$. The fact that $y \in [c_{s-1} - c_s, c_{s-1}]$ comes from the fact that $X(v')$ is a Schubert subvariety of $X(v)$ and that $PX(u'v') = X(w')$.

The Schubert subvariety $X(\theta)$ is contained in $X(v)$, contains $X(v')$ and is stable by the stabilizer of $X(v)$. It must have the same holes as v' except for those not corresponding to holes of v. In the A_n *case*, we have to fill the holes $k-1$ and $k+1$ of v' to obtain θ. In the D_n *case*, we have to fill the sth hole (in Case 2) or the $(s-1)$th hole (in Case 3) of v' to obtain θ. The associated quivers are shown below.





In the A_n *case*, the quiver Q_θ of θ has $s-1$ holes and the integers $(a_i(\theta))_{i\in[1,s]}$ and $(b_i(\theta))_{i\in[0,s-1]}$ are given by:

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \leqslant k-2, \\ a_{k-1}(v') + a_k(v') & \text{for } i = k-1, \\ a_{k+1}(v') + a_{k+2}(v') & \text{for } i = k, \\ a_{i+2}(v') & \text{for } i \geqslant k+1, \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i < k-2, \\ b_{k-2}(v') + b_{k-1}(v') & \text{for } i = k-2, \\ b_k(v') + b_{k+1}(v') & \text{for } i = k-1, \\ b_{i+2}(w) & \text{for } i \geqslant k. \end{cases}$$

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In the D_n *case*, for Case 2, the integers $(a_i(\theta))_{i\in[1,s+1]}$ and $(b_i(\theta))_{i\in[0,s]}$ associated to θ are given by:

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \neq s, \\ a_{s+1}(v') + a_s(v') & \text{for } i = s, \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i \neq s-1, \\ b_{s-1}(v') + b_s(v') & \text{for } i = s-1. \end{cases}$$

In Case 3, the integers $(a_i(\theta))_{i\in[1,s]}$ and $(b_i(\theta))_{i\in[0,s-1]}$ associated to θ are given by:

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \neq s-1, \\ a_{s-1}(v') + a_s(v') & \text{for } i = s-1, \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i \neq s-2, \\ b_{s-2}(v') + b_{s-1}(v') & \text{for } i = s-2. \end{cases}$$

It is now an easy calculation (and straightforward on the quiver) to find that, in the A_n *case*, the depth c'_i of θ in the holes of v is $c_{k-1} - y = c_k - x$ for the $(k-1)$th hole, and c_i for all the holes before the $(k-1)$th hole and c_{i+1} for all the holes after the $(k-1)$th hole. In the D_n *case* for Case 2, the depth c'_i of θ in the holes of v is $c_s - x$ for the sth hole and c_i for the other holes. In Case 3, the depth c'_i of θ in the holes of v is $y = c_s + x - b_{s-1}$ for the $(s-1)$th hole and c_i for the other holes.

To prove the theorem, we need to prove the inequality

$$2(\dim(X(u')) - \operatorname{codim}_{X(w')}(X(v'))) \leqslant \Gamma(w', w) - \Gamma(\theta, v).$$

We first compute the difference $\dim(X(u')) - \operatorname{codim}_{X(w')}(X(v'))$. It is equal to the following:

in the A_n *case*

$$a_k(w)b_{k-1}(w) - (a_k(w') - x)(b_{k-1}(w') - y);$$

in the D_n *case*, Case 2

$$\frac{(b_s(w) + \frac{1}{2})(b_s(w) - \frac{1}{2})}{2} - \frac{(b_s(w') - x + \frac{1}{2})(b_s(w') - x - \frac{1}{2})}{2};$$

in the D_n *case*, Case 3

$$\frac{x(x+1)}{2} + x(a_s(w) + b_{s-1}(w) - x) - \frac{x(x+1)}{2} - xa_s(w').$$

A simple calculation leads to the formulae (recall that $b_s(w') = b_s(w)$ in Case 2 of the D_n *case*):

$$\dim(X(u')) - \operatorname{codim}_{X(w')}(X(v')) = \begin{cases} xa_k(w) + yb_{k-1}(w) - xy & \text{in the } A_n \textit{ case}, \\ xb_s(w) - \frac{1}{2}x^2 & \text{in the } D_n \textit{ case}, \text{ Case 2}, \\ x(c_{s-1} - y) & \text{in the } D_n \textit{ case}, \text{ Case 3}. \end{cases}$$

On the other hand we have

$$\Gamma(w', w) - \Gamma(\theta, v) = \begin{cases} \displaystyle\sum_{i=1}^{s} c_i(a_i(w) + b_i(w)) - \sum_{i=1}^{s-1} c'_i(a_i(v) + b_i(v)) & \text{in the } A_n \textit{ case}, \\ \displaystyle\sum_{i=1}^{s} c_i(a_i(w) + b_i(w)) - \sum_{i=1}^{s} c'_i(a_i(v) + b_i(v)) & \text{in the } D_n \textit{ case}, \text{ Case 2}, \\ \displaystyle\sum_{i=1}^{s} c_i(a_i(w) + b_i(w)) - \sum_{i=1}^{s-1} c'_i(a_i(v) + b_i(v)) & \text{in the } D_n \textit{ case}, \text{ Case 3}. \end{cases}$$

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Simplifying we get

$$\Gamma(w',w)-\Gamma(\theta,v)$$
$$=\begin{cases} x(a_k(w)+b_k(w))+y(a_{k-1}(w)+b_{k-1}(w)) & \text{in the } A_n \textit{ case},\\ x(a_s(w)+b_s(w)) & \text{in the } D_n \textit{ case}, \text{ Case 2},\\ (c_{s-1}-y)(a_{s-1}(w)+b_{s-1}(w))+(b_{s-1}-x)(a_s(w)+b_s(w)) & \text{in Case 3.}\end{cases}$$

Now the required inequality follows from the following facts.

(a) In the A_n *case*, the kth peak of w is smaller than the adjacent peaks, meaning that we have $a_{k-1}(w) \geqslant b_{k-1}(w)$ and $b_k(w) \geqslant a_k(w)$. Furthermore x and y are non-negative.

(b) In Case 2 of the D_n *case*, the sth peak of w is smaller than the $(s-1)$th peak, meaning that $a_s(w) \geqslant b_s(w)$. Furthermore x is non-negative.

(c) And in Case 3 of the D_n *case*, the $(s-1)$th peak of w is smaller than the $(s-2)$th peak, meaning that we have $a_{s-1}(w) \geqslant b_{s-1}(w)$. Furthermore, we have $x \leqslant b_{s-1}(w') = b_{s-1}(w)$ and $a_s(w)+b_s(w) \geqslant 0$.

Theorem 7.3 is proved in all cases.

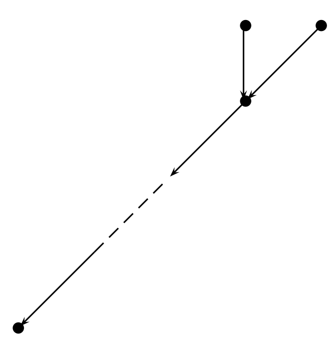
7.4 Exceptional cases

We are left to deal with three cases: quadrics and minuscule varieties for E_6 and E_7.

7.4.1 *Quadrics.* For quadrics, let us remark that all Schubert varieties except one are locally factorial (the quivers have a unique peak) so that in all cases except one we have $\widehat{X}(\widehat{w}) = X(w)$ and there is nothing to prove. The unique non-locally factorial Schubert variety (we are in $\mathbb{C}^{2p}$ with a non-degenerate quadratic form) is given by

$$X(w) = \{x \in \mathbb{P}(\mathbb{C}^{2p}) \mid x \text{ is isotropic and } x \in F_{p-2}^{\perp}\}$$

for a fixed isotropic subspace F_{p-2} of dimension $p-2$. The associated quiver has p vertices and is given below.



In particular, the resolution $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ is given by $p : \overline{PuH} \times^H X(v) \to X(w)$ where $\overline{PuH}/H$ is of dimension one. The fibers of the morphism $\widehat{\pi}$ have dimension at most one. On the other hand, as $\widehat{\pi}$ is P_w-equivariant, the fiber over a point $x \in X(w)$ has positive dimension only when x is contained in a strictly smaller Schubert variety stable under P_w. These subvarieties are of codimension at least three and the result follows.

More generally, if the morphism $\widehat{\pi}$ is of the form $p : \overline{PuH} \times^H X(v) \to X(w)$ and its fibers are of dimension at most one, then the morphism $\widehat{\pi}$ has to be *IH*-small.

7.4.2 *The E_6 case.* We have seen that if the Schubert variety is locally factorial (i.e. its quiver has a unique peak) or if the morphism $\widehat{\pi}$ is of the form $p : \overline{PuH} \times^H X(v) \to X(w)$ and its fiber is of dimension at most one, then the morphism $\widehat{\pi}$ has to be *IH*-small. We are now going to list the morphisms $\widehat{\pi}$ obtained from Construction 3 not satisfying these properties and verify that such morphisms $\widehat{\pi}$ are small.

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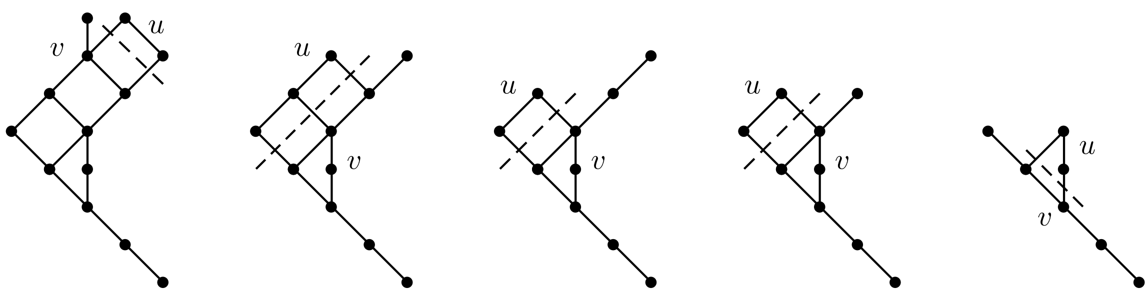
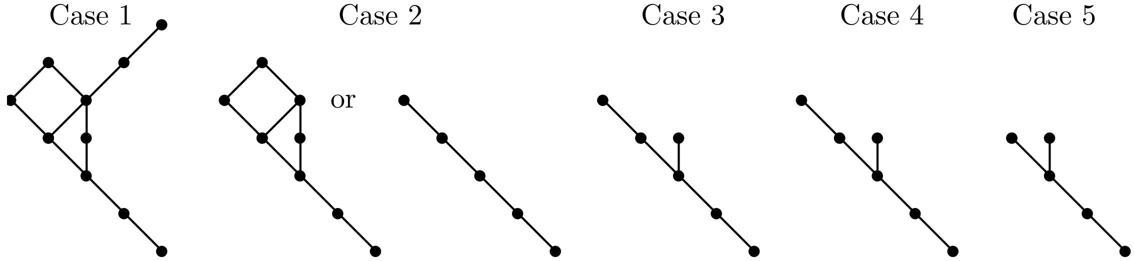
For E_6, all the morphisms $\widehat{\pi}$ not verifying the preceding properties are of the form $p : \overline{PuH} \times^H X(v) \to X(w)$. We list here the quiver of $X(w)$ together with the quivers of u and v.

Case 1 Case 2 Case 3 Case 4 Case 5



The dimension of the fiber in these morphisms is at most $f = 2$ except in the second case where it is at most $f = 3$. The Schubert subvarieties $X(w')$ stable under P_w of codimension not bigger than $2f$ are as shown below.

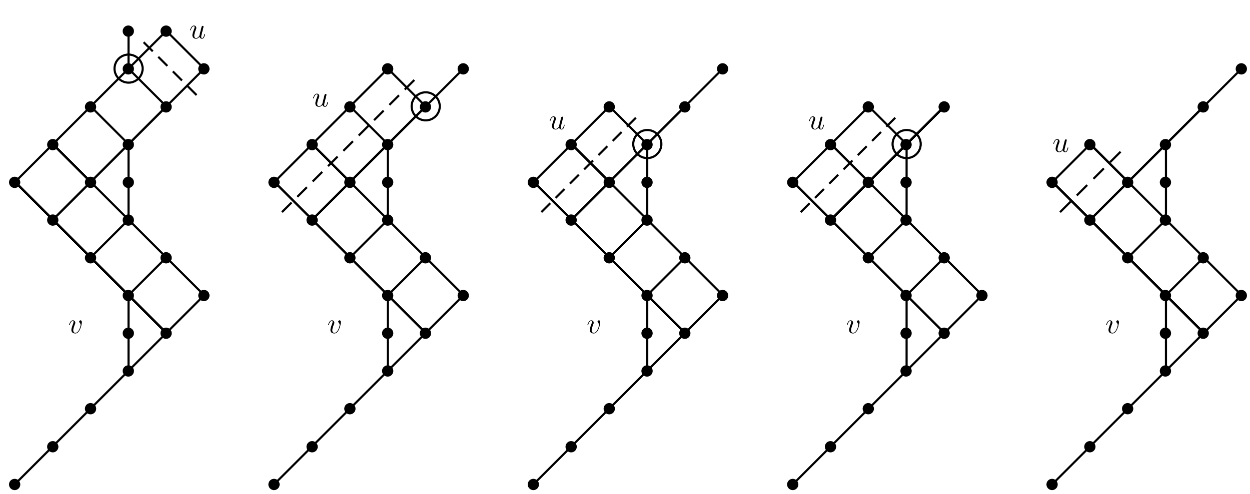
Case 1 Case 2 Case 3 Case 4 Case 5

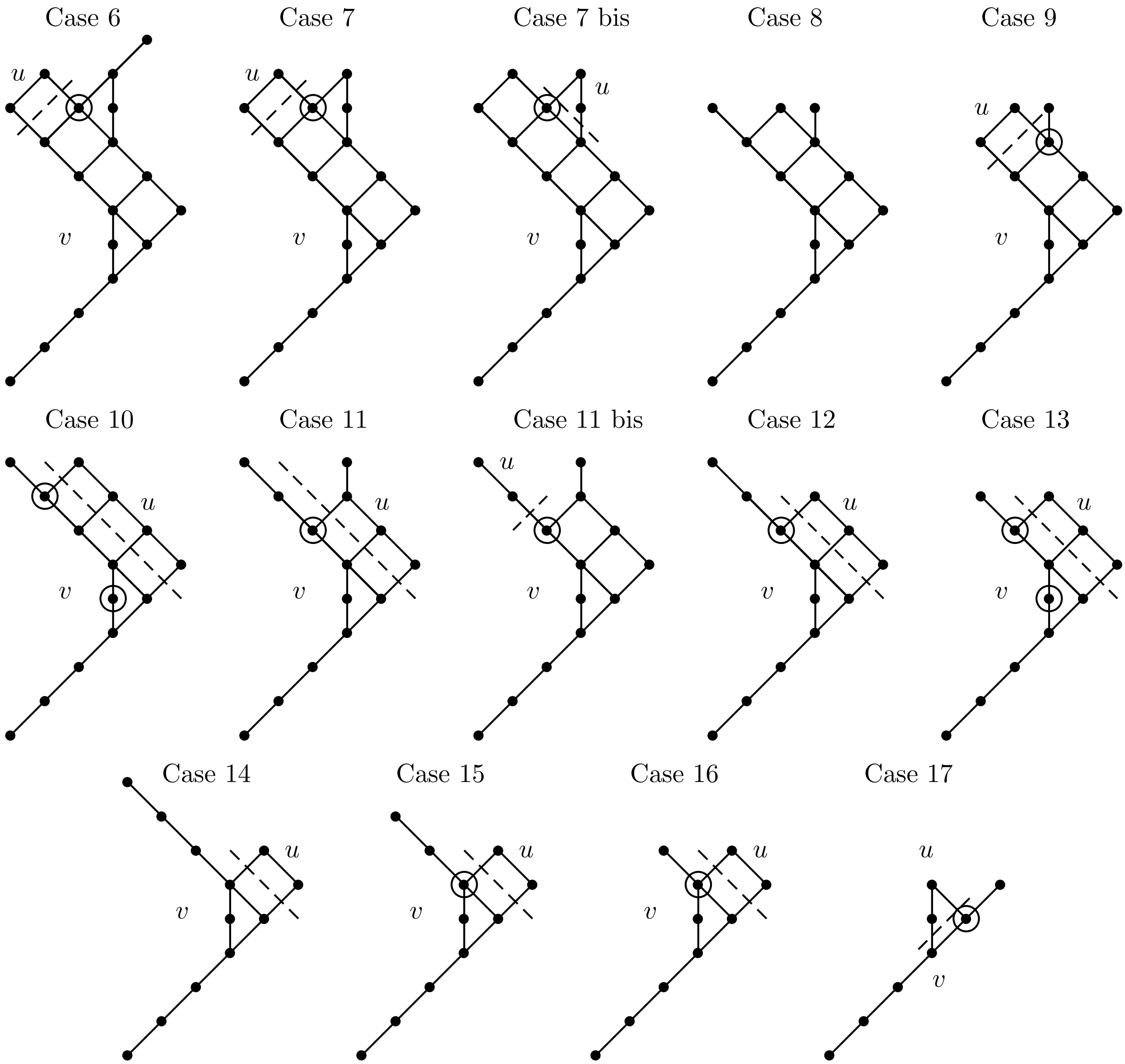


These quivers Q are obtained from the quiver Q_w by removing all the vertices smaller than a hole i of Q_w. It is now easy to see that any subvariety Z_K of the Bott–Samelson resolution $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to X(w)$ such that $\widetilde{\pi}(Z_K) = X(w')$ is contained in the divisor Z_i. We thus have $\widetilde{\pi}^{-1}(X(w')) = Z_i$ and $\widehat{\pi}^{-1}(X(w'))$ is contained in the image of Z_i in $\widehat{X}(\widehat{w})$. Seeing $\widehat{X}(\widehat{w})$ as a configuration variety, the image of Z_i in $\widehat{X}(\widehat{w})$ is the configuration variety $\overline{PuH} \times^H X(v')$ where $Q_{v'} = Q \cap Q_v$. In particular, the dimension of the fiber of $\widehat{\pi}$ above $X(w')$ is one, one, two, one, one, one in the different cases and the morphism $\widehat{\pi}$ is always *IH*-small.

7.4.3 *The E_7 case.* We proceed in the same way in this case and list the quivers having at least two peaks and for which the fiber is at least two.

Case 1 Case 2 Case 3 Case 4 Case 5





All the resolutions are of type $\overline{PuH} \times^H X(v')$ except Case 8. We will deal with this case later on. The maximal dimension f of the fiber in all other cases is given by:

Case	1	2	3	4	5	6	7	7 bis	9	10	11	11 bis	12	13	14	15	16	17
f	2	4	3	3	2	2	2	2	2	4	4	2	3	3	2	2	2	2

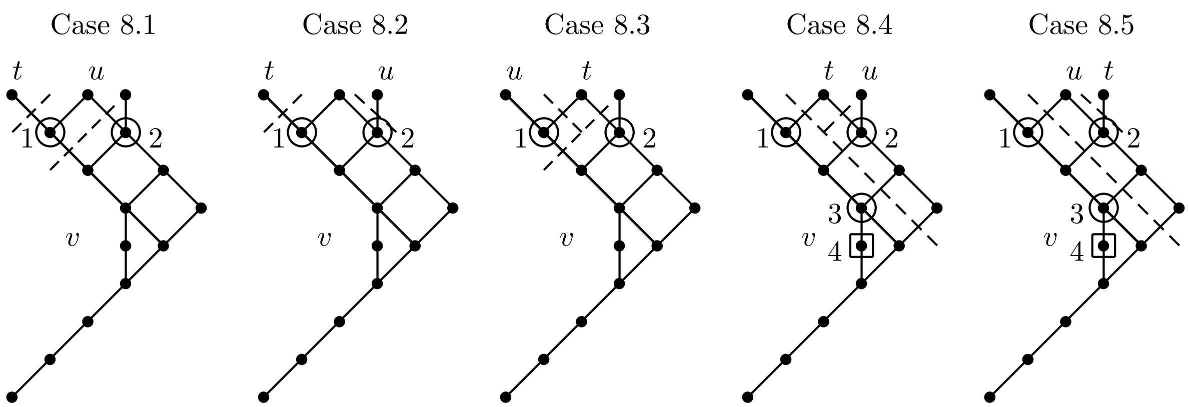
We have circled the vertices i such that the quivers $Q_{w'}$ obtained from the quiver Q_w by removing all the vertices smaller (for $\preccurlyeq$) than the hole i of Q_w are the quivers of the Schubert subvarieties $X(w')$ stable under P_w of codimension at most $2f$. The codimension of $X(w')$ in $X(w)$ is given by:

Case	1	2	3	4	6	7	7 bis	9	10	11	11 bis	12	13	15	16	17
codim	3	3	4	3	4	3	3	3	3 or 8	5	5	4	3 or 6	4	3	3

We remark that in Cases 5 and 14 there is no such Schubert subvariety so that the morphism is already small. In all the other cases, and as for the E_6 case, it is easy compute the dimension of the fiber of $\widehat{\pi}$ above $X(w')$. It is given by:

Case	1	2	3	4	6	7	7 bis	9	10	11	11 bis	12	13	15	16	17
dim	1	1	1	1	1	1	1	1	1 or 3	2	2	1	1 or 2	1	1	1

The morphism $\widehat{\pi}$ is always *IH*-small in these cases. We are left with Case 8 for which the resolution is of the form $\overline{PtH} \times^H \overline{RuS} \times^S X(v)$ and the partitions of the quivers are as given below.



The maximal dimension f of the fiber in all these cases is given by:

Case	8.1	8.2	8.3	8.4	8.5
f	3	2	3	5	5

We have circled and numbered the vertices i such that the quivers $Q_{w'}$ obtained from the quiver Q_w by removing all the vertices smaller (for $\preccurlyeq$) than a fixed subset of the holes of Q_w are the quivers of the Schubert subvarieties $X(w')$ stable under P_w of codimension at most $2f$. Let A be a non-empty subset of $\{1,2,3\}$ and let $Q_{w'}$ be the quiver obtained by removing the vertices smaller than the vertices in A. The codimension of $X(w')$ in $X(w)$ is given by (here A is $\{1\}$, $\{2\}$, $\{1,2\}$ or in the last two cases $\{3\}$):

Case	8.1	8.2	8.3	8.4	8.5
codim	$3, 3$ or 5	$3, 3$ or 5	$3, 3$ or 5	$3, 3, 5$ or 8	$3, 3, 5$ or 8

Suppose that A is a subset of $\{1,2\}$. It is easy to see that any subvariety Z_K of the Bott–Samelson resolution $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to X(w)$ such that $\widetilde{\pi}(Z_K) = X(w')$ is contained in the variety Z_A. The fiber $\widehat{\pi}^{-1}(X(w'))$ is thus contained in the image of Z_A in $\widehat{X}(\widehat{w})$. Seeing $\widehat{X}(\widehat{w})$ as a configuration variety, this image in $\widehat{X}(\widehat{w})$ is the configuration variety $\overline{PtH} \times^H \overline{Ru'S} \times^S X(v')$ where $Q_{u'}$ (respectively $Q_{v'}$) are obtained from Q_u (respectively Q_v) by removing the vertices smaller than one vertex in $A \cap Q_u$ (respectively $A \cap Q_v$). In particular, the dimension of the fiber of $\widehat{\pi}$ above $X(w')$ is given by:

Case	8.1	8.2	8.3	8.4	8.5
dim	$1,\ 1$ or 1	$1,\ 1$ or 2	$1,\ 1$ or 1	$1,\ 1$ or 1	$1,\ 1$ or 1

The morphism $\widehat{\pi}$ is always IH-small in these cases. We are left with the case where $A = \{3\}$. In this case it is not hard to see that any subvariety Z_K of the Bott–Samelson resolution $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \to X(w)$ such that $\widetilde{\pi}(Z_K) = X(w')$ is contained in the variety $Z_{\{2,3\}}$ or in Z_4. The fiber $\widehat{\pi}^{-1}(X(w'))$ is thus contained in the image of $Z_{\{2,3\}}$ or of Z_4 in $\widehat{X}(\widehat{w})$. Seeing $\widehat{X}(\widehat{w})$ as a configuration variety, the image of $Z_{\{2,3\}}$ in $\widehat{X}(\widehat{w})$ is the configuration variety $\overline{PtH} \times^H \overline{Ru'S} \times^S X(v')$ where $Q_{u'}$ (respectively $Q_{v'}$) are obtained from Q_u (respectively Q_v) by removing the vertices smaller than one vertex in $\{2,3\} \cap Q_u$ (respectively $\{2,3\} \cap Q_v$). The image of Z_4 is the configuration variety $\overline{PtH} \times^H \overline{RuS} \times^S X(v')$

where $Q_{v'}$ is obtained from Q_v by removing the vertices smaller than the vertex 4. In particular, the fiber of $\widehat{\pi}$ above $X(w')$ has two components whose dimensions are given by:

Case	8.4 ; $Z_{\{2,3\}}$	8.4 ; Z_4	8.5 ; $Z_{\{2,3\}}$	8.5 ; Z_4
dim	2	3	2	3

The morphism $\widehat{\pi}$ is always *IH*-small.

7.5 Small resolutions

Let us now describe all *IH*-small resolutions of minuscule Schubert varieties whenever they exist. We use the following result of Totaro [Tot00] using a key result of Wisniewski [Wis91].

Theorem 7.8. *Any IH-small resolution of X is a small relative minimal model for X.*

Furthermore, because of Theorem 7.3, the morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ from any minimal model to $X(w)$ is IH-small so that we get the following corollary.

Corollary 7.9. *The IH-small resolutions of $X(w)$ are given by the morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \to X(w)$ obtained from Construction 3 with $\widehat{X}(\widehat{w})$ smooth.*

Let us now give a combinatorial description of these varieties. Let Q_v be a quiver associated to a minuscule Schubert variety $X(v)$ and i a vertex of Q_v.

Definition 7.10. The vertex i of Q_v is called *minuscule* if $\beta(i)$ is a minuscule simple root of the sub-Dynkin diagram of G defined by $\mathrm{Supp}(v)$.

Construction 3 gives a partition of the quiver Q_w into subquivers Q_{w_i} which are quivers of minuscule Schubert varieties having a unique peak. We have the following theorem.

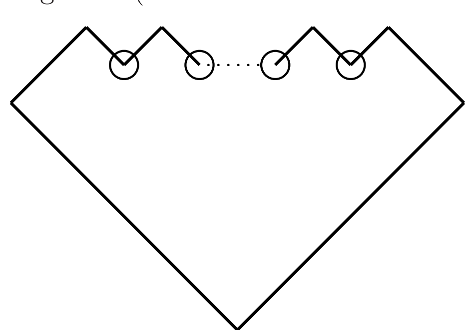
Theorem 7.11. *The variety $\widehat{X}(\widehat{w})$ obtained from Construction 1 is smooth if and only if, for all i, the unique peak p_i of Q_{w_i} is minuscule in Q_{w_i}.*

Proof. We have seen that the variety $\widehat{X}(\widehat{w})$ is a sequence of locally trivial fibrations with fibers Schubert varieties $X(w_i)$. Let us prove the following result.

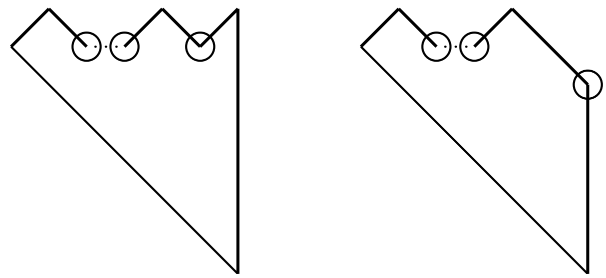
Proposition 7.12. *A minuscule Schubert variety $X(w)$ is smooth if and only if Q_w has a unique peak p and p is minuscule in Q_w.*

Proof. We know from [BP99] that a minuscule Schubert variety $X(w)$ is smooth if and only if it is homogeneous under its stabilizer. It is easy to verify that the quiver of any minuscule flag variety has a unique peak which is minuscule.

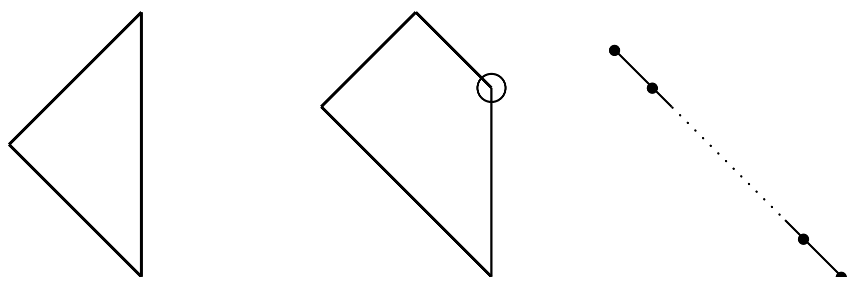
Conversely, according to Proposition 4.11, the variety is homogeneous under its stabilizer if and only if the quiver Q_w has no non-virtual hole. Now we have seen that for A_n the quiver of any Schubert variety is of the following form (we have circled the non-virtual holes of the quiver).



The only case where there is a unique peak is when there is no non-virtual hole. In this case the Schubert variety is isomorphic to a Grassmannian and hence is smooth. For the case of maximal isotropic subspaces (say associated to the simple root α_n with the notation of [Bou68]), the quiver is of the form (we have circled the non-virtual holes of the quiver)

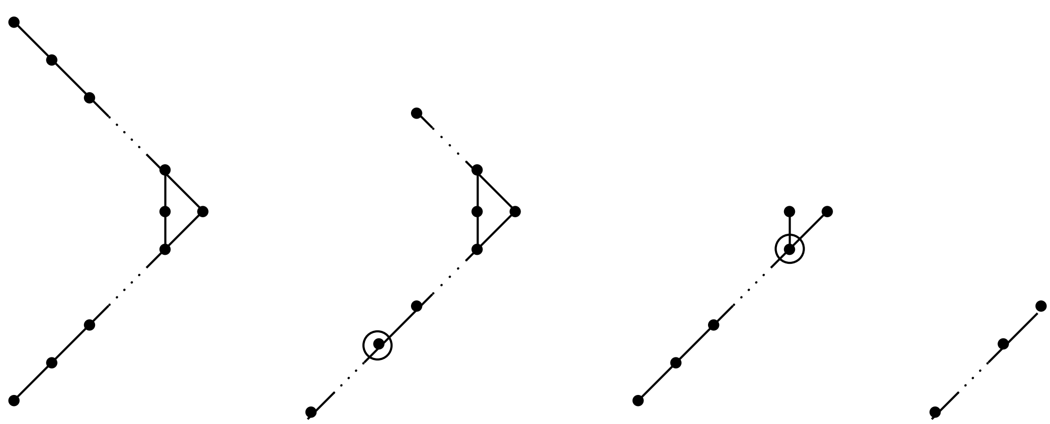


and there are three cases when there is a unique peak, as below.



In the second case, one of the two vertices i_{n-1} and i_n such that i_k is the smallest element (for $\preccurlyeq$) with $\beta(i_k) = \alpha_k$ with the notation of [Bou68] is a hole of the quiver. In the first case the quiver is the quiver of the isotropic Grassmannian and in the third case it is the quiver of a projective space.

For the quadric case, the quiver has one of the following four forms.



In the first and last cases we get respectively the quiver of a quadric or the quiver of a projective space. In the two intermediate cases, there is one hole in the quiver.

Finally, it is an easy verification on the quivers of E_6 and E_7 to check that Proposition 7.12 holds (cf. Appendix A). □

Theorem 7.11 follows directly from this proposition. □

7.6 Stringy polynomials

Another way of proving the non-existence of *IH*-small resolutions is the following. Any *IH*-small resolution is a relative minimal model (Theorem 7.8) and any relative minimal model factors through

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the relative canonical model (cf. [KMM87, Theorem 3-3-1]). In particular, any IH-small resolution $\pi : \widetilde{X} \to X$ of a variety X, if it exists, will give a resolution $\widetilde{\pi} : \widetilde{X} \to X_{\text{can}}$ of the relative canonical model X_{can} of X. The *IH*-smallness of π implies that $\widetilde{\pi}$ is crepant. We can thus use the 'stringy polynomial' $E(\widehat{X}, u, v)$ defined by Batyrev in [Bat98]. If X_{can} admits a crepant resolution then $E(\widehat{X}, u, v)$ (which in general is a formal power series) is a true polynomial. Thus to prove the non-existence of *IH*-resolution, it is enough to prove that $E(\widehat{X}, u, v)$ is not a polynomial.

Let us give an example where we make the full calculation. We first recall the following definitions (for more details and more general definitions, see [Bat98]).

Let X be a normal irreducible complex variety. We define

$$E(X, u, v) = \sum_{u,v} e^{p,q}(X) u^p v^q \quad \text{with } e^{p,q}(X) = \sum_i (-1)^i h^{p,q}(H^i_c(X, \mathbb{C})),$$

where $H^i_c(X, \mathbb{C})$ is the ith cohomology group with compact support and $h^{p,q}(H^i_c(X, \mathbb{C}))$ is the dimension of its (p, q)-type component. The polynomial $E(X, u, v)$ is what Batyrev calls the Euler polynomial (or E-polynomial).

Assume now that X is a Gorenstein normal irreducible variety with at worst terminal singularities. Let $\pi : Y \to X$ be a resolution of singularities such that the exceptional locus is a divisor D whose irreducible components $(D_i)_{i \in I}$ are smooth divisors with only normal crossings. We then have

$$K_Y = \pi^* K_X + \sum_{i \in I} a_i D_i \quad \text{with } a_i > 0.$$

For any subset $J \subset I$ we define

$$D_J = \begin{cases} \displaystyle\bigcap_{j \in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases} \quad \text{and} \quad D_J^o = D_J \Big\backslash \bigcup_{i \in I \setminus J} (D_J \cap D_i).$$

Definition 7.13. The stringy function associated with the resolution $\pi : Y \to X$ is the following:

$$E_{\text{st}}(X, u, v) = \sum_{J \subset I} E(D_J^o, u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1}.$$

Then Batyrev proves the following result.

Theorem 7.14. (i) *The function $E_{\text{st}}(X, u, v)$ is independent of the resolution $\pi : Y \to X$ with exceptional locus of pure codimension* 1 *given by smooth irreducible divisors with normal crossings.*

(ii) *If X admits a crepant resolution $\pi : Y \to X$ (that is to say $\pi^* K_X = K_Y$) then $E_{\text{st}}(X, u, v) = E(Y, u, v)$ and hence $E_{\text{st}}(X, u, v)$ is a polynomial.*

(iii) *In particular, if X admits a crepant resolution, the stringy Euler number*

$$e_{\text{st}}(X) = \lim_{u,v \to 0} E_{\text{st}}(X, u, v) = \sum_{J \subset I} e(D_J^o) \prod_{j \in J} \frac{1}{1 + a_j}$$

is an integer.

We now give an example of a minuscule Schubert variety which is singular non-locally factorial and does not admit an *IH*-small resolution.

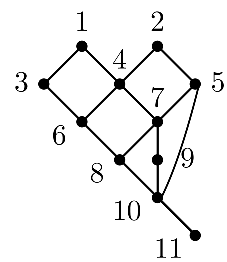
Example 7.15. Let G be $SO(12)$ and w given by the following reduced expression $\widetilde{w}$ (the symmetry s_i is the simple reflection associated to the ith simple root with the notation of [Bou68]):

$$w = s_2 s_4 s_1 s_3 s_6 s_2 s_4 s_3 s_4 s_4 s_6.$$

The associated Schubert variety is the following ($\mathbb{G}_{\mathrm{iso}}(k,12)$ is the isotropic Grassmannian, and we denote by $\mathbb{G}^1_{\mathrm{iso}}(6,12)$ and $\mathbb{G}^2_{\mathrm{iso}}(6,12)$ the flag varieties associated to the simple roots α_5 and α_6):

$$X(w) = \{V \in \mathbb{G}^2_{\mathrm{iso}}(6,12) \mid \dim(V \cap F_3) \geqslant 1 \text{ and } \dim(V \cap F_6) \geqslant 3\},$$

where $F_3 \in \mathbb{G}_{\mathrm{iso}}(3,12)$ and $F_6 \in \mathbb{G}^1_{\mathrm{iso}}(6,12)$. The quiver Q_w is as shown below.



The variety $X(w)$ admits the Bott–Samelson resolution $\widetilde{X}(\widetilde{w})$. Moreover, because the morphism $\pi : \widetilde{X}(\widetilde{w}) \to X(w)$ is B-equivariant, the exceptional locus has to be B-invariant and thus a union of Z_K. The unique non-contracted divisors Z_i of $\widetilde{X}(\widetilde{w})$ in $X(w)$ are Z_1 and Z_2. Furthermore, the variety $Z_{\{1,2\}}$ is not-contracted so that the exceptional locus D is the union

$$D = \bigcup_{i=3}^{11} Z_i.$$

All Z_i are smooth and intersect transversally. Denote by D_1 and D_2 the images of Z_1 and Z_2 in $X(w)$. The ample generator of the Picard group of $X(w)$ is given by $\mathcal{L} = D_1 + D_2$. We have

$$\pi^*\mathcal{L} = \sum_{i=1}^{11} Z_i.$$

Formulae of § 2.3.1 and Lemma 4.18 give us

$$-K_{\widetilde{X}(\widetilde{w})} = \sum_{i=1}^{11} (h(i)+1) Z_i$$

and Proposition 4.17 gives us

$$-K_{X(w)} = 7D_1 + 7D_2 = 7\mathcal{L}.$$

In particular, we have

$$\begin{aligned} K_{\widetilde{X}(w)} - \pi^* K_{X(w)} &= \sum_{i=1}^{11} (6 - h(i)) Z_i \\ &= (Z_3 + Z_4 + Z_5) + 2(Z_6 + Z_7) + 3(Z_8 + Z_9) + 4Z_{10} + 5Z_{11}. \end{aligned}$$

We remark that for $J \subset [3,9]$, the variety Z_J^o is a sequence of nine locally trivial fibrations in $\mathbb{A}^1$ or in points (there are exactly $|J|$ points) over $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we have $e(Z_J^o) = 4$ for all $J \subset [3,9]$.

Now we have the easy formula

$$\sum_{J \subset I} \prod_{j \in J} x_j = \prod_{i \in I} (1 + x_i).$$

We can thus calculate in our situation:

$$e_{\mathrm{st}}(X(w)) = 4(1+\tfrac{1}{2})^3(1+\tfrac{1}{3})^2(1+\tfrac{1}{4})^2(1+\tfrac{1}{5})(1+\tfrac{1}{6}) = \frac{105}{2}.$$

We conclude that $X(w)$ has no *IH*-small resolution as given by Theorem 7.11.

This kind of calculation can be generalized, and this will be done in a subsequent paper. For example, the same calculation in the general case where $X(w)$ is Gorenstein (or equivalently all the peaks $p \in \mathrm{Peaks}(Q_w)$ have the same height $h(w)$) gives the following result. Let us define for $i \in Q_w$ its coheight $\mathrm{coh}(i) = h(w) - h(i)$. Then we have

$$e_{\mathrm{st}}(X(w)) = \prod_{i \in Q_w} \left(1 + \frac{1}{1 + \mathrm{coh}(i)}\right).$$

Acknowledgements

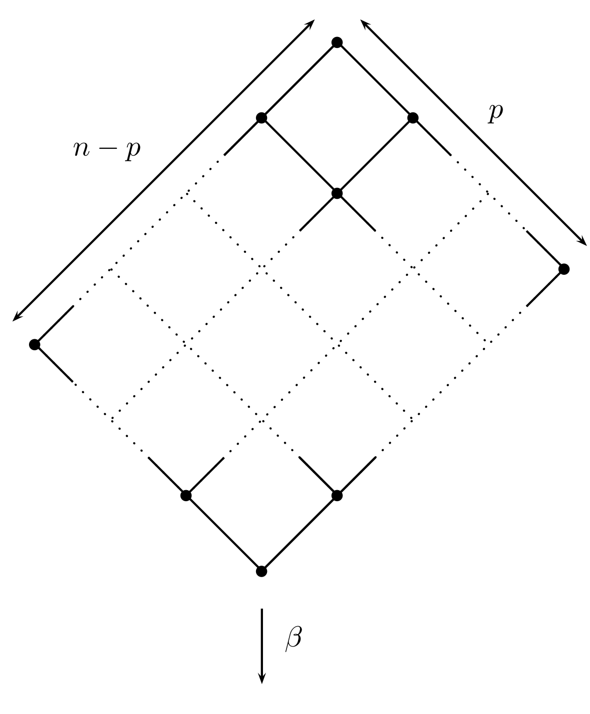
I would like to thank Michel Brion for his careful reading of preliminary versions of this paper and for many comments and improvements.

Appendix A

In this appendix we give the quivers of minuscule flag varieties and describe those of minuscule Schubert varieties. Note that we do not draw the arrows on the edges: all arrows are going down.

A.1. Quivers of minuscule flag varieties

The following quiver is the quiver of the Grassmannian of p-dimensional subvector spaces of an n-dimensional vector space. The morphism β associating to any vertex a simple root is simply the vertical projection on the Dynkin diagram.



It is easy to verify that this diagram satisfies the geometric conditions of Proposition 4.1 so that it corresponds to a Schubert variety of dimension $p(n - p)$ of the Grassmannian. It must be the quiver of the Grassmannian.

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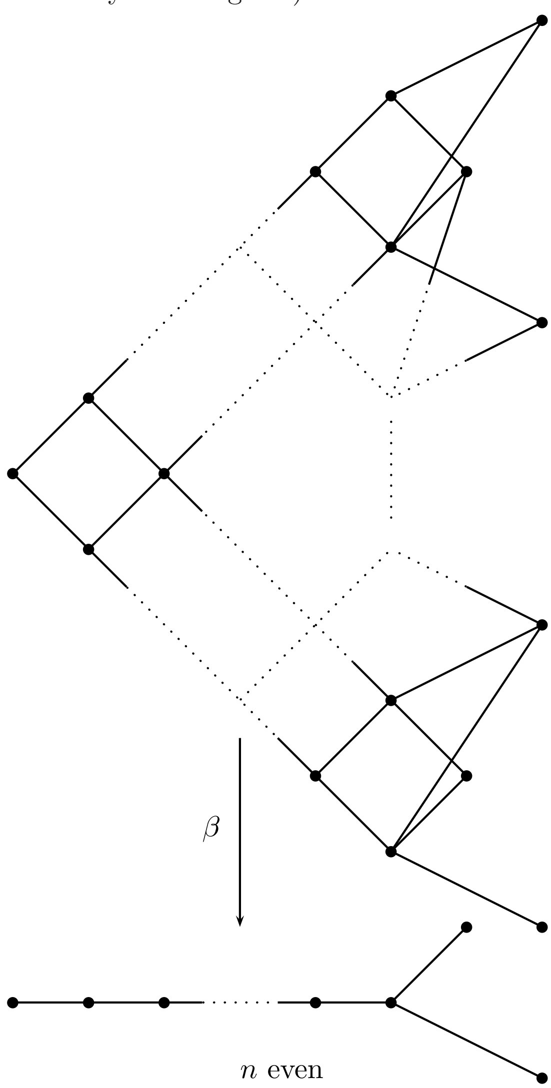
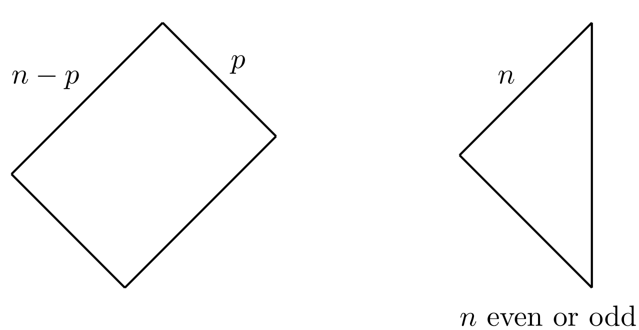
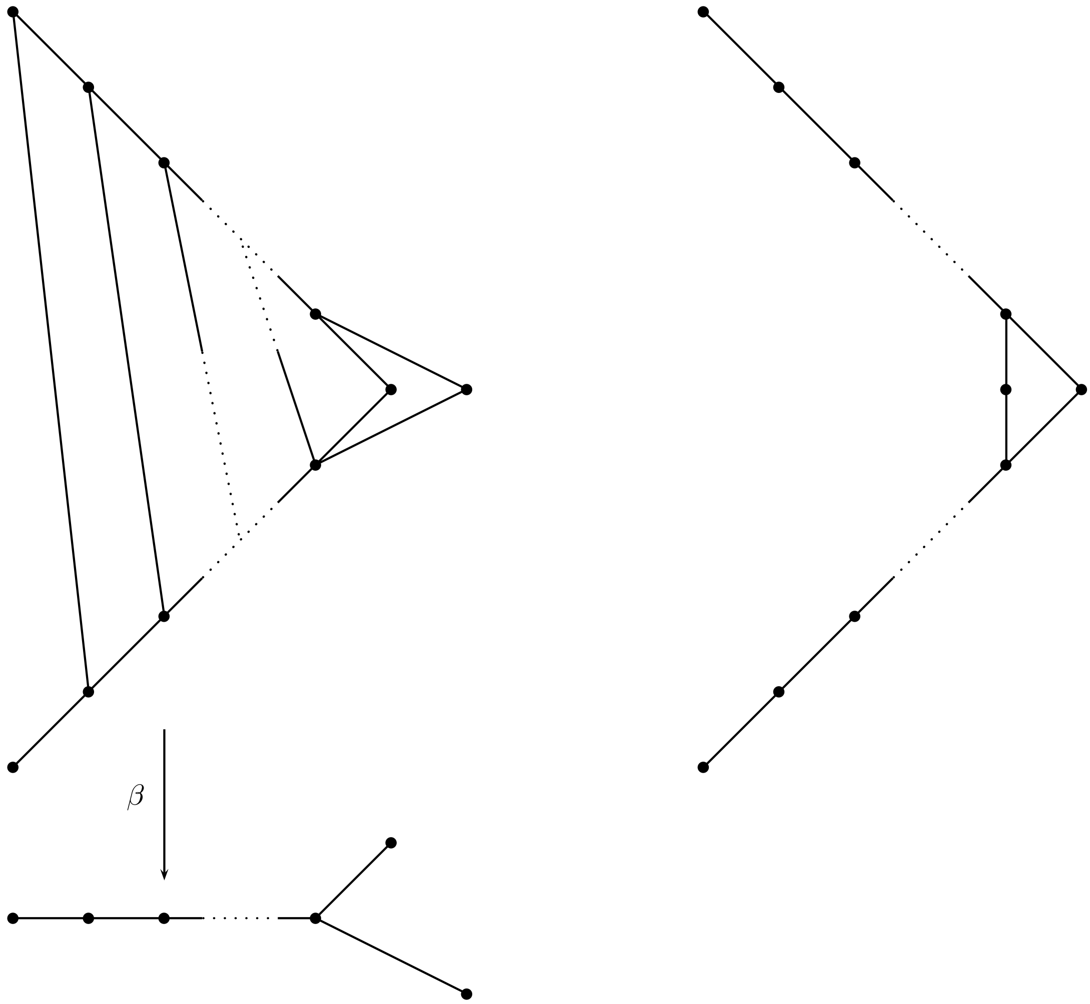
In the same way, the quiver of the Grassmannian of maximal isotropic subspaces in a $2n$-dimensional vector space endowed with a non-degenerate quadratic form is given by one of the following forms depending on the parity of n (the morphism β is again given by the vertical projection on the Dynkin diagram).



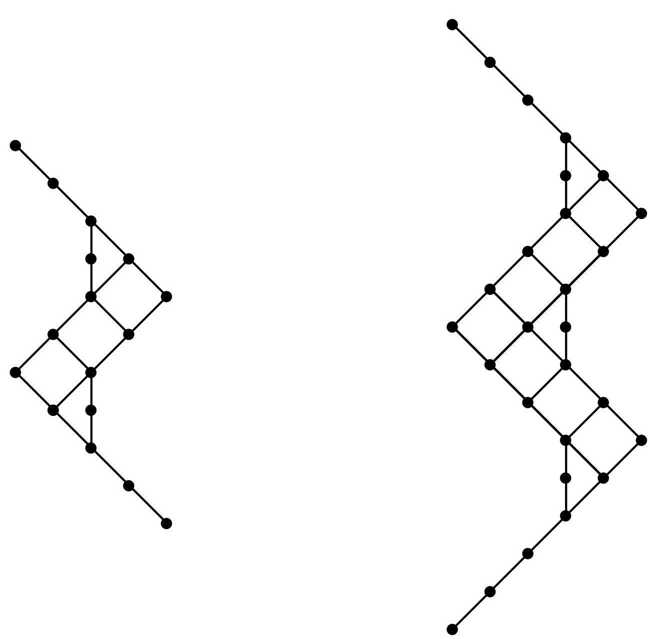
In the text, we have used the following schematic version of the quivers of the Grassmannian and of that of maximal isotropic subspaces.



For the even-dimensional quadrics, we get the quiver on the left of the following picture. On the right of this picture, we have drawn the same quiver without the arrows $i \to j$ between two vertices such that $h(i) - h(j) > 1$. One can easily recover one quiver from the other.



Finally for E_6 and E_7 we only draw below the simplified quivers where the arrows $i \to j$ between two vertices such that $h(i) - h(j) > 1$ have been removed.



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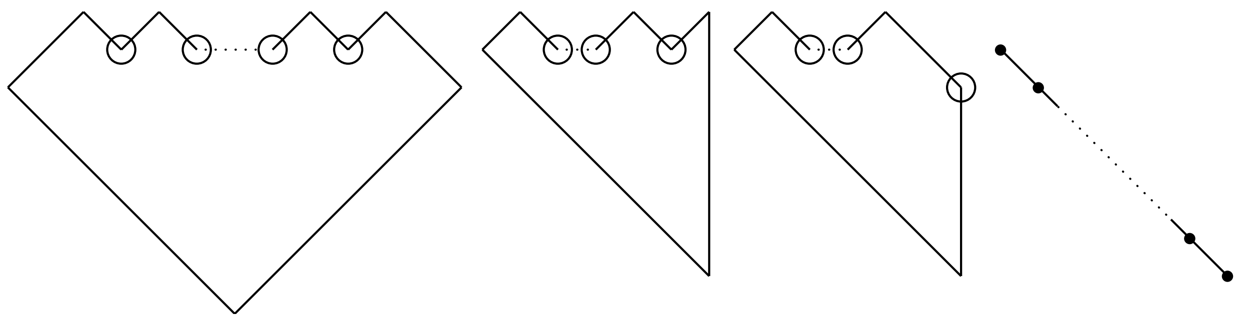
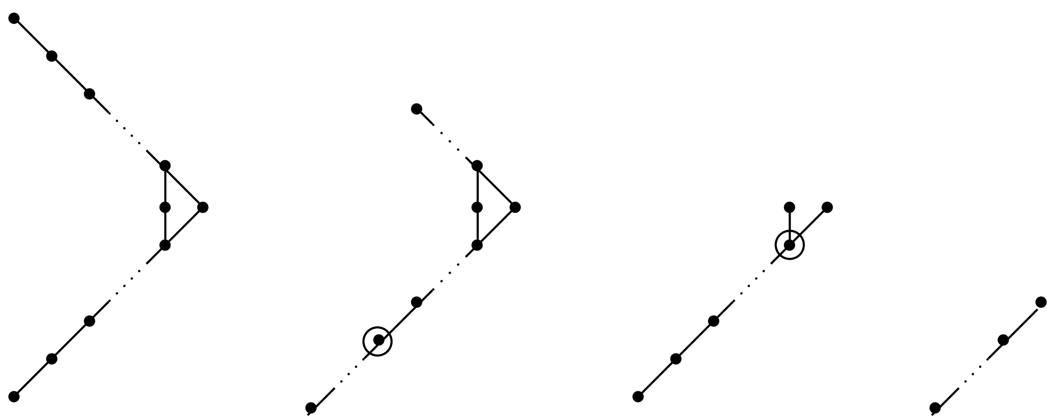
A.2. Quivers of minuscule Schubert varieties

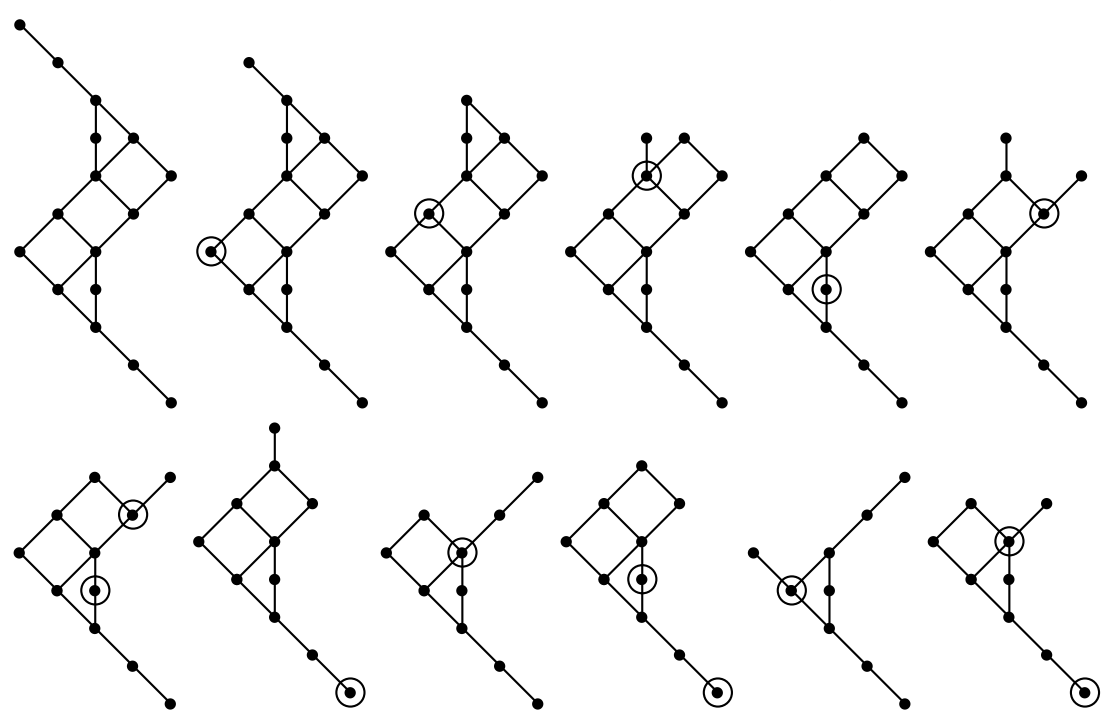
Thanks to the description of the quivers of minuscule flag varieties and Proposition 4.5, we know that the quiver of a minuscule Schubert variety is of the following form (we have circled the successors of elements in the set A described in Proposition 4.5).



For the quadrics we have the following forms.



Finally for E_6 and E_7 we give below a complete list of the quivers (except the empty quiver).



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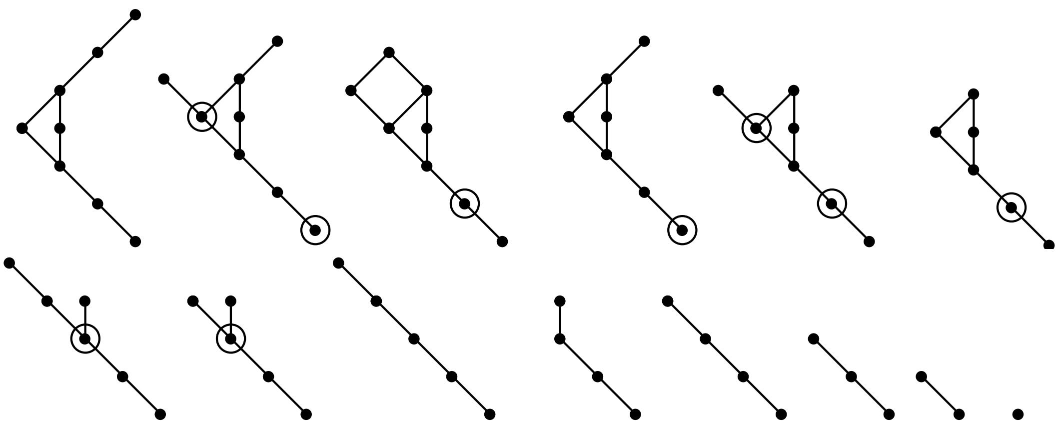
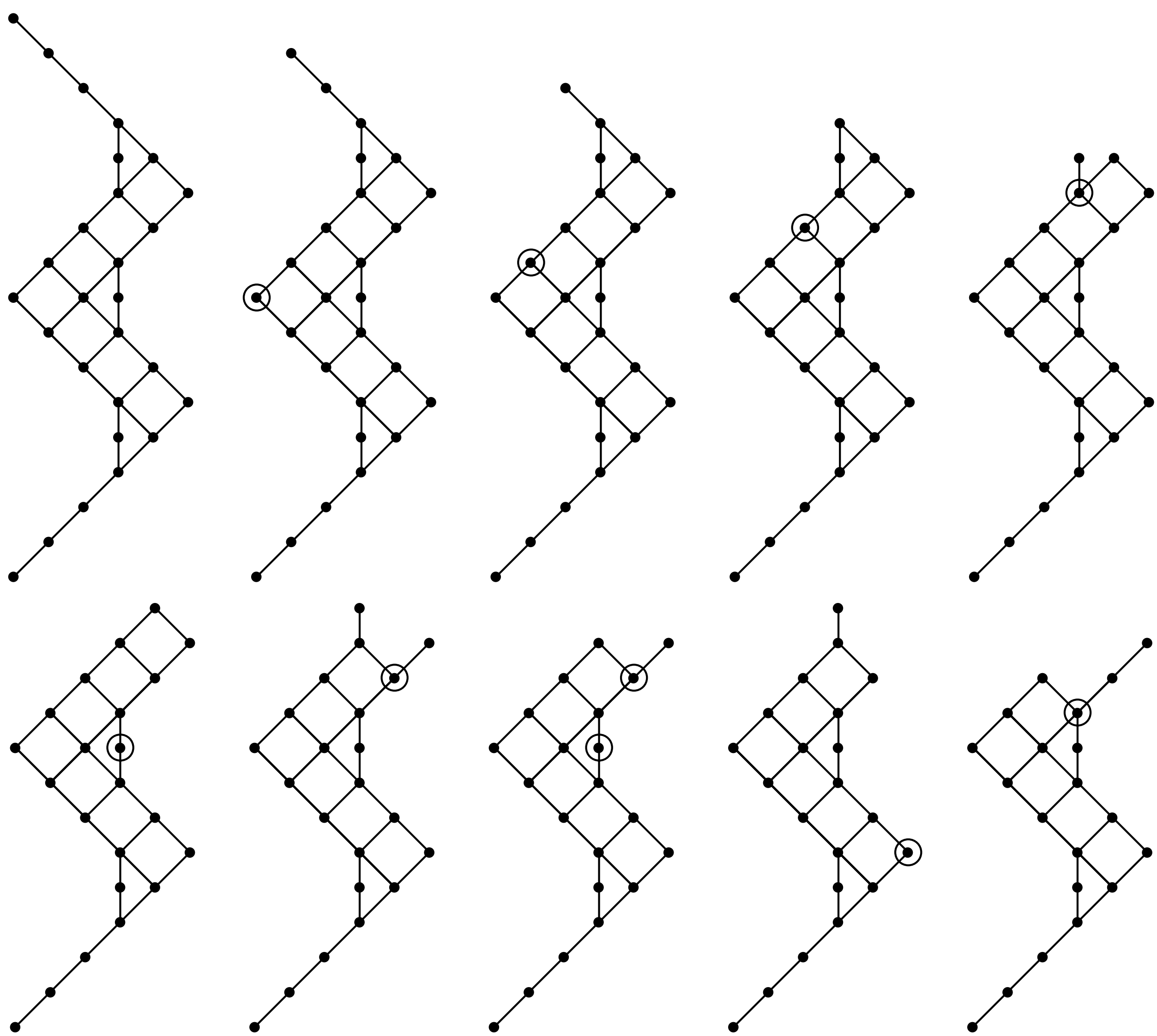
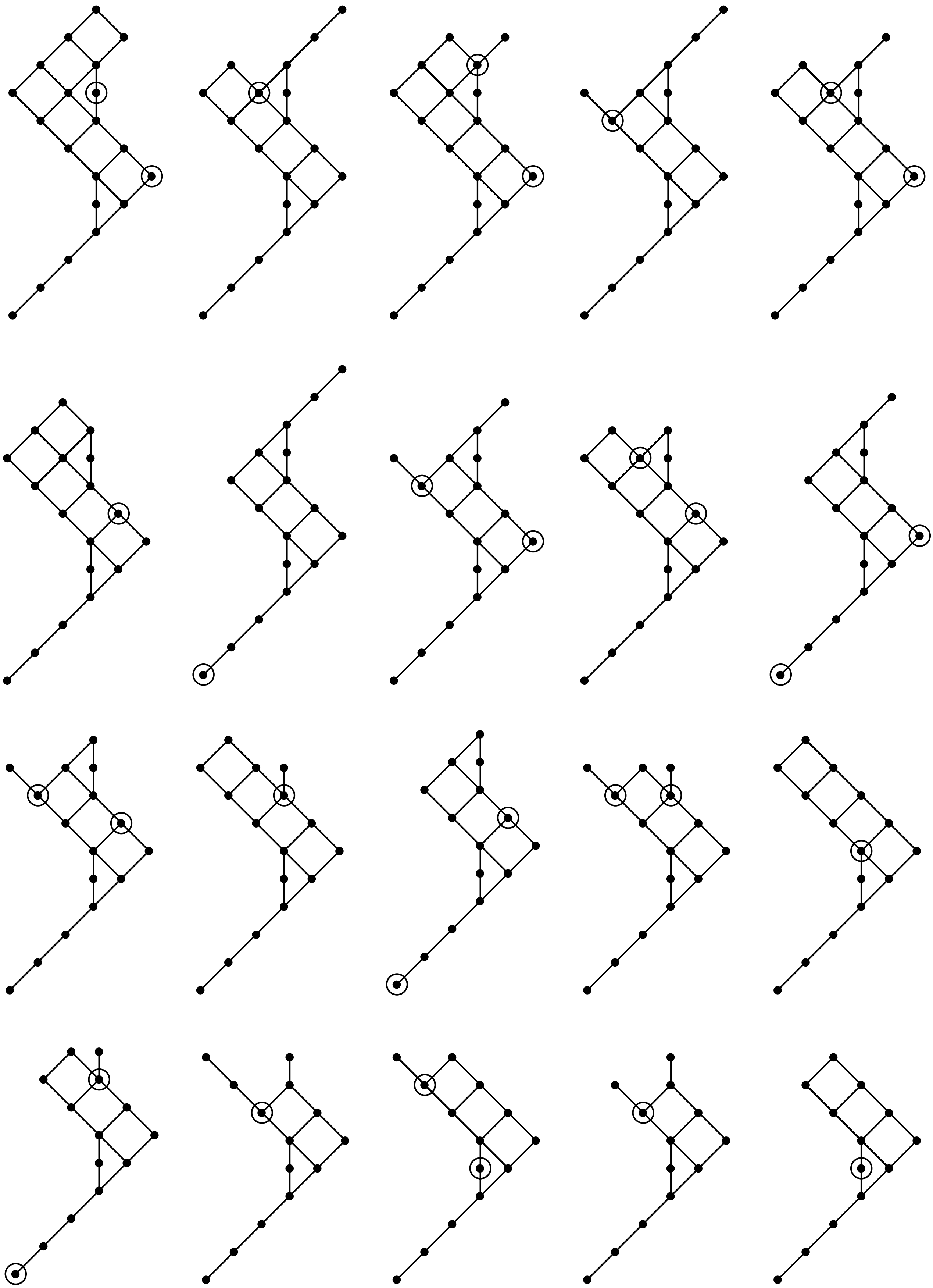


It is easy to verify (thanks to our results) that the only Schubert varieties admitting a *IH*-small resolution are the following (we only list their number in the previous list): 1, 7, 9, 11, 13, 17, 19, 20, 21, 22, 23, 24, 25, 26 and the zero-dimensional one.

We now list below the Schubert varieties for the E_7 case.



https://doi.org/10.1112/S0010437X07002734 Published online by Cambridge University Press



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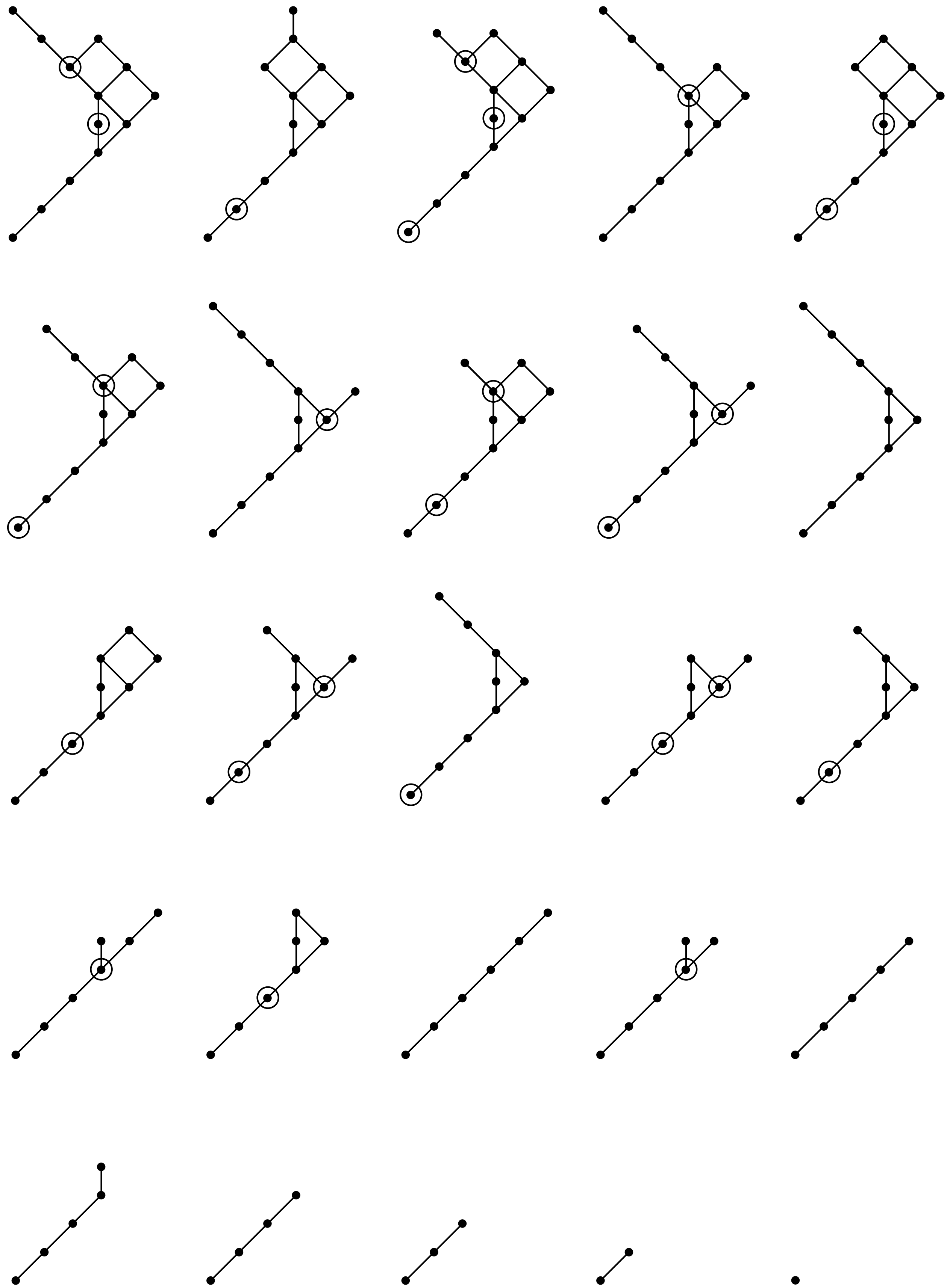

https://doi.org/10.1112/S0010437X07002734 Published online by Cambridge University Press

Likewise, one checks that the only Schubert varieties admitting a *IH*-small resolution are the following (we only list their number in the previous list): 1, 24, 27, 28, 31, 34, 37, 40, 44, 46, 48, 49, 50, 51, 52, 53, 54, 55 and the zero-dimensional one.

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Nicolas Perrin nperrin@math.jussieu.fr
Université Pierre et Marie Curie, Paris 6, UMR 7586, Institut de Mathématiques de Jussieu, 175 rue du Chevaleret, 75013 Paris, France