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POOR MODULES: THE OPPOSITE OF INJECTIVITY

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Abstract. A module M is called poor whenever it is N-injective, then the module N is semisimple. In this paper the properties of poor modules are investigated and are used to characterize various families of rings.

Warmly dedicated to Patrick F. Smith on the occasion of his 65th birthday.

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1. Introduction. The purpose of this paper is to initiate the study of *poor* modules and their various related concepts. As usual, Mod-R denotes the category of all right modules over a ring R and $SSMod$-R the class of all semisimple right R-modules. Following [**1**], for any $M \in Mod$-R, we denote by $\mathfrak{In}^{-1}(M)$ the class $\{N \in Mod\text{-}R : M \text{ is } N\text{-injective}\}$. Clearly, M is injective if $\mathfrak{In}^{-1}(M) = Mod$-$R$. In other words, M is injective if its injectivity domain is *as large as it can be*. In this paper, we are interested in the opposite situation. We will focus on modules that have a domain of injectivity which is as small as possible. We will refer to such modules as being 'poor' (as opposed to injective modules, which are 'wealthy' in terms of their injectivity domains). It is easy to see that if $M \in Mod$-R and $N \in SSMod$-R, then $N \in \mathfrak{In}^{-1}(M)$. In fact, even more is true. Proposition 3.1 shows that $\bigcap_{M\in Mod\text{-}R} \mathfrak{In}^{-1}(M) = SSMod$-$R$. For that reason, it makes sense to define a module M to be 'poor' if for every $N \in Mod$-R, M is N-injective only if N is semisimple.

Immediately several questions come to mind. For example, can the limit in Proposition 3.1 be attained? In other words, does every ring R have (at least) one poor module? In the hypothetical case of a ring R that does not have any poor right modules, we say that R is a right *utopia*. In Section 3 we consider when a ring itself or the direct sum of its simple modules may be poor.

Given a family $\mathcal{A}$ of modules, possibly the easiest question one may ask is what happens when $\mathcal{A}$ contains at least one poor module. The significance of such an

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event should not be underestimated; for example, if a ring R has a nonsingular poor module, then R is an SI-ring in the sense that all singular R-modules are injective (Proposition 3.7) Along these lines, Section 4 explores the significance of the existence of semisimple projective poor modules.

Another line of enquiry is considering when all modules of the family $\mathcal{A}$ are poor (a situation we describe with the expression that $\mathcal{A}$ is *destitute*). Section 5 focuses on simple-destitute rings (i.e. the case when $\mathcal{A} = \{\text{simple modules}\}$ is destitute.) Alternatively, one may investigate the significance of $\mathcal{A}$ not having any poor modules at all (a situation we denote as $\mathcal{A}$ being a *utopia*). Note that the ring R is a right utopia ring if and only if the class $\mathcal{A} = Mod\text{-}R$ is a utopia. As before, the expressions R is destitute and R has no middle-class mean that the class $\mathcal{A} = Mod\text{-}R$ has those properties. Also, one may consider the possibility that all modules of $\mathcal{A}$ are either injective or poor (a situation we refer to as $\mathcal{A}$ *not having a middle class*). Such questions are among those considered in Sections 4, 5 and 6.

Clearly, a module M is poor if and only if for every cyclic module $xR \in \mathfrak{In}^{-1}(M)$, xR is semisimple. This fact will be used freely throughout the paper.

2. Definitions, notation and preliminary remarks. In this paper, all rings have an identity and all modules are right and unital. Our terminology and notation adheres to that of the major references in the theory of rings and modules such as [**1**]. Other good references are [**6**], [**11**] and [**20**]. We here highlight a few specific facts, notation and terminology because they have been used in this paper. The socle, Jacobson radical and singular submodules of a module M will be denoted as is customary by $Soc(M)$, $J(M)$ and $Z(M)$, respectively.

While, due to a classic result of Osofsky, a ring for which all cyclic modules are injective must be semisimple artinian, a slight modification of this definition asking only for proper cyclic modules to be injective yields a larger family. A ring R is said to be right PCI ring if every cyclic module, which is not isomorphic to R, is injective. The notion of right PCI rings was introduced by Faith in ref. [**5**]. Right PCI domains play a central role in the study of right QI rings (those rings in which all quasi-injective modules are injective). For example,it has been shown that every hereditary right QI ring is Morita equivalent to a right PCI domain. Right PCI domains and right QI rings are examples of right V-rings, namely, rings for which all simple modules are injective. The notion of right V-rings may be generalized to that of right GV-rings (Generalized V-rings), in which every simple module is either injective or projective. Right GV rings were introduced in [**10**] and are discussed in [**4**]. References on these various related topics include [**2, 3, 4, 7, 12**].

Yet another concept of interest is that of a right SI-ring. A ring R is right SI if every singular module is injective. It is shown in Proposition 3.1 of [**9**] that a ring R is right SI if and only if every singular module is semisimple. This result is of importance for our Proposition 3.7. Furthermore, if R is a domain, then the notion of right PCI and right SI are indeed equivalent. Right PCI domains are right Öre and, therefore, in particular, every proper cyclic R-module is singular.

We recall that a module U is called uniserial if it has a unique composition series of finite length. Furthermore, a module is generalized uniserial (or serial) if it is a direct sum of finitely many uniserial modules. A ring R is said to be right generalized uniserial if the module R_R is generalized uniserial. The following lemma summarizes various results from [**16**] and [**17**] that are instrumental in our search for poor modules over hereditary Noetherian prime rings in Proposition 3.4. A related and useful reference is [**18**].

LEMMA 2.1.
(a) ([**16**, Lemma 1]) *Any finitely generated torsion module over a hereditary Noetherian prime ring R is a direct sum of finitely many uniserial modules.*
(b) ([**16**, Lemma 2, part(i)]) *If in an R-module M, an element x is a torsion element, then xR is a torsion submodule with nonzero annihilator.*
(c) ([**16**, Theorem 1]) *Let R be a generalized uniserial ring. Then every R-module is a direct sum of uniserial modules.*
(d) ([**17**, Theorem 1]) *Every proper factor ring of a hereditary Noetherian prime ring is generalized uniserial.*

Uniserial modules of length two are conspicuous in the paper. Therefore, it seems reasonable to highlight the following fact.

REMARK 2.2. Given a ring R, the following are equivalent:
(a) There exists a uniserial R-module U of length two
(b) There exists a local module L with simple radical.
(c) There exists a simple module S which is not injective such that $Soc(E(S)/S) \neq 0$.

A semiperfect ring R has an indecomposable decomposition $R = \bigoplus_{i=1}^{n} e_iR$ such that for every i, e_iR/e_iJ is simple. In that case, $\{e_iR/e_iJ | i = 1 \dots n\}$ is a complete list of simple modules (up to isomorphism) and for every i, j, $e_iR/e_iJ \cong e_jR/e_jJ$ if and only if $e_iR \cong e_jR$. In other words, when R is semiperfect, it is made up of projective indecomposables that are uniquely determined by their tops.

A couple of elementary properties of poor modules are worth mentioning; their proofs are left as an easy warm-up exercise for the reader. The first remark points out that the conditions '*all modules are wealthy*' and '*all modules are poor*' are equivalent.

REMARK 2.3. For an arbitrary ring R, the following conditions are equivalent:
(a) R is semisimple artinian.
(b) Mod-R is destitute.
(c) There exists an injective poor module E.
(d) $\{0\}$ is a poor module.

The second remark serves, in particular, to reject the notion that direct summands of poor modules must necessarily be poor.

REMARK 2.4. If M is a poor module, then for all $N \in Mod$-R, $M \oplus N$ is poor.

3. Do all rings have poor modules?.

We begin with the following proposition which justifies our definition of a poor module and establishes that $SSMod$-R is a *limit* for the domains of injectivity of the modules over a ring R.

PROPOSITION 3.1. $\bigcap_{M \in Mod\text{-}R} \mathfrak{In}^{-1}(M) = SSMod\text{-}R$.

Proof. Clearly, $SSMod\text{-}R \subset \bigcap_{M \in Mod\text{-}R} \mathfrak{In}^{-1}(M)$. Now, let N be a element of $\bigcap_{M \in Mod\text{-}R} \mathfrak{In}^{-1}(M)$ and T be a submodule of N. Then T is N-injective. So, T is a direct summand of N and, hence, N is semisimple. So $N \in SSMod$-R, and this completes the proof. □

The obvious question now is whether, given an arbitrary ring R, the limit $SSMod$-$R$is always attained by some module over R; in other words, whether poor modules over arbitrary rings do indeed exist.

Let us start with a situation in which the regular module R_R is itself poor.

PROPOSITION 3.2. *Let R be a right PCI domain. Then R has no middle class and R_R is a poor module.*

Proof. If R is a division ring, then the result follows by Remark 2.3. Assume that R is not a division ring, then the only cyclic R-modules that are not injective are those that are isomorphic to R. The injective cyclics, on the other hand, are all semisimple. So, if an R-module M is not injective, then it is poor. □

Next we show that our search for utopia cannot start with right artinian rings.

THEOREM 3.3. *No right artinian ring is a right utopia; in fact, over a right artinian ring R, it is always the case that the cyclic module $M = R/J$ is poor.*

Proof. Let $M = R/J$ and $N = aR$ be a nonzero cyclic module in $\mathfrak{In}^{-1}(M)$. Let L_1 be a simple submodule of aR. Then L_1 is isomorphic to a submodule S_0 of M and so S_0 is aR-injective. But then $aR = L_1 \oplus K_1$ for some submodule K_1 of aR. If $K_1 = 0$, then aR is semisimple. Otherwise, let L_2 be a simple submodule of K_1. Arguing as before, $K_1 = L_2 \oplus K_2$ and so $aR = L_1 \oplus L_2 \oplus K_2$, where $L_1 \oplus L_2$ is semisimple and $K_1 \supseteq K_2$. This process must stop after finite steps. Hence, aR is semisimple. Thus, R/J is poor. □

In light of Theorem 3.3, as we ponder further the existence of poor modules, it makes sense to focus on semisimple modules $M \in SSMod$-R that contain at least one copy of each simple R-module. We will consider the case when R is a hereditary Noetherian prime ring R. In light of Remark 2.4, this is strongly related to considering when the specific module M that is the direct sum of exactly one member from each isomorphism class of simple modules is poor.

PROPOSITION 3.4. *Let R be a hereditary Noetherian domain and let M be a semisimple module that contains exactly one copy of each simple R-module. Then M is either poor or injective. In particular, if R has only one simple module (up to isomorphism), then that module is either injective or poor. It also follows that for a ring R and module M satisfying these hypotheses, M is poor unless R is a V-ring.*

Proof. Let R be hereditary and Noetherian; and let M be as given in the Proposition. Assume that M is not injective. Let $xR \in \mathfrak{In}^{-1}(M)$. Because M is not injective, then $ann_R(x)$ is nonzero. Then by (a) and (b) of Lemma 2.1, xR is serial. So, write $xR = U_1 \oplus \cdots \oplus U_n$, where each U_i is uniserial. Then M is U_i-injective for each i. Next we show that each U_i is simple, for if U_i is not simple, then it contains a simple submodule, say $S < U_i$. But then the embedding map of S into M could be extended to a monomorphism of U_i into M, which would be a contradiction. Hence, U_i must be simple. It follows that xR is semisimple. □

COROLLARY 3.5. *Let R be a hereditary Noetherian domain. If there exists a non-simple and nonzero uniserial module U, then every semisimple module M that contains every simple R-module is poor.*

Proof. Assume that M is injective and U is a non-simple uniserial R-module. Then U contains a simple submodule, say $S < U$ is simple. It is clear that S is embedded in M, and hence S is U-injective. But then S is a direct summand of U, a contradiction. □

The following example illustrates that both possibilities in Proposition 3.4 are indeed possible.

EXAMPLE 3.6.
(i) Let $R = \mathbb{Z}$, the ring of integers. Then $M = \oplus_{\text{p is prime}}(\mathbb{Z}/p\mathbb{Z})$ is a poor $\mathbb{Z}$-module, while no proper summand of M is poor.
(ii) Let R be a right PCI domain. Then $\oplus_{P \subset_{max} R}(R/P)$ is injective. Notice however that, in this case, R_R itself is poor by Proposition 3.2 and, therefore, this is not an example of a right utopia ring.

We close this section with an interesting proposition.

PROPOSITION 3.7. *If a nonsingular ring R has a nonsingular poor module, then R is an SI-ring.*

Proof. Let M be a nonsingular poor R-module. As all singular modules belong to $\mathfrak{In}^{-1}(M)$, they are semisimple. The result then follows from Proposition 3.1 of [**9**]. □

4. No middle class: Families of modules where every module is either poor or injective.

Let $\mathcal{A}$ be a class of R-modules. Then we say R has no $\mathcal{A}$-*middle class* if every element of the class $\mathcal{A}$ is either poor or injective. In particular, if $\mathcal{A} = Mod$-R (= the class of simple modules, the class of projective modules etc.) we say that R has no *middle class*(*simple-middle class, projective-middle class* etc.).

THEOREM 4.1. *Let R be a ring such that $J(R)$ is a simple and essential right ideal of R. If you further assume that $R/J(R)$ is semisimple, then R has no middle class. In particular, $J(R)$ is a poor R-module.*

Proof. Let M be an R-module which is not injective and aR be a cyclic module. If M is aR-injective, then $ann_R(a)$ is a nonzero right ideal because M is not injective. So $J(R) \subset ann_R(a)$ because $J(R)$ is essential and simple. As $R/J(R)$ is semisimple, so is its quotient module aR. □

EXAMPLE 4.2.
(i) A right PCI domain has no middle class.
(ii) A right V-ring has no simple-middle class.
(iii) A QF ring R has no projective-middle class.
(iv) If $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is the upper triangular matrix ring over a field F, then $S = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ is a simple, projective and poor R-module. The only other simple R-module is $\frac{R}{\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}}$, which is injective. Hence, R has no simple-middle class.
This example motivates the next three results and is explained in Corollary 4.5.

Two modules are called *orthogonal* if they have no nonzero isomorphic submodules [**13**]. The following is a key lemma to obtain other results.

THEOREM 4.3. *Let M be a projective semisimple poor module. Then any semisimple module B orthogonal to M is injective.*

Proof. We prove that for every $X \subseteq E(B)$, $Hom(X, M) = 0$. Thus M is $E(B)$-injective and therefore $E(B) = B$. Let X be a submodule of $E(B)$ and let f be a homomorphism from X into M. As $f(X)$ is projective, we get that $X = Y \oplus Kerf$, with $Y \cong f(X)$. In order to show that $f(X) = 0$, we will first show that $f(X \cap B) = 0$. If $f(X \cap B) \neq 0$, then, being a projective submodule of M, it follows that $X \cap B \cong f(X \cap B) \oplus (Kerf \cap (X \cap B))$. This would contradict the hypothesis that B and M are orthogonal. So $f(X \cap B) = 0$ and hence, $X \cap B \subset Kerf$. As $X \cap B <_e X, f(X) = 0$. So M is $E(B)$-injective. Because M is poor, $E(B)$ is semisimple and so $E(B) = B$. This completes the proof. □

COROLLARY 4.4. *Let R be an arbitrary ring. If R has a simple projective poor module M, then R is a GV-ring.*

COROLLARY 4.5. *Let R be a ring which is not semisimple artinian. If there is a simple projective poor module M, then*

(a) *every direct sum of simple injective modules is injective,*
(b) *R has no simple-middle class.*

Recall that a ring R is called right Kasch if every simple R-module is embedded in R.

THEOREM 4.6. *If a right Kasch ring R has a nonzero semisimple projective poor R-module, then R is semisimple artinian.*

Proof. It is enough to show that every maximal right ideal is a direct summand of R. By Corollary 4.4, every minimal right ideal is either projective or injective. Let T be a maximal right ideal. Then R/T is either injective or projective. If R/T is projective, then we are done. So, assume that R/T is injective. Then R/T is isomorphic to a minimal right ideal S because R is right Kasch and S is a direct summand of R. But then T is also a direct summand because R/T is projective. □

THEOREM 4.7. *Let R be a semiperfect ring. If R has a projective simple poor module, then $R = R_1 \oplus R_2$, as rings direct sum, where R_1 is semisimple artinian and R_2 is semiperfect with projective poor homogeneous socle.*

Proof. Let S be a projective simple poor module. As R is semiperfect, there are local idempotents e_i, $i \in F$, such that F is a finite set and $R = \oplus_{i \in F} e_i R$. Let R_1 be the sum of $e_i R$ that are sums of injective minimal right ideals. Then R_1 is semisimple because each such e_i is a local idempotent. Let R_2 be the sum of the remaining $e_i R$. Then $R = R_1 \oplus R_2$, is a ring direct sum. Moreover, R_2 is a ring with nonzero socle and all minimal right ideal of R_2 are isomorphic, projective and poor (by Lemma 4.3). □

As a result from Theorem 4.7 and Corollary 4.4, we get the following corollary.

COROLLARY 4.8. *If there is a projective semisimple poor R-module M, then $Soc(R)$ is projective. Indeed the socle of any projective R-module under this hypothesis is projective.*

Proof. Let S be a minimal right ideal and so it is either projective or injective. If S is injective, then S is direct summand of R and so S is also projective. Therefore, all minimal right ideals are projective and so $Soc(R)$ is also projective. The second conclusion is clear from the first one. □

THEOREM 4.9. *A semiprime ring with a finite right uniform dimension and a projective simple poor module is semisimple artinian.*

Proof. As R has a projective simple poor module, it has a minimal right ideal summand, which is projective and poor. Write $R = M_1 \oplus S_1$, where M_1 is a maximal right ideal and S_1 is a minimal right ideal, which is projective and poor. If $Hom_R(M_1, S_1) = 0$, then we claim that S_1 is M_1-injective. It is enough to show that $Hom_R(X, S_1) = 0$ for any submodule X of M_1. Assume to the contrary that there exists a nonzero homomorphism from X onto S_1. As S_1 is projective, $X = X^* \oplus S^*$, where S^* is isomorphic to S_1. As S^* is a minimal right ideal of R and R is semiprime, $S^* = eR$, where e is an idempotent element of R. But then S^* is a direct summand of M_1 and so $Hom_R(M_1, S_1) \neq 0$, a contradiction. Hence, the only homomorphism from any submodule of M_1 into S_1 is the zero homomorphism. Therefore, S_1 is M_1-injective. In this case M_1 is semisimple because S_1 is poor and we are done. If $Hom_R(M_1, S_1) \neq 0$, then M_1 has a summand which is projective, poor and simple; say $M_1 = M_2 \oplus S_2$, then $R = M_2 \oplus S_2 \oplus S_1$. Repeating the same argument to the decomposition of $M_1 = M_2 \oplus S_2$, we either have Ras semisimple artinian or $M_2 = M_3 \oplus S_3$, where S_3 is projective, poor and simple. In the later case $R = M_3 \oplus S_3 \oplus S_2 \oplus S_1$. Continuing this process, we must stop at a point where R is semisimple artinian becuase R has a finite uniform dimension. $\square$

5. Destituteness: When all modules in a family are poor. Let $\mathcal{A}$ be a class of R-modules. Then R is called $\mathcal{A}$-*destitute ring* if every element of class $\mathcal{A}$ is poor. In particular, if $\mathcal{A} = Mod$-R (= the class of simple modules, the class of projective modules etc.), we say that R is *destitute*(*simple-destitute ring, projective-destitute ring* etc.).

EXAMPLE 5.1.

(i) Let $R = \mathbb{Z}/4\mathbb{Z}$ be the ring of integers modulo 4. Then $S = 2\mathbb{Z}/4\mathbb{Z}$ is the only minimal ideal of R, S is poor and every simple $\mathbb{Z}/4\mathbb{Z}$-module is isomorphic to $2\mathbb{Z}/4\mathbb{Z}$. So, $\mathbb{Z}/4\mathbb{Z}$ is a simple-destitute ring.

(ii) More generally, let $m = p_1^{e_1} \cdots p_n^{e_n} \in \mathbb{Z}$. Then $\mathbb{Z}/m\mathbb{Z}$ is simple-destitute if and only if $n = 1$ or $e_i = 1$ for all i.

(iii) Let R be a semiperfect ring. It is known that any projective R-module is a direct summand of indecomposable projective modules. Then R is a projective-destitute ring if and only if all projective indecomposable R-modules are poor. If this hold good, then R is poor as a right R-module.

(iv) Let R be a local ring. Then R is poor if and only if R is a projective-destitute ring.

THEOREM 5.2. *If a right artinian ring R has only one simple module (up to isomorphism), then that module is poor (and thus R is a simple-destitute ring.)*

Proof. Suppose R satisfies the hypotheses in the statement. Let M be a simple and N-injective module for some nonzero cyclic module $N = aR$. Since aR contains a simple submodule L_1 which is isomorphic to M, let φ denote an isomorphism between L_1 and M. Then φ extends to a homomorphism ψ from aR onto M. Thus, $\psi(aR) = M$. Hence, $aR = L_1 \oplus K_1$ for some submodule K_1 of aR. If K_1 is semisimple, then the proof is complete because L_1 is simple. Otherwise, K_1 has a simple submodule $L_2 \cong M$ and, using the same argument, $K_1 = L_2 \oplus K_2$ and $aR = L_1 \oplus L_2 \oplus K_2$, where $L_1 \oplus L_2$ is semisimple. Continuing this process, we must stop at a point where aR is semisimple because aR has a finite uniform dimension. This completes the proof. □

It is well known that a simple module is either projective or singular. In the following theorem, we show that all simple modules over a simple-destitute ring are not projective unless the ring is semisimple artinian.

THEOREM 5.3. *Let R be a simple-destitute ring which is not semisimple artinian. Then every simple module is singular.*

Proof. Let R be a simple-destitute ring which is not semisimple artinian. Then R has no injective simple poor R-module. Furthermore, we will show that no simple module is projective. If it were the case that there is a projective simple R-module, then by Lemma 4.3, all simple modules would be (isomorphic to one another and) projective. Take a maximal ideal T of R so that R/T is projective. It follows that T is a direct summand of R. This means that every maximal right ideal is direct summand and so R is semisimple artinian, a contradiction. Therefore, there is no projective simple R-module. For any simple module is either projective or singular, we get that all simple modules are singular, as claimed. □

Following [**14**], Module M is said to be an ACS-module if for every element $a \in M$, $aR = P \oplus S$, where P is projective and S is singular. ACS stands for annihilator-CS and this property is named so because (by Lemma 2.9 of [**14**]) the condition is equivalent to saying that for every $a \in M$, the right annihilator ideal $r(a)$ is essential in a direct summand of M.

COROLLARY 5.4. *Let R be a simple-destitute ring such that $R/SocR$ is a semisimple R-module. Then*

(i) *R is semiprimary,*
(ii) *$Soc(R) = J(R) = Z(R)$ unless R is semisimple artinian,*
(iii) *$R^{(k)}$ satisfies the $C2$ and ACS conditions for all $k > 0$,*
(iv) *$Soc(R)$ is an essential ideal of R.*

Proof. (i) Let R be a simple-destitute ring. By theorem 5.3, all simple modules are singular and so all maximal right ideals are essential. Therefore, $Soc(R) \subset J(R)$ and $Rad(R/Soc(R)) = J(R)/Soc(R)$. On the other hand, as $R/Soc(R)$ is semisimple, it follows that $J(R) = Soc(R)$ and so $J(R)^2 = 0$. Therefore, R is semiprimary.

(ii) By (i) and theorem 5.3, we have $J(R) = Soc(R) = Z(R)$.

(iii) We have $T = R^{(n)}$, which satisfies the $C2$ condition by [**21**, Lemma 1.1] for $n \in \mathbb{Z}$.

From (ii), we get that R is Soc-semiperfect and so by [**15**, Theorem 2.10], T is Soc-semiperfect and so by the definitions of ACS-module, T is ACS-module because all simple modules are singular.

(iv) If possible, let K be a nonzero right ideal of R such that $Soc(R) \cap K = 0$. Then $K \cong (K + Soc(R))/Soc(R)$, and is semisimple because $R/Soc(R)$ is semisimple, a contradiction. □

Following [**19**], module M is called a weak CS-module provided that for each semisimple submodule S of M there exists a direct summand K of M such that S is essential in K. We generalize this definition slightly; an R-module M is called SCS if every closed simple module is a direct summand of M. Then, clearly a weak CS-module is SCS and any summand of an SCS module is SCS. Then, as a result of Theorem 5.3, we have the following proposition.

PROPOSITION 5.5. *Let R be a simple-destitute ring with the condition SCS and not semisimple artinian. Then there is no closed simple submodule in R_R.*

Proof. Let S be a closed simple submodule in R_R. Then S is a direct summand of R and so it is projective. But from Theorem 5.3, since R is a simple-destitute ring, S is singular, a contradiction because S is projective. □

THEOREM 5.6. *Let R be a simple-destitute ring such that $R/SocR$ is a semisimple R-module. If $R \oplus R$ is SCS, then R is a QF-ring with $J(R)^2 = 0$.*

Proof. First, we show that $R \oplus R$ is a CS module. Let K be a closed submodule of $R \oplus R$. Then by Corollary 5.4(i), R is semiperfect and hence $R \oplus R$ has a projective cover. Therefore, there is a decomposition $R \oplus R = A \oplus B$ such that $K = A \oplus (B \cap K)$ and $B \cap K$ is small in B and is therefore contained in $Rad(R \oplus R)$. Consequently, by Corollary 5.4(ii), $B \cap K$ is semisimple. Assume that $B \cap K$ is nonzero. Then if S is a simple submodule of $B \cap K$, by Lemma 5.5, S is not closed and so it has an essential extension submodule $\widehat{S}$ of $R \oplus R$. Hence, K is essential in $A \oplus \widehat{S} \oplus N$, where $B \cap K = N \oplus S$. As K is closed, we get that $K = A \oplus S \oplus N = A \oplus \widehat{S} \oplus N$. Then $S = \widehat{S}$ is also closed, which is a contradiction. Hence, $B \cap K = 0$ and K is a direct summand of $R \oplus R$. Therefore, $R \oplus R$ is a CS module and so $R \oplus R$ is continuous by Corollary 5.4(iii). Then by [**21**, Proposition 1.21], R is right self-injective. On the other hand, since $R/SocR$ is semisimple, R has ACC on essential right ideals and so by [**4**, Theorem 18.12], R is QF-ring. □

The conclusion of Theorem 5.6 fails in the absence of some of its hypotheses, as is shown by the following example.

EXAMPLE 5.7. Let R be the localization of $\mathbb{Z}$ with respect to $2\mathbb{Z}$. Then R is a simple-destitute ring but not a QF-ring.

COROLLARY 5.8. *Let R be a simple-destitute ring such that $R/SocR$ is a semisimple R-module. If $R \oplus R$ is SCS, then R is a finite direct sum of local modules R_i such that $Soc(R_i) = J(R_i)$ is simple.*

Proof. Let R be a simple-destitute ring such that $R/SocR$ is a semisimple R-module. Then by Corollary 5.4, R is a semiperfect ring and so there are local idempotents $\{e_i\}_{i=1}^n$ such that $R = e_1R \oplus \cdots \oplus e_nR$. Then $Soc(e_iR) = J(e_iR)$ is maximal in e_iR. We also get that R is a semiperfect, right continuous ring with essential right socle and so by [**8**, Corollary 2.3], $Soc(e_iR)$ is simple. □

6. Utopia: When a family does not contain any poor modules. Let $\mathcal{A}$ be a class of R-modules. Then $\mathcal{A}$ is called a utopia (and, in the case when $\mathcal{A}$ is easily identifiable, R is called an $\mathcal{A}$-*utopia* (ring)) if no element of the class $\mathcal{A}$ is poor. In particular, in the cases where $\mathcal{A} = Mod$-R, the class of simple modules, the class of projective modules, the class of artinian modules etc., we, respectively, say that R is a *utopia ring*, a *simple-utopia ring, a projective-utopia ring, an artinian-utopia ring* etc.).

EXAMPLE 6.1.

(i) A right PCI domain which is not a field is a singular-utopia ring.
(ii) A non-semisimple artinian right V-ring is a simple-utopia ring.
(iii) A non-semisimple artinian QF ring is a projective-utopia ring.

(iv) Let F be a field. Set $R = \prod_{\alpha\in\Lambda} F_\alpha$, where Λ is an infinite index set and $F_\alpha = F$ for every $\alpha \in \Lambda$. Then $R \simeq R \oplus R$. Hence, by Theorem 6.3, R is an artinian-utopia ring.

THEOREM 6.2. *Let $R = R_1 \oplus R_2$ be a ring decomposition. If M is a poor R-module, then $M_i = MR_i$ is a poor R_i-module for each $i = 1, 2$. However, M_i need not to be poor as an R-module. For instance, if R_2 is not semisimple artinian, then $M_1 = MR_1$ is not a poor R-module.*

Proof. For the first result, note that $N \in \mathfrak{In}^{-1}(M) \subset Mod\text{-}R$ if and only if $N = N_1 \oplus N_2$, where $N_i = NR_i \in \mathfrak{In}^{-1}(M_i) \subset Mod\text{-}R_i$ for each $i = 1, 2$. Now, suppose A is an R_i-module and M_i is A-injective as an R_i-module, then M is A-injective as an R-module. So $A \in \mathfrak{In}^{-1}(M) \subset Mod\text{-}R$. But M is poor and, therefore, A is a semisimple R-module. Clearly, $AR_i = A$ is a semisimple R_i-module and thus M_i is a poor R_i-module.

Now concerning the second result, note that for any semisimple R_1-module A, M_1 is $A \oplus R_2$-injective as an R-module. Since R_2 is not a semisimple artinian ring, we conclude that M_1 is not as poor as an R-module. □

THEOREM 6.3. *Let R be a ring that decomposes as a direct sum $R_1 \oplus S_1$ of rings $R_1 \simeq R$ and S_1, which is not semisimple artinian. Then R is an artinian-utopia.*

Proof. Let M be a poor R-module. We will show that M contains an infinite descending chain of R-submodules. By Theorem 6.2 $M_1 = MR_1$ is a poor R_1-module, which is not as poor as an R-module. Hence, M_1 is a nonzero, proper R-submodule of M. As $R_1 \simeq R$, write $R_1 = R_2 \oplus S_2$, where $R_2 \simeq R_1$ and S_2 is not semisimple artinian. Repeating the same argument, we get that $M_2 = M_1R_2$ is a poor R_2-module and a nonzero proper R-submodule of M_1. Proceeding inductively we obtain an infinite chain of R-submodules $\{M_n\}$ of M. □

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