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Analytic models of pseudo-Anosov maps

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Abstract. We give a new proof of the existence of analytic models of pseudo-Anosov maps. The persistence properties of Thurston's maps ensure that any C^0-perturbation of them presents all their dynamical features. Using Lyapunov functions of two variables we are able to choose certain analytic perturbations which do not add any new dynamical behaviour to the original pseudo-Anosov map.

0. *Introduction*

In [**2**] Gerber gives a proof of the existence of analytic Bernoulli models of pseudo-Anosov maps [**1**], [**10**] as an application of a conditional stability result for the smooth models [**3**] of these maps. The main tool in Gerber's proofs is Markov partitions. In this paper we prove the same results using Lyapunov functions of two variables and the results of [**7**].

Our construction is based on the fact that pseudo-Anosov maps f are persistent (see [**7**, p. 568]). If for some $\rho > 0$ we can find diffeomorphisms g arbitrarily C^0-close to f with expansivity constant ρ then, for those g such that for every $x \in M$ there exists $y \in M$ with the property that

$$\operatorname{dist}(f^n(x), g^n(y)) \leq \varepsilon, \qquad n \in \mathbb{Z}, \quad 0 < \varepsilon \leq \rho/2,$$

we have that there is only one such y and the mapping that sends x to y is a conjugacy between f and g. We construct a family of smooth diffeomorphisms g satisfying the above conditions perturbing f locally in a neighbourhood of the singular points as in [**3**]. In order to check the assertion regarding the expansivity constant, we construct a function W defined on a fixed neighbourhood of the diagonal in $M \times M$, that is a Lyapunov function [**6**, p. 192] for all g close enough to f, i.e. W is such that $W(g(x), g(y)) - W(x, y) > 0$ for x, y in the fixed neighbourhood, $x \neq y$, and all those g.

Since W has the same property with respect to any diffeomorphism h close to g in the C^r-topology (for suitable r) which preserves some finite jets of g at singular points, we obtain analytic models of pseudo-Anosov maps.

Whenever such an h preserves a normalized measure ν with a smooth positive density, it is Bernoulli with respect to that measure. This follows via Pesin's theory from the fact that $\nu(P) = 1$, P being the Pesin's region of h, and this, in turn, is a consequence of the fact that $h^{\#}(B) - B$ is a positive definite quadratic form in the complement of the set of singular points; here B denotes the quadratic part of W (see [6, p. 194]).

The analytic models of pseudo-Anosov maps that are Bernoulli with respect to the normalized Lebesgue measure are then obtained as in [2], as an application of results of Moser and Katok concerning diffeomorphisms preserving certain volume forms.

With a similar procedure we also prove the existence of analytic Bernoulli homeomorphisms of S^2.

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1. *Smooth models*

Let M be a compact riemannian smooth surface which supports pseudo-Anosov maps.

THEOREM 1.1. [3] *Let f be a pseudo-Anosov map of M and N a C^0-neighbourhood of f. Then there exist smooth diffeomorphisms $g \in N$ conjugate to f.*

Proof. We assume, as we may, that f fixes all the singular points and all their prongs. Moreover we work at a single singular point; the arguments needed to deal with them all are similar. Let x_0 be a singular point, U a neighbourhood of x_0 and

$$\psi : U \to \{z \in \mathbb{C} : |z| < 1\}$$

a coordinate homeomorphism such that $\psi \circ f \circ \psi^{-1}$ is the time-one map of $\dot{z} = k(\bar{z}^{p-1}/|z|^{p-2})$, where $k > 0$, $p = 3, 4, \ldots$, and the transversal measures μ, $\hat{\mu}$ are given by $|\text{Re}\, dz^{p/2}|$ and $|\text{Im}\, dz^{p/2}|$.

Let v be the vector field on $\mathbb{C}$ defined by

$$\dot{z} = \frac{p}{2} k\bar{z}|z|^{r(|z|)},$$

where r: $[0, \infty) \circlearrowleft$ is a smooth decreasing function such that, for a suitable fixed ε,

$$\begin{aligned} r(t) &= 2(p-2)/p && \text{if } t \in [0, \varepsilon], \\ r(t) &= 0 && \text{if } t \in [1-\varepsilon, \infty), \\ |\dot{r}(t)| &\leq 2 && \text{if } t \in [0, \infty). \end{aligned}$$

Notice that the vector field u, $(p/2)z^{p/2-1}u(z) = v(z^{p/2})$, equals $k\bar{z}^{p-1}$ if $0 \leq |z|^{p/2} \leq \varepsilon$ and $k(\bar{z}^{p-1}/|z|^{p-2})$ if $1-\varepsilon \leq |z|^{p/2}$.

LEMMA 1.2. *Let $V = \text{Re}\,(z-w)^2$. Then $\dot{V}$, the derivative of V with respect to v [6], is positive on $0 < |z-w| < \infty$.*

Proof. Except for positive constant factors,

$$\begin{aligned} \tfrac{1}{2}\dot{V}(z, w) = {} & \text{Re}\,\{(z-w)(\bar{z}|z|^{r(|\bar{z}|)} - \bar{w}|w|^{r(|z|)})\} \\ & + \text{Re}\,\{(z-w)\bar{w}(|w|^{r(|z|)} - |w|^{r(|w|)})\}, \end{aligned}$$

and by [7, p. 576]

$$\begin{aligned} \tfrac{1}{2}\dot{V} \geq {} & |z-w|^2(|z|^{r(|z|)+1} + |w|^{r(|z|)+1})(|z| + |w|)^{-1} \\ & - |z-w||w||w|^{r^*}|\log|w|(r(|z|) - r(|w|))|, \end{aligned}$$

where r^* is a real number between $r(|z|)$ and $r(|w|)$.

If $|w| \leq 1$,

$$\tfrac{1}{2}\dot{V} \geq |z-w|^2\left((|z|^{r(|z|)+1}+|w|^{r(|z|)+1})(|z|+|w|)^{-1}-\frac{2}{e}|w|^{r^*}\right),$$

and if $|z| \geq |w|$, $r(|z|) \leq r(|w|)$, and $r(|z|) \leq r^* \leq r(|w|)$.

Therefore, in this case, if $z \neq w$

$$\tfrac{1}{2}\dot{V} \geq |z-w|^2\left((|z|^{r(|z|)+1}+|w|^{r(|z|)+1})(|z|+|w|)^{-1}-\frac{2}{e}|w|^{r|z|}\right) > 0.$$

If $|z| \leq |w|$, $|z| \leq 1$, and we have

$$\begin{aligned}\tfrac{1}{2}\dot{V} &= \mathrm{Re}\,(z-w)(\bar{z}|z|^{r(|w|)}-\bar{w}|w|^{r(|w|)})+\mathrm{Re}\,(z-w)(\bar{z}|z|^{r(|z|)}-\bar{z}|z|^{r(|w|)})\\ &\geq |z-w|^2\left((|z|^{r(|w|)+1}+|w|^{r(|w|)+1})(|z|+|w|)^{-1}-\frac{2}{e}|z|^{\hat{r}}\right),\end{aligned}$$

where this time, $r(|z|) \geq \hat{r} \geq r(|w|)$ and consequently

$$\frac{1}{2}\dot{V} = |z-w|^2\left((|z|^{r(|w|)+1}+|w|^{r(|w|)+1})(|z|+|w|)^{-1}-\frac{2}{e}|z|^{r(|w|)}\right) > 0$$

if $z \neq w$.

When $|w| \geq 1$ and $|z| \geq 1$, the result is clear; the fact that the roles of z and w may be interchanged completes the proof of the lemma. □

Through a change of scale by a factor $\alpha > 0$ we may get the modification of $\dot{z} = \bar{z}$ to take place in a very small neighbourhood of 0 of radius α; the resulting vector field v_α will be defined, except factors, by $\dot{z} = \bar{z}(|z|/\alpha)^{r(|z|/\alpha)}$. Pulling back through $\Psi: z \to z^{p/2}$ the time-one map of v_α we get a smooth diffeomorphism on $\psi(U)$ and if we pull back this diffeomorphism through ψ and extend it to M in the obvious way, we get smooth diffeomorphisms, say g_α, which tend to f in the C^0-topology when $\alpha \to 0$.

For $x, y \in M$, y close to x, define $W(x, y)$ by $V(\Psi(\psi(x)), \Psi(\psi(y)))$ when x lies in a suitable neighbourhood of a singular point (ψ being the corresponding coordinate homeomorphism and $\Psi: z \to z^{p/2}$, where we assume that p, the number of prongs of the singularity, is even) and by $\mu^2(x, y) - \hat{\mu}^2(x, y)$ when x is outside the neighbourhood. Here $\mu(x, y)$ (resp. $\hat{\mu}(x, y)$) denotes the transversal measure of the segment $[x, Z]$ ($[y, Z]$) of the unstable (stable) leaf through x (resp. y), Z being the intersection of the unstable leaf through x with the stable leaf through y. On account of the way μ, $\hat{\mu}$ are given near singular points in terms of the homeomorphisms ψ, we may readily check that W is a smooth function defined in a neighbourhood of the diagonal in $M \times M$ and that there exists $\rho > 0$ such that if $\mathrm{dist}\,(x, y) \leq \rho$

$$W(g_\alpha(x), g_\alpha(y)) - W(x, y) \geq 0.$$

Indeed, since lemma 1.2 holds for every z, w, $0 < |z-w|$, we get at once that ρ does not depend on α. In spite of the fact that when x and y are close to a singularity, $x \neq y$, $W(g_\alpha(x), g_\alpha(y)) - W(x, y)$ may not be strictly positive, we conclude, on account of the hyperbolic behaviour of g_α near singular points, that ρ is an expansivity constant for all of them.

In case p is odd, we take a disk $\tilde{U}$, map it on $\psi(U)$ through $z \to z^2$ and define $\tilde{W}(z, w) = V(z^p, w^p)$, $z, w \in \tilde{U}$. Since g/U may be lifted to $\tilde{U}$, we prove expansivity using g and W when we are far from singularities and using the lifting of g/U and $\tilde{W}$ close to them.

Since f is persistent, for any $\varepsilon>0$, we may choose $g=g_\alpha\in N$ such that for every $x\in M$ there exists $y\in M$ with the property dist $(f^n(x), g^n(y))<\varepsilon$, $n\in\mathbb{Z}$. If $\varepsilon<\rho/2$, the mapping which sends x to the unique y with the above property is a conjugacy between f and g. □

Remark 1.3. All these diffeomorphisms g preserve a probability measure ν with a positive smooth density (see [**3**, § 4]).

Remark 1.4. The same methods may be applied to the so called generalized pseudo-Anosov maps. For instance, if we consider the branched two covering of S^2 by T^2 and the homeomorphism f of S^2 obtained as the projection of a linear Anosov map $\bar{f}$, as in [**4**, § 2], we may construct smooth models g of f by projecting to S^2 suitable perturbations $\bar{g}$ of $\bar{f}$. $\bar{g}$ may be a diffeomorphism of T^2 that coincides (modulo coordinates) with the time one map of $\dot{z}=\bar{z}|z|^2$ on some small neighbourhoods of the branch points, and with $\bar{f}$ outside of some larger neighbourhoods, or, as in [**2**], [**3**], $\bar{g}$ may be obtained, from a homeomorphism of T^2 that coincides with the time-one map of $\dot{z}=\bar{z}|z|^{2/3}$ on small neighbourhoods of the branch points and with $\bar{f}$ outside of some larger neighbourhoods, by conjugation with a suitable homeomorphism χ (see [**2**]) (with this last procedure we get a smooth diffeomorphism g that preserves a probability measure ν with a smooth positive density).

To prove conjugacy between f and g we show that their liftings are conjugate. This follows, as before, as an application of lemma 1.2 and the persistence properties of $\bar{f}$.

2. *Conditional stability and analytic models*

LEMMA 2.1. *Let q be a positive integer. Consider the equation $\dot{z}=\bar{z}^q$ and the function $V_1(z,w)=\operatorname{Re}\{(z-w)(z^q-w^q)\}$; then*

$$\dot{V}_1(z,w)\geq q|z-w|^2(|z|^{2q-1}+|w|^{2q-1})(|z|+|w|)^{-1}.$$

Proof. $\dot{V}_1=\operatorname{Re}\{|z^q-w^q|^2+q(z-w)(\bar{z}^q z^{q-1}-\bar{w}^q w^{q-1})\}$. Thus the lemma will be proved if we show that

$$\operatorname{Re}\{(z-w)(\bar{z}^q z^{q-1}-\bar{w}^q w^{q-1})\}\geq|z-w|^2(|z|^{2q-1}+|w|^{2q-1})(|z|+|w|)^{-1}$$

or that

$$|z-w|^2(|z||w|^{2q-2}+|w||z|^{2q-2})+\operatorname{Re}\{(z-w)(|z|+|w|)(\bar{w}|z|^{2q-2}-\bar{z}|w|^{2q-2})\}\geq 0.$$

But the left hand side equals

$$\begin{aligned}&\operatorname{Re}\{(z-w)(\bar{w}|z|+|w|\bar{z})(|z|^{2q-2}-|w|^{2q-2})\\&=\operatorname{Re}(|z|^{2q-2}-|w|^{2q-2})(|z|-|w|)(|wz|+\operatorname{Re}\bar{w}z),\end{aligned}$$

which is non-negative. □

LEMMA 2.2. *Let ϕ be the flow defined by $\dot{z}=\bar{z}^q$; then if $g=\phi_1$,*

$$V_1(g(x),g(y))-V_1(x,y)\geq H|z-w|^2(|z|^{2q-1}+|w|^{2q-1})(|z|+|w|)^{-1},$$

for some $H>0$.

Proof. It is the same as that of lemma 3.4 in [7].

In the sequel we shall work as if all the singularities had an even number of prongs p; in case of presence of singularities with p odd, we may use arguments similar to those in the proof of theorem 1.1.

LEMMA 2.3. *There exists $\delta > 0$ and a continuous W_1 defined for $x, y \in M$*, dist $(x, y) < \delta$ *such that $W_1(g(x), g(y)) - W_1(x, y) > 0$ if $0 <$ dist $(x, y) < \delta$. Moreover for some $m > 0$, and x, y in some neighbourhood $U_1 \subset U$ of x_0,*

$$W_1(g(x), g(y)) - W_1(x, y) \geq m|z-w|^2(|z|^{2q-1} + |w|^{2q-1})(|z|+|w|)^{-1},$$

where $z = \psi(x)$, $w = \psi(y)$ and $q = p - 1$.

Proof. Take $U = W$ and $V(x, y) = V_1(\psi(x), \psi(y))$ in the proof of [7, lemma 3.2, p. 575]. □

Let x_i, $i = 1, \ldots, m$ be the singular points of f (or g), $\varepsilon > 0$ and q a positive integer. As in [2], if $1 \leq i \leq m$, we define $H_\gamma^q(g, x_i)$ as the set of smooth diffeomorphisms of M such that, identifying $x \in U$ with the point in $\mathbb{R}^2$ obtained by application of the coordinate homeomorphism ψ, we have

(i) $h(x_i) = x_i$;

(ii) $d^j h(x_i) = d^j g(x_i)$, $j = 1, \ldots, q-1$;

(iii) $\|d^q h(x) - d^q g(x)\|$, $\|d^q(h^{-1})(x) - d^q(g^{-1})(x)\| < \gamma$.

Here we regard $dg(x)$, $dh(x)$ as elements of $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ and $d^2 g(x)$, $d^2 h(x)$ as elements of $\mathcal{L}(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2))$, etc.; the norms are the usual ones.

Let $p(i)$ denote the number of prongs at x_i.

THEOREM 2.4. [2] *For any $\gamma > 0$, there exists a neighbourhood N of g in the C^1-topology such that any $h \in N \cap (\bigcap_1^m H_\gamma^{p(i)}(g, x_i))$ is conjugate to f.*

Proof. In suitable neighbourhoods U of singular points we may calculate $W(h(x), h(y)) - W(x, y)$ as $W_1(\bar{h}(z), \bar{h}(w)) - W_1(z, w)$, where we have denoted by $\bar{h}$ the diffeomorphism induced by h in the coordinate region $\psi(U)$. With the same notation

$$\begin{aligned} &W_1(\bar{h}(z), \bar{h}(w)) - W_1(z, w) \\ &= W_1(\bar{g}(z), \bar{g}(w)) - W_1(z, w) + \rho(z, \bar{z}, w, \bar{w}); \end{aligned}$$

if $h \in \bigcap_1^m H_\gamma^{p(i)}(g, x_i)$, we obtain, after a simple estimate, that for some $A > 0$ depending on γ

$$|\rho(z, \bar{z}, w, \bar{w})||z-w|^{-2}(|z|^{2p-3} + |w|^{2p-3})^{-1} \leq A.$$

Therefore, by lemma 2.3, we conclude that there are fixed neighbourhoods, say U_i, of the singular points such that for all those h, $W_1(h(x), h(y)) - W_1(x, y) > 0$ for $(x, y) \in U_i \times U_i$, $i = 1, \ldots, m$, $x \neq y$.

Since for any x in the complement of $\bigcup_1^m U_i$, y close to x, $W_1(g(x), g(y)) - W_1(x, y)$ is greater than a positive definite quadratic form we may choose a C^1-neighbourhood N' of g such that, for some $\sigma > 0$, all the $h \in N' \cap (\bigcap_1^m H_\gamma^{p(i)}(g, x_i))$ have expansivity constant σ. Since g is persistent we may find a C^0-neighbourhood

N_0 of g such that for any $x \in M$ and any $h \in N_0$, there exists $y \in M$ with the property

$$\operatorname{dist}(g^n(x), h^n(y)) \leq \tfrac{1}{2}\sigma, \qquad n \in \mathbb{Z}.$$

Take $N = N_0 \cap N'$. □

We consider again the diffeomorphism g of S^2 constructed with the last procedure mentioned in remark 1.4. Consider a set of smooth diffeomorphisms h, of S^2, that preserve the 2-jets of g at the projections of branch points and such that their third derivatives are uniformly bounded on fixed neighbourhoods of these points. We lift h to a homeomorphism $\bar{h}$ of T^2, undo the conjugation by χ, and get in this way another homeomorphism $\bar{j}$ of T^2 that on small neighbourhoods of the branch points differs from the time one map of $\dot{z} = \bar{z}|z|^{2/3}$ by $\rho(z, \bar{z})$ where $\lim_{z\to 0} \rho(z, \bar{z})|z|^{-5/3} = 0$ uniformly in h. Therefore the function $\operatorname{Re}(z-w)^2$ whose derivative with respect to $\dot{z} = \bar{z}|z|^{2/3}$ is greater than

$$|z-w|^2(|z|^{5/3} + |w|^{5/3})(|z|+|w|)^{-1}$$

is also a Lyapunov function for $\bar{j}$ on small neighbourhoods of the branch points.

If the h are in a suitable C^1-neighbourhood of g we conclude as above, through a global Lyapunov function $W = \operatorname{Re}(z-w)^2$, the uniform expansivity of the corresponding $\bar{j}$, and $\bar{h}$. On account of the persistence of $\bar{g}$, this implies in the usual way the conjugacy between $\bar{g}$ and $\bar{h}$ provided h is close enough to g in the C^0-topology; this implies, in turn, the conjugacy of their projections g and h. We get in this way a conditional stability theorem for g.

Remark 2.5. It is easy to check that all the models h of pseudo-Anosov or generalized pseudo-Anosov maps we constructed, have the following property: There is a continuous quadratic form $B: TM \to R$ such that B_x is non-degenerate and $(h^{\#}B - B)_x$ is positive definite if x is non-singular (see [6]). In fact, it is enough to take B as the quadratic part of W.

3. *The Bernoulli property*

Let $f: M \to M$ be a C^∞-diffeomorphism, conjugate to a pseudo-Anosov map, that preserves a smooth measure ν. We assume, moreover, that there exists a continuous quadratic form $B: TM \to \mathbb{R}$, which is non-degenerate except at singular points, and such that $(f^{\#}B - B)_x$ is a positive definite quadratic form at every $x \in M$ which is not singular.

THEOREM 3.1. *f is Bernoulli with respect to ν.*

Since the proof follows essentially the same lines as those in [**4**, § 4], we sketch the proof of only a couple of lemmas leading to the construction of a $(\delta(x), 1)$ continuous invariant foliation for f, which in turn permits us to apply theorems 7.7, 7.8 and 8.1 of [**9**] to get the assertion of the theorem.

LEMMA 3.2. *Let μ be a probability measure on the Borel sets of M such that $\mu(\{x\}) = 0$ for each singular point x, and let Λ be the Pesin's region of f. Then $\mu(\Lambda) = 1$.*

Proof. Since the fact that $P = f^{\#}B - B$ is positive a.e. implies the semi-invariance property for a suitable family of cones, this result follows from previous work by

Katok [4]. A proof using Lyapunov functions directly may be given as follows: let $S_x(U_x)$ be the set of $u \in T_xM$ such that $B((f^n)'u) < 0$, (resp. > 0), $n \in \mathbb{Z}$, $x \in M$. If $D \subset M$ is compact and contains no singular points, there exists $\alpha > 0$ such that $P(u) > \alpha B(u)$, $u \in U_x$, $x \in D$, and therefore, for those x and u,

$$B(f'u) \geq (1+\alpha)B(u).$$

If $f^{n_k}(x) \in D$ for some increasing sequence $\{n_k\}$, we have that, for $u \in U_x$,

$$B((f^{n_k+1})'u) \geq (1+\alpha)^k B(u).$$

Thus,

$$B((f^n)'u) \geq (1+\alpha)^{N_n} B(u),$$

where $N_n = \#\{j : f^j(x) \in D;\ 0 \leq j \leq n-1\}$. Let $\beta > 0$ be such that $|B(u)| \leq \beta \|u\|^2$ for any $u \in TM$; then

$$\|(f^n)'u\|^2 \geq \frac{1}{\beta}(1+\alpha)^{N_n} B(u), \qquad n \geq 0,$$

and consequently, applying Birkhoff's theorem to the right hand side of

$$\underline{\lim}\, \frac{1}{n} \log \|(f^n)'u\| \geq \tfrac{1}{2} \log(1+\alpha)\, \underline{\lim}\, \frac{N_n}{n},$$

we get that the upper Lyapunov exponent is positive on a set D', $\mu(D') \geq \mu(D)$. □

LEMMA 3.3. *There exists an invariant subset $H \subset M$, $\nu(H) = 1$, such that the whole stable manifold of any $x \in H$ is smooth.*

Proof. Let D_n be the set of those $x \in M$ such that their local stable manifold V_x is the image of a smooth embedding $\rho : [-1, 1] \to V_x$, $\rho(0) = x$, and dist $(\rho(-1), x)$, dist $(\rho(1), x) > 1/n$. Then, lim $\nu(D_n) = 1$. Let D'_n be the set of x such that $f^k(x) \in D_n$, for infinitely many $k > 0$. Since D'_n is invariant and the whole stable manifold of any $x \in D'_n$ is smooth, we may let $H = \bigcup_1^\infty D'_n$, since $\nu(D'_n) \geq \nu(D_n)$. □

The following corollaries are proved exactly as in [2] on the basis of the results of [8] and [5] referring to diffeomorphisms acting on volume forms.

COROLLARY 3.4. *Let f be a pseudo-Anosov map and ℓ a probability measure with a positive analytic density. Then there exist analytic homeomorphisms preserving ℓ, that are conjugate to f and Bernoulli with respect to ℓ.*

COROLLARY 3.5. *For any probability measure ℓ of S^2 with positive analytic density, there exist analytic homeomorphisms of S^2 that are Bernoulli with respect to ℓ.*

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