# CAUCHY COMPLETION OF PARTIALLY ORDERED GROUPS

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A number of completions have been applied to p.o.-groups — the Dedekind-Macneille completion of archimedean l.o. groups; the lateral completion of l.o. groups (Conrad [2]); and the orthocompletion of l.o. groups (Bernau [1]). Fuchs in [3] has considered a completion of p.o. groups having a non-trivial open interval topology — the only l.o. groups of this form being fully ordered. He applies an ordering, which arises from the original partial order, to the group of round Cauchy filters over this topology; Kowalsky in [6] has shown that group, imbued with a suitable topology, is in fact the topological completion of the original group under its open interval topology. In this paper a slightly different ordering, also arising from the original order, is proposed for the group of round Cauchy filters; Fuchs' ordering can be obtained from this one as the associated order.

§1 introduced the necessary underlying concepts, whilst §2 describes a slight short cut to the results needed from Kowalsky. §3 brings in new ordering described above, and shows that it has some desirable properties—the open interval topology corresponding to it is in fact the topology of the topological completion; the completion of the completion is $o$-isomorphic to the completion; and the completion is the ubique maximal extension of the original p.o. group in which this latter is sub-dense. In §4 the connection with Fuchs' completion is established, and it is noted that the tight Riesz property is preserved by completion.

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## 1. Preliminaries

A group $(A, +)$ is a p.o. group if it is a p.o. set under a binary relation $\geqq$, such that if $a, b$ and $c \in A$, then $a \geqq b$ implies each of $a + c \geqq b + c$ and $c + a \geqq c + b$. The sets $P = \{a \in A \mid a \geqq 0\}$ and $P^* = \{a \in A \mid a > 0\}$ are respectively the positive and strictly positive cones of $A$, each completely determining the order $\geqq$ by the rule "$a \geqq b$ if and only if $a - b \in P$". Any subset $T^*$ of $A$ will form the strict positive cone of an order on $A$ if and only if each of (i) $T^*$ is normal; (ii) $T^* + T^* \subseteq T^*$; and (iii) $T^* \cap (-T^*) = \phi$ holds.

A p.o. group $A$ is directed if for each $a, b \in A$ there is a $c \in A$ such that $a < c$ and $b < c$ (abbreviated to $a, b < c$). A directed p.o. group $A$ is called a Riesz group if, whenever $a, b \leqq c, d$ in $A$, there is an $e \in A$ such that $a, b \leqq e \leqq c, d$; it is called a tight Riesz group if, whenever $a, b < c, d$ in $A$, there is an $e \in A$ such that $a, b < e < c, d$.

An antilattice $A$ is a Riesz group with no non-trivial meets and joins. It is dense if, whenever $a < c$ in $A$, there is a $b \in A$ such that $a < b < c$. It can in fact be shown that dense antilattices are the same thing as tight Riesz groups.

An isomorphism $\theta$ from one p.o. group $(A, \geqq)$ to another, $(B, \geqq)$, is called an *o-isomorphism* whenever $\theta(a) > 0$ in $B$ if and only if $a > 0$ in $A$.

On any p.o. group $A$ we can form the open interval topology $\mathscr{U}$ by using as a subbase the set of open intervals — i.e., those sets of the form $(a, b) = \{x \in A \mid a < x < b\}$. The convention that $\mathscr{U}$ is the set of open sets, and that a neighbourhood of a point is an open set containing the point, will be used. $(A, \mathscr{U})$ is a topological group if both maps $+ : A \times A \to A$ and $- : A \to A$ are continuous; the nature of the subbasic sets makes — continuous, and so the following conditions are equivalent:

(1) *$(A, \mathscr{U})$ is a topological group;*

(1A) *If $F \in \mathscr{U}$ and $a + b \in F$, then there are neighbourhoods $G$ and $H$ of $a$ and $b$ respectively such that $G + H \subseteq F$; and*

(1B) *If $F$ is a neibourhood of $0$, then there are neighbourhoods $G$ and $H$ of $0$ such that $G + H \subseteq F$.*

Some authors also require the topology to be Hausdorff; in this case, this is equivalent to the condition that $A$ has no non-zero elements $w$ for which $x > w \Leftrightarrow x > 0$. Such elements are called pseudozeros. Similarly elements $w$ for which $x > w \Rightarrow x > 0$ holds, but which are not positive, are called pseudopositives. If $A$ has no pseudozeros, then the positive and pseudopositive sets together form the positive set for a new ordering $\succcurlyeq$ of $A$, called the associated order of $\geqq$. Thus $a \succcurlyeq b$ if and only if $x > a \Rightarrow x > b$.

Generally, operations and relations, when applied to sets, denote complex ones, for example, $F + G = \{f + g \mid f \in F,\ g \in G\}$; and $F > G$ means that $f > g$ for each $f \in F$ and $g \in G$. The exceptions to this rule are filters; the appropriate operations and relations will be defined later.

A subset $B$ of $A$ will be called sub-dense if, whenever $a_1, \cdots, a_n < b < c_1, \cdots, c_m$ in $A$ there is a $d \in B$ such that $a_1, \cdots, a_n < d < c_1, \cdots, c_m$; such a subset is topologically dense in $A$ under the open interval topology, and vice versa.

These concepts may be found in Fuchs [3], [4] and [5], and Loy and Miller [7]; and the terminology is mixture of these and others.

## 2. 0-filters

In establishing the completion of a uniform space, Kowalsky [6] has used the concept of round Cauchy filters. Because of his different definition of a topology, and also of a neighborhood, it has been more convenient to use the idea of an 0-filter rather than a filter.

A filter $\mathscr{F}$ on $(A, \mathscr{U})$ is a non-empty set of subset of $A$ satisfying (i) if $F \in \mathscr{F}$ and $G \subseteq A$, with $F \subseteq G$, then $G \in \mathscr{F}$; (ii) if $F, G \in \mathscr{F}$, then $F \cap G \in \mathscr{F}$; and (iii) $\phi$ (the empty set) $\notin \mathscr{F}$.

*An 0-filter $\mathscr{F}$ on $(A, \mathscr{U})$ satisfies these three conditions, but each element of $\mathscr{F}$ must also be an open set on $A$,* (i) *being altered appropriately.*

For any set $\mathscr{B}$ of subsets of $A$ we can define

$$\overline{\mathscr{B}} = \{X \subseteq A \mid Y \subseteq X \text{ for some } Y \in \mathscr{B}\}.$$

We call $\mathscr{B}$ a filter base if $\overline{\mathscr{B}}$ is a filter; an open filter is then a filter which has a base consisting of open sets. There is an obvious correspondence between open filters and 0-filters; if $\mathscr{F}$ is an 0-filter, then $\overline{\mathscr{F}}$ is an open filter, and $\overline{\mathscr{F}} \cap \mathscr{U} = \mathscr{F}$; and if $\mathscr{G}$ is an open filter, then $\mathscr{G} \cap \mathscr{U}$ is an 0-filter, and $\overline{\mathscr{G} \cap \mathscr{U}} = \mathscr{G}$.

The sum of two 0-filters $\mathscr{F}$ and $\mathscr{G}$ is defined to be

$$\mathscr{F} + \mathscr{G} = \{H \in \mathscr{U} \mid F + G \subseteq H \text{ for some } F \in \mathscr{F} \text{ and } G \in \mathscr{G}\};$$

whilst $-\mathscr{F} = \{F \in \mathscr{U} \mid -F \in \mathscr{F}\}$. Both are clearly 0-filters.

Cauchy filters and round filters can both be defined for a uniform space; topological groups, however, have two associated uniformities, called the **l**-uniformity and the ***r***-uniformity. Thus on topological groups we have **l**- and ***r***-Cauchy filters, and **l**- and ***r***-round filters. Filters which are both **l**- and ***r***-Cauchy are called simply Cauchy, and ones which are **l**- and ***r***-round are called round. However, we shall see that the conjunction of ***r***-Cauchy and **l**-round implies the other two, so that, after this section, the distinction will not be made.

Rather than to define these concepts for a uniform space, and then to apply these definitions to the uniformities of a topological group, it is easier to give them in group terminology.

*Thus the 0-filter $\mathscr{F}$ is **r**-Cauchy if, for any neighbourhood $V$ of $0$, there is an $F \in \mathscr{F}$ such that $F - F \subseteq V$; further, it is called **l**-round if, for any $F \in \mathscr{F}$, there is a neighbourhood $V$ of $0$ and a $G \in \mathscr{F}$ such that $G + V \subseteq F$.*

For any $a \in A$, we can form $\mathscr{V}(a)$, the set of neighbourhoods of $a$; $\mathscr{V}(a)$ is clearly an 0-filter. We can thus translate each of the above definitions into terms of 0-filters:-

An 0-filter $\mathscr{F}$ is ***r**-Cauchy* if $\mathscr{F} - \mathscr{F} = \mathscr{V}(0)$; and $\mathscr{F}$ is ***l**-round* if $\mathscr{F} + \mathscr{V}(0) = \mathscr{F}$.

If $(A, \mathscr{U})$ is a topological group, we have

$$1\text{A}: \text{— for any } a, b \in A,\ \mathscr{V}(a) + \mathscr{V}(b) = \mathscr{V}(a+b);$$

and

$$1\text{B}: \text{—}\ \mathscr{V}(0) + \mathscr{V}(0) = \mathscr{V}(0).$$

Thus, as $\mathscr{V}(0) = -\mathscr{V}(0)$, $\mathscr{V}(0)$ is both **r**-Cauchy and **l**-round, and so the translated definitions become the **r**-inverse and **r**-indentity laws for the set of **r**-Cauchy **l**-round 0-filters. Therefore:

THEOREM. (Kowalsky, [6], p.250) *The set $A^*$ of r-Cauchy* **1**-*round 0-filters on $(A, \mathscr{U})$ is a group under the given operations if and only if $(A, \mathscr{U})$ is a topological group.*

(In a personal communication, R. Ramsay has provided an example of a p.o. group which is not a topological group under its open interval topology.)

As the **l**-inverse and **l**-indentity laws also hold for the group $A^*$, the 0-filters must be **l**-Cauchy and **lr**-round as well. Moreover, the translation of 1A assures us of a canonical homomorphism $\phi: A \to A^*$, $a\phi = \mathscr{V}(a)$. Further, as Fuchs has noted in [3], if $\mathscr{F}, \mathscr{G} \in A^*$ and $\mathscr{F} \subseteq \mathscr{G}$, then $\mathscr{F} = \mathscr{G}$.

## 3. The completion

In this section we shall suppose $(A, \geqq)$ to be a p.o. group which is a topological group under its open interval topology. Its (Cauchy) completion $(A^*, \geqq)$ is the group $A^*$ of round Cauchy 0-filters under the ordering $\geqq$ whose strictly positive cone $\boldsymbol{P}^* = \{\mathscr{F} \in A^* \mid P^* \in \mathscr{F}\}$. Fuchs' ordering, which will be denoted by $\preccurlyeq$, has as its positive set

$$\boldsymbol{Q} = \{\mathscr{F} \in A^* \mid f \geqq 0 \text{ for some } f \in F, \text{ for each } F \in \mathscr{F}\}.$$

(Actually, Fuchs gives a condition which also brings in the topological group definition of a Cauchy filter, which he has not specifically given.) *Thus we have $\mathscr{F} > \mathscr{G}$ in $A^*$ if there is an $F \in \mathscr{F}$ and a $G \in \mathscr{G}$ such that $F > G$; $\mathscr{V}(a) > \mathscr{F}$ in $A^*$ if thhere is an $F \in \mathscr{F}$ such that $a > F$; and $\mathscr{V}(a) > \mathscr{V}(b)$ if and only if $a > b$.*

If $(A, \geqq)$ has pseudozeros, then, with 0, they form a subgroup $Z$. If $w$ is a pseudozero, then $x > w \Leftrightarrow x > 0$, so that each neighbourhood of 0 also contains $w$; hence $\mathscr{V}(w) = \mathscr{V}(0)$. Therefore $Z$ is the kernel of the canonical homomorphism $\phi$. Thus, if $(A, \geqq)$ has no pseudozeros (that is), if $(A, \mathscr{U})$ is Hausdorff), then $\phi$ is a monomorphism; and so, as $\mathscr{V}(a) > \mathscr{V}(0)$ if and only if $\mathscr{V}(a) \in \boldsymbol{P}^*$, if and only if $P^* \in \mathscr{V}(a)$, if and only if $a \in P^*$, if and only if $a > 0$, $A$ is $o$-isomorphic to its image in $A^*$ in this case.

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We form the open interval topology $\mathscr{U}^*$ on $A^*$ in the usual fashion; if we take $X \in \mathscr{U}$, then we can define $S(X) = \{\mathscr{F} \in A^* \mid X \in \mathscr{F}\}$, and thus form another topology $\mathscr{W}^*$ on $A^*$ by taking $\{S(X) \mid X \in \mathscr{U}\}$ as a basic. If $(A, \mathscr{U})$ is Hausdorff, then $(A^*, \mathscr{W}^*)$ is the topological completion of $(A, \mathscr{U})$ (Kowalsky [6], pp. 166, 232, 251), and $(A^*, \mathscr{W}^*)$ is a topological group (also p. 251 of Kowalsky).

LEMMA 1. *If $\mathscr{F} \in A^*$ and $F \in \mathscr{F}$, then there is a basic open set $J \in \mathscr{F}$ such that $J \subseteq F$.*

PROOF. $\mathscr{F}$ is round, so there is a $G \in \mathscr{F}$ and *a* $V \in \mathscr{V}(0)$ such that $G + V \subseteq F$. As $V$ is open, there is a basic open neighbourhood I of 0 such that $I \subseteq V$. As $\mathscr{F}$ is also Cauchy, there is an $H \in \mathscr{F}$ such that $-H + H \subseteq I$; so $G \cap H \in \mathscr{F}$, $(G \cap H) + I \subseteq F$, and $-(G \cap H) + (G \cap H) \subseteq I$.

Take $g \in G \cap H$; then $-g + (G \cap H) \subseteq I$, so that $G \cap H \subseteq g + I$. Further, $g + I \subseteq F$; so, if we put $J = g + 1$, then $J \in \mathscr{F}$, $J$ is a basic open set, and $J \subseteq F$.

LEMMA 2. *$(A^*, \geqq)$ has no pseudozeros.*

PROOF. Suppose $\mathscr{W} \in A^*$ is such that $\mathscr{F} > \mathscr{W} \Leftrightarrow \mathscr{F} > \mathscr{V}(0)$ in $A^*$, and take $W \in \mathscr{W}$. By lemma 1 there is a basic open set

$$J = \cap \{(a_i, b_i) \mid i = 1, \cdots, n\}$$

such that $J \in \mathscr{W}$ and $J \subseteq W$. Thus $a_i < J < b_i$ for each $i$, and so $\mathscr{V}(a_i) < \mathscr{W} < \mathscr{V}(b_i)$ for each $i$. By definition of $\mathscr{W}, \mathscr{V}(a_i) < \mathscr{V}(0) < \mathscr{V}(b_i)$ for each $i$, and thus $a_i < 0 < b_i$ for each $i$; i.e., $0 \in J$, so $W \in \mathscr{V}(0)$. Hence $\mathscr{W} = \mathscr{V}(0)$.

LEMMA 3. *If $X \in \mathscr{U}$, then $S(X) \in \mathscr{U}^*$.*

PROOF. Take $\mathscr{F} \in S(X)$, so that $X \in \mathscr{F}$. By lemma 1 there is a basic open set $I \in \mathscr{F}$ such that $I \subseteq X$. Then $I = \cap \{a_i, b_i) \mid i = 1, \cdots, n\}$.

Form $\boldsymbol{J} = \cap \{\mathscr{V}(a_i), \mathscr{V}(b_i)) \mid i = 1, \cdots, n\}$; now $a_i < I < b_i$ for each $i$, so that $\mathscr{V}(a_i) < \mathscr{F} < \mathscr{V}(b_i)$ for each $i$, and hence $\mathscr{F} \in \boldsymbol{J}$. Therefore $\boldsymbol{J}$ is a neighbourhood of $\mathscr{F}$.

Take $\mathscr{G} \in \boldsymbol{J}$, so that $\mathscr{V}(a_i) < \mathscr{G} < \mathscr{V}(b_i)$ for each $i$. Thus for each $i$ we have sets $G_i$ and $G_i' \in \mathscr{G}$ such that $a_i < G_i$ and $G_i' < b_i$. So, if

$$G = (\cap \{G_i\}) \cap (\cap \{G_i'\}),$$

then $a_i < G < b_i$ for each $i$, and $G \in \mathscr{G}$. Thus $G \subseteq I \subseteq X$, so that $X \in \mathscr{G}$; that is $\mathscr{G} \in S(X)$. Hence $\boldsymbol{J}$ is a neighbourhood of $\mathscr{F}$ in $S(X)$.

LEMMA 4. *Suppose $X, Y \in \mathscr{U}$. Then;-*

(i) $S(X) \cap S(Y) = S(X \cap Y)$.
(ii) *If $X \subseteq Y$, then $S(X) \subseteq S(Y)$.*
(iii) $S(X) + S(Y) \subseteq S(X + Y)$.

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PROOF. Trivial.

LEMMA 5. *$\mathscr{U}^* = \mathscr{W}^*$ if $(A, \mathscr{U})$ is Hausdorff.*

PROOF. *By lemma* 3, $\mathscr{W}^* \subseteq \mathscr{U}^*$. Let $F$ be a basic open set of $(A^*, \mathscr{U}^*)$; then $F = \cap\{(\mathscr{F}_i, \mathscr{G}_i) \mid i = 1, \cdots, n\}$.

But $P^* = \{\mathscr{F} \in A^* \mid P^* \in \mathscr{F}\} = S(P^*)$, and so $P^* \in \mathscr{W}^*$; thus, as $(A^*, \mathscr{W}^*)$ is a topological group, for each $i$,

$$(\mathscr{F}_i, \mathscr{G}_i) = (\mathscr{F}_i + P^*) \cap (\mathscr{G}_i - P^*) \in \mathscr{W}^*,$$

and so $F \in \mathscr{W}^*$.

Thus we may conclude:-

THEOREM 1. *$(A^*, \mathscr{U}^*)$ is the topological completion of $(A, \mathscr{U})$ if $(A, \mathscr{U})$ is Hausdorff; if not, then it is the topological completion of $(A, \mathscr{U})/Z$, where $Z$ is the group of pseudozeros.*

THEOREM 2. *$(A^*, \geqq)$ is o-isomorphic to its own completion.*

PROOF. By lemma 2, $(A^*, \mathscr{U}^*)$ is Hausdorff. As its completion $(A^*)^*$ is thus its topological completion, it is complete, i.e., every Cauchy filter has a limit; so every round Cauchy 0-filter has a limit, so that $\phi$ will be an epimorphism. Hence $\phi$ is the required $o$-isomorphism.

THEOREM 3. *If $(A, \geqq)$ has no pseudozeros, then any extension of $(A, \geqq)$ in which $(A, \geqq)$ is sub-dence can be extended to a maximal such extension, and this latter will be o-isomorphic to $(A^*, \geqq)$.*

PROOF. Applying Kowalsky [6] (p. 234), we see that each complete extension in which $(A, \geqq)$ is sub-dense is $o$-isomorphic to $(A^*, \geqq)$. If $(B, \geqq)$ is any extension in which $(A, \geqq)$ is sub-dense, then the image of $(A, \geqq)$ under the canonical homomorphism $\phi: B \to B^*$ will be sub-dense in $(B^*, \geqq)$, and this latter is complete; thus $(B^*, \geqq)$ is $o$-isomorphic to $(A^*, \geqq)$, and so $(B, \geqq)$ is $o$-isomorphic to a sub-dense subgroup of $(A^*, \geqq)$.

## 4. Tight Riesz groups

Fuchs [3] gives his completion for commutative isolated divisible antilattices. These conditions mean that such a group should be a dense antilattice, and hence a tight Riesz group. As Loy and Miller [7] have point out, the open intervals form a base for the open interval topology on a tight Riesz group, and, further, any tight Riesz group is a topological group under this topology; so we may form the completion $(A^*, \geqq)$.

In this section, then, we shall assume $(A, \geqq)$ to be a tight Riesz group. We shall first need a refinement of lemma 1:–

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LEMMA 6. *If* $\mathscr{F} \in A^*$ *and* $F \in \mathscr{F}$, *then there are* $u, v \in F$ *such that* $(u, v) \subseteq F$ *and* $(u, v) \in \mathscr{F}$.

PROOF. By lemma 1 there is an interval $(a, b) \in \mathscr{F}$ such that $(a, b) \subseteq F$; so we can find $(u, v) \in \mathscr{F}$ and $(c, d) \in \mathscr{V}(0)$ such that $(u, v) + (c, d) \subseteq (a, b)$; as $(A, \geqq)$ is a tight Riesz group, $(u, v) + (c, d) = (u + c, v + d)$; so, as $c < 0 < d$, we find that $u, v \in (u + c, v + d)$, and hence $u, v \in F$, $(u, v) \in \mathscr{F}$ and $(u, v) \subseteq F$.

THEOREM 4. $(A^*, \geqq)$ *is a tight Riesz group.*

PROOF. For directedness, take $\mathscr{F}, \mathscr{G} \in A^*$, and $F \in \mathscr{F}$ and $G \in \mathscr{G}$. By lemma 6 we can find $a, f \in F$ and $b, g \in G$ such that $(a, f) \in \mathscr{F}$, $(b, g) \in \mathscr{G}$, $(a, f) \subseteq F$, and $(b, g) \subseteq G$. Taking $h > f, g$ by the directedneess of $(A, \geqq)$, we see that $h > (a, f)$ and $h > (b, g)$; so $\mathscr{V}(h) > \mathscr{F}, \mathscr{G}$.

For the tight Riesz interpolation property, take $\mathscr{F}, \mathscr{G} < \mathscr{H}, \mathscr{K}$ in $A^*$; so we can find $F_1 < H_1$, $F_2 < K_2$, $G_1 < K_1$, $G_2 < K_2$, where $F_1, F_2 \in \mathscr{F}$, etc.; thus, putting $F = F_1 \cap F_2$, $G = G_1 \cap G_2$ etc., we have that $F, G < H, K$. By lemma 6, we can find $f_1, f_2 \in F$, $g_1, g_2 \in G$, etc. such that $(f_1, f_2) \in \mathscr{F}$ and $(f_1, f_2) \subseteq F$, etc. Hence $f_2, g_2 < h_1, k_1$ in $A$; so by the tight Riesz interpolation property of $(A, \geqq)$, then, there is an $a \in A$ such that $f_2, g_2 < a < h_1, k_1$.

Hence $(f_1, f_2)$, $(g_1, g_2) < a < (h_1, h_2)$, $(k_1, k_2)$; so $\mathscr{F}, \mathscr{G} < \mathscr{V}(a) < \mathscr{H}, \mathscr{K}$.

Thus we may now establish the connection with Fuchs' ordering $\succcurlyeq$ (cf. §3).

THEOREM 5. *Fuchs' ordering* $\succcurlyeq$ *on* $A^*$ *is the associated order of* $\geqq$ *on* $A^*$; *i.e.,* $\mathscr{F} > \mathscr{G} \Rightarrow \mathscr{F} > \mathscr{V}(0)$ *if and only if* $\mathscr{G} \succcurlyeq \mathscr{V}(0)$.

PROOF. By lemma 2, $(A^*, \geqq)$ has an associated order. Suppose $\mathscr{F} > \mathscr{G} \Rightarrow \mathscr{F} > \mathscr{V}(0)$, and take $G \in \mathscr{G}$; then, by lemma 6, there are $g, h \in G$ such that $(g, h) \in \mathscr{G}$ and $(g, h) \subseteq G$. So, as $h > (g, h)$, we have $\mathscr{V}(h) > \mathscr{G}$. Hence $\mathscr{V}(h) > \mathscr{V}(0)$, and so $h > 0$; thus $\mathscr{G} \succcurlyeq \mathscr{V}(0)$.

Next suppose that $\mathscr{G} \succcurlyeq \mathscr{V}(0)$ and $\mathscr{F} > \mathscr{G}$; then there are $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $F > G$, and, also, $g \in G$ such that $g \geqq 0$. Hence $F > g \geqq 0$, so that $\mathscr{F} > \mathscr{V}(0)$.

Fuchs [3] has investigated the problem of finding tight Riesz groups $(A, \geqq)$ whose Fuchs completion $(A^*, \succcurlyeq)$ is lattice-ordered; these he calls approximation antilattices. He, Reilly [8] and Wirth [9] have characterised tight Riesz groups whose associated order is lattice-ordered, in terms of the subset forming the positive cone for the tight Riesz order. One of his results may be extended to show that such a tight Riesz group has its Fuchs completion lattice-ordered; hence so does any dense subgroup of it.

Noticing that a sub-dense subgroup of a tight Riesz group is in fact a dense subgroup, and is therefore also a tight Riesz group itself, we can state: –

THEOREM 6. *A tight Riesz group without pseudozeros will have its Fuchs completion lattice-ordered if and only if it is o-isomorphic to a dense subgroup of a tight Riesz group with lattice-ordered associated order.*

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