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# Weighted Inequalities for Hardy–Steklov Operators

A. L. Bernardis, F. J. Martín-Reyes, and P. Ortega Salvador

*Abstract.* We characterize the pairs of weights $(v, w)$ for which the operator $Tf(x) = g(x)\int_{s(x)}^{h(x)} f$ with $s$ and $h$ increasing and continuous functions is of strong type $(p, q)$ or weak type $(p, q)$ with respect to the pair $(v, w)$ in the case $0 < q < p$ and $1 < p < \infty$. The result for the weak type is new while the characterizations for the strong type improve the ones given by H. P. Heinig and G. Sinnamon. In particular, we do not assume differentiability properties on $s$ and $h$ and we obtain that the strong type inequality $(p, q)$, $q < p$, is characterized by the fact that the function

$$\Phi(x) = \sup\Big(\int_c^d g^q w\Big)^{1/p}\Big(\int_{s(d)}^{h(c)} v^{1-p'}\Big)^{1/p'}$$

belongs to $L^r(g^q w)$, where $1/r = 1/q - 1/p$ and the supremum is taken over all $c$ and $d$ such that $c \le x \le d$ and $s(d) \le h(c)$.

## 1 Introduction and Results

Let us consider the Hardy–Steklov operator defined by

$$Tf(x) = g(x)\int_{s(x)}^{h(x)} f, \quad f \ge 0,$$

where $g$ is a positive measurable function and $s$ and $h$ are functions defined on an interval $(a, b)$ such that $s(x) \le h(x)$ for all $x \in (a, b)$. Particular cases of this operator are the Hardy operator $Tf(x) = \int_0^x f$, the Hardy averaging operators $Tf(x) = x^\eta \int_0^x f$ and the Steklov operator $Tf(x) = \int_{x-1}^{x+1} f$ which have been studied intensively (see [5] and the references given therein).

Weighted weak and strong type $(p, q)$ inequalities for the operator $T$ were studied by several authors. In the case $1 < p \le q < \infty$ and considering the functions $s$ and $h$ strictly increasing and differentiable, Heining and Sinnamon [4] have characterized the weighted strong type inequality

$$\Big(\int_a^b [Tf]^q w\Big)^{\frac{1}{q}} \le C\Big(\int_{s(a)}^{h(b)} f^p v\Big)^{\frac{1}{p}}, \quad f \ge 0, \tag{1.1}$$

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where $s(a) = \lim_{x\to a^+} s(x)$ and $h(b) = \lim_{x\to b^-} h(x)$ (analogously we write $s(b) = \lim_{x\to b^-} s(x)$ and $h(a) = \lim_{x\to a^+} h(x)$) by means of the condition

$$\text{(1.2)} \qquad \sup\Big(\int_t^x g^q w\Big)^{1/q}\Big(\int_{s(x)}^{h(t)} v^{1-p'}\Big)^{1/p'} < \infty,$$

where the supremum is taken over all $x$ and $t$ such that $t \le x$ and $s(x) \le h(t)$. Gogatishvili and Lang [3] obtained the same result, but assume a weaker hypothesis on the functions $s$ and $h$. They only assume that these functions are increasing ($x < y \Rightarrow s(x) \le s(y), h(x) \le h(y)$). Their result is proved in the more general setting of the Banach function spaces [3, Theorem 3.2].

The case $0 < q < p$ and $1 < p < \infty$ is different. Heinig and Sinnamon [4] obtained the following result.

***Theorem 1.3*** *Let $s$ and $h$ be strictly increasing differentiable functions defined on $(0,\infty)$ satisfying $s(0) = h(0) = 0$, $s(x) < h(x)$ for $x \in (0,\infty)$ and $s(\infty) = h(\infty) = \infty$. Let $g$ be a positive measurable function. Let $0 < q < p$, $1 < p < \infty$, $1/r = 1/q - 1/p$, and let $w$ and $v$ be nonnegative measurable functions defined on $(0,\infty)$. Let $(h^{-1}\circ s)^k$ be the $k$ times repeated composition and let $\{M_k\}_{k\in\mathbb{Z}}$ be a sequence defined by $M_0 = h^{-1}(1)$, $M_{k+1} = s^{-1}(h(M_k))$, if $k \ge 0$ and $M_k = (h^{-1}(s(M_{k+1}))$, if $k < 0$. Then there is a constant $C$ such that* (1.1) *holds if and only if*

$$\Big(\int_0^\infty \int_{h^{-1}(s(t))}^t \Big(\int_{s(t)}^{h(x)} v^{1-p'}\Big)^{r/p'}\Big(\int_x^t g^q w\Big)^{r/p} g^q(x)w(x)\,dx\,\sigma(t)\,dt\Big)^{1/r} < \infty$$

*and*

$$\Big(\int_0^\infty \int_t^{s^{-1}(h(t))} \Big(\int_{s(x)}^{h(t)} v^{1-p'}\Big)^{r/p'}\Big(\int_t^x g^q w\Big)^{r/p} g^q(x)w(x)\,dx\,\sigma(t)\,dt\Big)^{1/r} < \infty,$$

*where the "normalizing function" $\sigma$ is defined by*

$$\sigma(t) = \sum_{k\in\mathbb{Z}} \chi_{(M_k,M_{k+1})}(t)\frac{d}{dt}(h^{-1}\circ s)^k(t).$$

The results in [4] are stated for $g(x) = 1$ and not for general $g$. However, we notice that if $g(x) \ne 1$, then the characterizations of the strong type inequalities follow easily from the case $g(x) = 1$.

Characterizations of the weighted strong type inequality for the case $1 < p \le q < \infty$ and the case $0 < q < p$, $1 < p < \infty$ were obtained also by Chen and Sinnamon [2] under the hypothesis of the existence of a "discrete normalizing measure" for $s$ and $h$. A measure $\xi$ on the real line is called a normalizing measure for $s$ and $h$ provided there exist positive constants $C_1$ and $C_2$ such that

$$C_1 \le \xi([s(t), h(t)]) \le C_2,$$

for all $t$. If $\xi$ is a counting measure on a subset of the real line, then $\xi$ is called a discrete normalizing measure (see [2]).

We point out that the existence of the discrete normalizing measure is a hypothesis weaker than the monotonicity of the functions $s$ and $h$. We remark also that the measure $\xi$ is not involved in the characterizing conditions given in [2] for the case $1 < p \leq q < \infty$. However, in the case $0 < q < p$, $1 < p < \infty$, the normalizing measure appears explicitly in the conditions playing the role of $\sigma$ in Theorem 1.3. Furthermore, the construction of a discrete normalizing measure can be somewhat complicated (see [2]).

The first goal of this paper is to improve Theorem 1.3 by providing new characterizing conditions which do not involve either the normalizing measure or the double integral. We will suppose that the functions $s$ and $h$ are increasing and continuous. This level of generality allows us to obtain easily a convenient decomposition of $\Omega = \{x \in (a, b) : s(x) < h(x)\}$ which we need to state the result. The decomposition appears in the next lemma.

***Lemma 1.4*** *Let $s, h\colon (a, b) \to \mathbb{R}$ be increasing and continuous functions such that $s(x) \leq h(x)$ for all $x \in (a, b)$. Let $\{(a_j, b_j)\}_j$ be the connected components of the open set $\Omega = \{x \in (a, b) : s(x) < h(x)\}$. Then*

(i) *$(s(a_j), h(b_j)) \cap (s(a_i), h(b_i)) = \varnothing$ for all $j \neq i$.*

(ii) *For every $j$ there exists a (finite or infinite) sequence $\{m_k^j\}$ of real numbers such that:*

  (a) *$a_j \leq m_k^j < m_{k+1}^j \leq b_j$ for all $k$ and $j$;*

  (b) *$(a_j, b_j) = \bigcup_k (m_k^j, m_{k+1}^j)$ almost everywhere for all $j$;*

  (c) *$s(m_{k+1}^j) \leq h(m_k^j)$ for all $k$ and $j$ and $s(m_{k+1}^j) = h(m_k^j)$ if $a_j < m_k^j < m_{k+1}^j < b_j$.*

Now we are prepared to state our first theorem.

***Theorem 1.5*** *Let $s$ and $h$ be increasing and continuous functions defined on an interval $(a, b)$ satisfying $s(x) \leq h(x)$ for $x \in (a, b)$. Let $g$ be a positive measurable function on $(a, b)$. Let $w$ and $v$ be nonnegative measurable functions defined on $(a, b)$ and $(s(a), h(b))$, respectively. Let $q$, $p$ and $r$ be such that $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. The following statements are equivalent.*

(i) *There exists a positive constant $C$ such that* (1.1) *holds,* i.e.,

$$\Big(\int_a^b [Tf]^q w\,dx\Big)^{1/q} \leq C\Big(\int_{s(a)}^{h(b)} f^p v\Big)^{1/p}$$

*for all $f \geq 0$.*

(ii) *There exists a positive constant $C$ such that*

$$\sum_i \Big(\int_{c_i}^{d_i} g^q w\Big)^{r/q} \Big(\int_{s(d_i)}^{h(c_i)} v^{1-p'}\Big)^{r/p'} \leq C,$$

*for every sequence $\{(c_i, d_i)\}$ of disjoint intervals of $(a, b)$ such that $s(d_i) \leq h(c_i)$.*

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(iii) *For every connected component $(a_j, b_j)$ of the set $\Omega = \{x \in (a,b) : s(x) < h(x)\}$, there exists a sequence $\{m_k^j\}$ in the conditions of Lemma 1.4(ii) such that the functions*

$$\psi_1(x) = \sum_{j,k} \Big( \int_{m_k^j}^{x} g^q w \Big)^{\frac{1}{p}} \Big( \int_{s(x)}^{h(m_k^j)} v^{1-p'} \Big)^{\frac{1}{p'}} \chi_{(m_k^j, m_{k+1}^j)}(x)$$

*and*

$$\psi_2(x) = \sum_{j,k} \Big( \int_{x}^{m_{k+1}^j} g^q w \Big)^{\frac{1}{p}} \Big( \int_{s(m_{k+1}^j)}^{h(x)} v^{1-p'} \Big)^{\frac{1}{p'}} \chi_{(m_k^j, m_{k+1}^j)}(x)$$

*belong to $L^r(g^q w)$.*

(iv) *The function*

$$\Phi(x) = \sup \Big( \int_c^d g^q w \Big)^{1/p} \Big( \int_{s(d)}^{h(c)} v^{1-p'} \Big)^{1/p'}$$

*belongs to $L^r(g^q w)$, where the supremum is taken over all $c$ and $d$ such that $a < c \le x \le d < b$ and $s(d) \le h(c)$.*

Notice that condition (iv) in this theorem is the natural one if we compare it with condition (1.2). In fact, if $p = q$, then (1.2) can be written as the function $\Phi$ belongs to $L^\infty$; observe that if $p = q$ and $1/r = 1/q - 1/p$, then $r$ should be $\infty$. We remark also that condition (iv) is independent of the possible decompositions $\{m_k^j\}$ of the set $\Omega$.

Let us point out that condition (ii) is analogous to the one in [9], while the characterizing condition (iii) is inspired by Mazja's condition for the Hardy operator [8]. In fact, if we consider the particular case of the Hardy operator we have the following result (a direct proof appears in [1]).

***Corollary 1.6*** *Let $Tf(x) = \int_0^x f$, $x > 0$. Let $w$ and $v$ be nonnegative measurable functions defined on $(0, \infty)$. Let $q$, $p$ and $r$ be such that $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. The following statements are equivalent.*

(i) *There exists a positive constant $C$ such that* (1.1) *holds,* i.e.,

$$\Big( \int_0^\infty [Tf]^q w \Big)^{1/q} \le C \Big( \int_0^\infty f^p v \Big)^{1/p}$$

*for all $f \ge 0$.*

(ii) *The function*

$$\Psi(x) = \Big( \int_x^\infty w \Big)^{1/p} \Big( \int_0^x v^{1-p'} \Big)^{1/p'}$$

*belongs to $L^r(w)$.*

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(iii) *The function*

$$\Phi(x)=\sup\Big(\int_c^\infty w\Big)^{1/p}\Big(\int_0^c v^{1-p'}\Big)^{1/p'}$$

*belongs to* $L^r(w)$, *where the supremum is taken over all* $c$ *such that* $0<c\leq x$.

Observe that condition (ii) is Mazja's condition, so we recover the well-known condition, while (iii) is a new characterizing condition. In the next corollary we shall consider another particular case.

***Corollary 1.7*** *Let* $0<A<B<\infty$ *and* $\lambda=B/A$. *Let us define*

$$Tf(x)=\int_{Ax}^{Bx} f,\quad x>0.$$

*Let* $w$ *and* $v$ *be nonnegative measurable functions defined on* $(0,\infty)$. *Let* $q$, $p$ *and* $r$ *be such that* $0<q<p$, $1<p<\infty$ *and* $1/r=1/q-1/p$. *The following statements are equivalent.*

(i) *There exists a positive constant* $C$ *such that* (1.1) *holds,* i.e.,

$$\Big(\int_0^\infty [Tf]^q w\Big)^{1/q}\leq C\Big(\int_0^\infty f^p v\Big)^{1/p}$$

*for all* $f\geq 0$.

(ii) $\max\{K_1,K_2\}<\infty$, *where*

$$K_1=\Big(\int_0^\infty \frac{1}{t}\int_{\frac{A}{B}t}^{t}\Big(\int_{At}^{Bx} v^{1-p'}\Big)^{r/p'}\Big(\int_x^t w\Big)^{r/p} w(x)\,dxdt\Big)^{1/r}$$

*and*

$$K_2=\Big(\int_0^\infty \frac{1}{t}\int_t^{\frac{B}{A}t}\Big(\int_{Ax}^{Bt} v^{1-p'}\Big)^{r/p'}\Big(\int_t^x w\Big)^{r/p} w(x)\,dxdt\Big)^{1/r}.$$

(iii) *There exists* $t>0$ *such that the functions*

$$\Psi_1(x)=\sum_{k\in\mathbb{Z}}\Big(\int_{\lambda^k t}^{x} w\Big)^{1/p}\Big(\int_{Ax}^{B\lambda^k t} v^{1-p'}\Big)^{1/p'}\chi_{(\lambda^k t,\lambda^{k+1}t)}(x)$$

*and*

$$\Psi_2(x)=\sum_{k\in\mathbb{Z}}\Big(\int_x^{\lambda^{k+1}t} w\Big)^{1/p}\Big(\int_{A\lambda^{k+1}t}^{Bx} v^{1-p'}\Big)^{1/p'}\chi_{(\lambda^k t,\lambda^{k+1}t)}(x)$$

*belong to* $L^r(w)$.

(iv) *The function*

$$\Phi(x) = \sup\Big(\int_c^d w\Big)^{1/p}\Big(\int_{Ad}^{Bc} v^{1-p'}\Big)^{1/p'}$$

*belongs to $L^r(w)$, where the supremum is taken over all $c$ and $d$ such that $c \le x \le d \le \lambda c$.*

We notice that the equivalence between (ii) and (i) was obtained in [4] as a corollary of Theorem 1.3 [4, Theorem 2.5].

The last goal of this paper is to search the weighted weak type inequalities. In this case the results cannot be deduced from the corresponding for $g(x) = 1$. In [3], for the case $1 < p \le q < \infty$ and for $s$ and $h$ increasing functions, the authors characterize the weighted weak-type inequality

$$[w(\{x \in (a,b) : Tf(x) > \lambda\})]^{\frac{1}{q}} \le \frac{C}{\lambda}\Big(\int_{s(a)}^{h(b)} f^p v\Big)^{\frac{1}{p}} \tag{1.8}$$

for all $\lambda > 0$ and all $f \ge 0$ by means of the following condition: there exists $C > 0$ such that

$$\|g\chi_{(x,y)}\|_{q,\infty;w}\Big(\int_{s(y)}^{h(x)} v^{1-p'}\Big)^{\frac{1}{p'}} < C,$$

for all $a < x < y < b$ such that $s(y) \le h(x)$. Here $\|f\|_{q,\infty;w}$ denotes the norm in the space $L^{q,\infty}(w)$ defined by $\|f\|_{q,\infty;w} = \sup_{\lambda>0} \lambda(w(\{x : |f(x)| > \lambda\}))^{\frac{1}{q}}$ and $w(E)$ stands for $\int_E w$. This result can be obtained as a particular case of [3, Theorem 3.2].

We shall study this problem in the case $q < p$ by assuming that the function $g$ is monotone. The result is the next one.

***Theorem 1.9*** *Let $s$ and $h$ be increasing continuous functions defined on an interval $(a,b)$ satisfying $s(x) \le h(x)$ for $x \in (a,b)$. Let $g$ be a positive monotone function defined on $(a,b)$. Let $q$, $p$ and $r$ be such that $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. Let $w$ and $v$ be nonnegative measurable functions defined on $(a,b)$ and $(s(a), h(b))$, respectively. The following statements are equivalent.*

(i) *There exists a positive constant $C$ such that* (1.8) *holds,* i.e.,

$$\Big[w(\{x \in (a,b) : Tf(x) > \lambda\})\Big]^{1/q} \le \frac{C}{\lambda}\Big(\int_{s(a)}^{h(b)} f^p v\Big)^{1/p}$$

*for all $f \ge 0$ and all positive real number $\lambda$.*

(ii) *The function*

$$\Phi(x) = \sup\Big(\big(\inf_{y\in(c,d)} g(y)\big)\Big(\int_c^d w\Big)^{1/p}\Big(\int_{s(d)}^{h(c)} v^{1-p'}\Big)^{1/p'}\Big)$$

*belongs to $L^{r,\infty}(w)$, where the supremum is taken over the numbers $c$ and $d$ such that $a < c \le x \le d < b$ and $s(d) \le h(c)$.*

Observe that condition (iv) in Theorem 1.5 and condition (ii) in the above theorem are given in terms of, essentially, the same function. In fact, if $g(x) = 1$ then the functions $\Phi$ in both theorems are the same and the weighted strong type inequality is characterized by $\Phi \in L^r(w)$ while the weighted weak type inequality is characterized by $\Phi \in L^{r,\infty}(w)$. For the case of the modified Hardy operator $Tf(x) = g(x)\int_0^x f$, $x > 0$, this theorem was obtained in [7].

The organization of the paper is as follows. In section 2 we shall prove Lemma 1.4 and the relationships among some functions that we need in the proof of Theorem 1.5. Section 3 is devoted to prove Theorem 1.5. In section 4 we shall prove the Corollaries 1.6 and 1.7, while Theorem 1.9 is proved in Section 5.

## 2 Technical Results

**Proof of Lemma 1.4** In order to prove (i) let us observe first that the continuity of $s$ and $h$ imply that if $a_j > a$ then $s(a_j) = h(a_j)$ and if $b_j < b$ then $s(b_j) = h(b_j)$. Now, notice that $(a_j, b_j) \cap (a_i, b_i) = \varnothing$ and therefore $a_j < b_j \le a_i < b_i$ or $a_i < b_i \le a_j < b_j$. We may assume without lost of generality that $a_j < b_j \le a_i < b_i$. Then $a_i > a$ and $b_j < b$. By the monotonicity of $s$ we have $h(b_j) = s(b_j) \le s(a_i)$ and (i) follows.

Now, we shall prove (ii). We may suppose that there is only one connected component, *i.e.,* $\Omega = (a, b)$.

If $s(b) \le h(a)$, we choose $m_0 = a$, $m_1 = b$. The finite sequence $\{m_k\}_{k=0,1}$ satisfies conditions (a), (b) and (c).

Let us suppose now that $h(a) < s(b)$. By the continuity of the functions $s$ and $h$, there exist $m_0, m_1$ such that $a < m_0 < m_1 < b$ and $s(m_1) = h(m_0)$. If $m_0, m_1, \dots, m_k$ $(k \ge 1)$ have been chosen in such a way that $m_0 < m_1 < \cdots < m_k < b$ and $s(m_i) = h(m_{i-1})$ for $1 \le i \le k$, we select $m_{k+1}$ as follows:

- If $s(b) \le h(m_k)$), then $m_{k+1} = b$ and the process of selection of $m_k$, $k \ge 1$ finishes.
- If $h(m_k) < s(b)$, then we choose $m_{k+1}$ such that $m_k < m_{k+1} < b$ and $s(m_{k+1}) = h(m_k)$. This number $m_{k+1}$ can be chosen since $s(m_k) < h(m_k) < s(b)$.

In this way, we get $\{m_k\}_{k\ge 0}$ such that $(m_0, b) = \bigcup_{k\ge 0}(m_k, m_{k+1})$ almost everywhere. This is obvious if the sequence $\{m_k\}_{k\ge 0}$ is finite. If it is infinite and $c = \lim_{k\to\infty} m_k$, the fact that $s(m_{k+1}) = h(m_k)$ and the continuity of $s$ and $h$ imply $s(c) = h(c)$, which is possible only if $c = b$.

In an analogous way, we choose $m_k$ for $k < 0$ verifying $(a, m_0) = \bigcup_{k<0}(m_k, m_{k+1})$ almost everywhere. So, we have $(a, b) = \bigcup_k (m_k, m_{k+1})$ almost everywhere and we are done. ■

In the rest of this section we state and prove some results that we shall use in the proof of Theorem 1.5. Taking into account the partition given in Lemma 1.4 and calling $I_{j,k} = (m_k^j, m_{k+1}^j)$, $I_{j,k}^-(x) = (m_k^j, x)$ and $I_{j,k}^+(x) = (x, m_{k+1}^j)$, we define the following functions:

$$G_1(x) = \sum_{j,k} \sup_{d\in I_{j,k}^+(x)} \Big(\int_x^d w\Big)^{\frac{1}{p}} \Big(\int_{s(d)}^{h(m_k^j)} v^{1-p'}\Big)^{\frac{1}{p'}} \chi_{I_{j,k}}(x);$$

$$G_2(x)=\sum_{j,k}\sup_{c\in I^-_{j,k}(x)}\Big(\int_c^x w\Big)^{\frac{1}{p}}\Big(\int_{s(m^j_{k+1})}^{h(c)} v^{1-p'}\Big)^{\frac{1}{p'}}\chi_{I_{j,k}}(x);$$

$$H_1(x)=\sum_{j,k}\sup_{d\in I^+_{j,k}(x)}\Big(\int_{m^j_k}^d w\Big)^{\frac{1}{p}}\Big(\int_{s(d)}^{h(m^j_k)} v^{1-p'}\Big)^{\frac{1}{p'}}\chi_{I_{j,k}}(x);$$

$$H_2(x)=\sum_{j,k}\sup_{c\in I^-_{j,k}(x)}\Big(\int_c^{m^j_{k+1}} w\Big)^{\frac{1}{p}}\Big(\int_{s(m^j_{k+1})}^{h(c)} v^{1-p'}\Big)^{\frac{1}{p'}}\chi_{I_{j,k}}(x).$$

If $\mathcal{K}_1(j)=\{k: a_j<m^j_k\}$ and $\mathcal{K}_2(j)=\{k: m^j_{k+1}<b_j\}$, we also define:

$$U_1(x)=\sum_j\sum_{k\in\mathcal{K}_1(j)}\sup_{t\in I_{j,k-1}}\Big(\int_t^{m^j_k} w\Big)^{1/p}\Big(\int_{s(x)}^{h(t)} v^{1-p'}\Big)^{1/p'}\chi_{I_{j,k}}(x)$$

and

$$U_2(x)=\sum_j\sum_{k\in\mathcal{K}_2(j)}\sup_{t\in I_{j,k+1}}\Big(\int_{m^j_{k+1}}^t w\Big)^{1/p}\Big(\int_{s(t)}^{h(x)} v^{1-p'}\Big)^{1/p'}\chi_{I_{j,k}}(x).$$

The next lemmas establish some relationships among these functions.

***Lemma 2.1*** *For all $\lambda>0$ and $i=1,2$,*

$$w(\{x\in(a,b): G_i(x)>\lambda\})\le 2w(\{x\in(a,b):\psi_i(x)>\lambda\}),$$

*where $\psi_1$ and $\psi_2$ are the functions defined in Theorem* 1.5 *with $g(x)=1$.*

**Proof** We only prove the case $i=1$, the other one follows in a similar way. Notice that

$$w(\{x: G_1(x)>\lambda\})=w(\{x: G_1(x)>\lambda,\psi_1(x)>\lambda\})+\sum_{k,j}w(E_{k,j}),$$

where $E_{k,j}=\{x\in I_{j,k}: G_1(x)>\lambda,\psi_1(x)\le\lambda\}$. Clearly, we only have to show that

$$w(E_{k,j})\le w(\{x\in I_{j,k}:\psi_1(x)>\lambda\}).$$

To prove this inequality it suffices to establish that

$$\int_{m^j_k}^z w\le w(\{x\in I_{j,k}:\psi_1(x)>\lambda\}) \tag{2.2}$$

for all $z\in E_{k,j}$. Let $z\in E_{k,j}$. Then there exist $d$, $z<d<m^j_{k+1}$, such that

$$\Big(\int_{m^j_k}^z w\Big)^{1/p}\Big(\int_{s(z)}^{h(m^j_k)} v^{1-p'}\Big)^{1/p'}\le\lambda<\Big(\int_z^d w\Big)^{1/p}\Big(\int_{s(d)}^{h(m^j_k)} v^{1-p'}\Big)^{1/p'},$$

and since $d > z$ we get that

$$\int_{m_k^j}^{z} w \le \int_{z}^{d} w.$$

If $(z,d) \subset \{x \in I_{j,k} : \psi_1(x) > \lambda\}$ then (2.2) follows immediately. Assume now that the set $F = \{t \in (z,d) : \psi_1(t) \le \lambda\}$ is nonempty and let $\alpha = \sup F$. If $t \in F$ then

$$\Big(\int_{m_k^j}^{t} w\Big)^{1/p}\Big(\int_{s(t)}^{h(m_k^j)} v^{1-p'}\Big)^{1/p'} \le \lambda < \Big(\int_{z}^{d} w\Big)^{1/p}\Big(\int_{s(d)}^{h(m_k^j)} v^{1-p'}\Big)^{1/p'}.$$

Since $t < d$, we obtain $\int_{m_k^j}^{t} w \le \int_z^d w$ and consequently

$$\int_{m_k^j}^{z} w \le \int_{t}^{d} w.$$

Letting $t$ tend to $\alpha$ we get that

$$\int_{m_k^j}^{z} w \le \int_{\alpha}^{d} w.$$

This inequality implies (2.2) since $(\alpha, d) \subset \{x \in I_{j,k} : \psi_1(x) > \lambda\}$. ■

The following lemma follows easily from the definitions.

***Lemma 2.3*** *For all $x \in (a,b)$ and $i = 1,2$,*

$$H_i(x) \le \psi_i(x) + G_i(x),$$

*where $\psi_1$ and $\psi_2$ are the functions defined in Theorem 1.5 with $g(x) = 1$.*

***Lemma 2.4*** *For all $\lambda > 0$ and $i = 1,2$*

$$w(\{x : U_i(x) > \lambda\}) \le \sum_{i=1}^{2} w(\{x : H_i(x) > \lambda\}).$$

**Proof** We shall prove it only for $i = 1$. Observe that

$$w(\{x : U_1(x) > \lambda\}) \le w(\{x : H_1(x) > \lambda\}) + \sum_{j,k} w(E_{j,k}),$$

where $E_{j,k} = \{x \in I_{j,k} : U_1(x) > \lambda, H_1(x) \le \lambda\}$. It will suffice to show that

$$w(E_{j,k}) \le w(\{x \in I_{j,k-1} : H_2(x) > \lambda\})$$

If $E_{j,k} = \varnothing$, there is nothing to prove. Assume that $E_{j,k} \neq \varnothing$ and let $z \in E_{j,k}$. Notice that $k \in \mathcal{K}_1(j)$, since $U_1(z) > \lambda$. Then there exists $t \in I_{j,k-1}$ with $s(z) < h(t)$ such that

$$\Big(\int_{m_k^j}^{z} w\Big)^{1/p}\Big(\int_{s(z)}^{h(m_k^j)} v^{1-p'}\Big)^{1/p'} \leq \lambda < \Big(\int_{t}^{m_k^j} w\Big)^{1/p}\Big(\int_{s(z)}^{h(t)} v^{1-p'}\Big)^{1/p'}.$$

Since $h(t) \leq h(m_k^j)$ we get that $\int_{m_k^j}^{z} w \leq \int_t^{m_k^j} w$. Observe that from the second inequality above we get that $(t, {m_k}^j) \subset \{x \in I_{j,k-1} : H_2(x) > \lambda\}$ and therefore

$$\int_{m_k^j}^{z} w \leq \int_{\{x \in I_{j,k-1}: H_2(x) > \lambda\}} w,$$

for all $z \in E_{j,k}$. Then if $z \nearrow \sup E_{j,k}$ we get that

$$w(E_{j,k}) \leq w(\{x \in I_{j,k-1} : H_2(x) > \lambda\})$$

and we are done. ■

***Lemma 2.5*** *For almost every $x \in (a, b)$*

$$\Phi(x) \leq 2\sum_{i=1}^{2}[\psi_i(x) + H_i(x) + U_i(x)],$$

*where $\Phi$, $\psi_1$ and $\psi_2$ are the functions defined in Theorem* 1.5 *with* $g(x) = 1$.

**Proof** It is clear that $\Phi(x) \leq \Phi_1(x) + \Phi_2(x)$, where

$$\Phi_1(x) = \sup_{c \leq x: s(x) \leq h(c)} \Big(\int_c^x w\Big)^{1/p}\Big(\int_{s(x)}^{h(c)} v^{1-p'}\Big)^{1/p'}$$

and

$$\Phi_2(x) = \sup_{x \leq d: s(d) \leq h(x)} \Big(\int_x^d w\Big)^{1/p}\Big(\int_{s(d)}^{h(x)} v^{1-p'}\Big)^{1/p'}.$$

We can prove that $\Phi_1(x) \leq \psi_1(x) + H_2(x) + U_1(x)$ and $\Phi_2(x) \leq \psi_2(x) + H_1(x) + U_2(x)$. As in the above lemmas, we only prove the first case since the other one follows in a similar way. Notice that $\Phi_1(x) = \Phi_2(x) = 0$ if $x \notin \Omega = \{x \in (a, b) : s(x) < h(x)\}$. Let us suppose that $x \in (m_k^j, m_{k+1}^j)$. If $c < x$ is such that $s(x) < h(c)$ and $m_k^j \in (a_j, b_j)$, then we get that $c > m_{k-1}^j$. In such a case, if $m_k^j < c < x$ then

$$\Big(\int_c^x w\Big)^{1/p}\Big(\int_{s(x)}^{h(c)} v^{1-p'}\Big)^{1/p'} \leq \psi_1(x) + H_2(x),$$

and if $m_{k-1}^j < c \leq m_k^j$

$$\Big(\int_c^x w\Big)^{1/p}\Big(\int_{s(x)}^{h(c)} v^{1-p'}\Big)^{1/p'} \leq \psi_1(x) + U_1(x),$$

obtaining $\Phi_1(x) \leq \psi_1(x) + H_2(x) + U_1(x)$. If $m_k^j = a_j$, then we only have the possibility $m_k^j < c < x$ which, as we have just seen, gives the desired inequality. ■

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## 3 Proofs of Theorem 1.5

It is enough to prove the theorem for $g(x) = 1$.

(i) $\Rightarrow$ (ii). Let $\{(c_i, d_i)\}$ be a sequence of disjoint intervals such that $s(d_i) \le h(c_i)$. Using standard approximation arguments we may assume that

$$\int_{c_i}^{d_i} w < \infty \quad \text{and} \quad \int_{s(d_i)}^{h(c_i)} v^{1-p'} < \infty.$$

Then (ii) follows easily from (i) applied to the function

$$f(x) = \Big[\sum_i \Big(\int_{c_i}^{d_i} w\Big)^{r/q} \Big(\int_{s(d_i)}^{h(c_i)} v^{1-p'}\Big)^{r/q'} v^{-p'}(x)\chi_{(s(d_i),h(c_i))}(x)\Big]^{1/p}$$

Since

$$Tf(x) \ge \int_{s(d_i)}^{h(c_i)} f \ge \Big(\int_{c_i}^{d_i} w\Big)^{r/pq} \Big(\int_{s(d_i)}^{h(c_i)} v^{1-p'}\Big)^{r/pq'+1},$$

for all $x \in (c_i, d_i)$.

(ii) $\Rightarrow$ (iii). First, notice that (ii) and Lemma 1.4(ii) imply that there exists $C > 0$ such that for every connected component $(a_j, b_j)$ of the set $\Omega$, there exists a sequence $\{m_k^j\}$ in the required conditions such that

$$\sum_j \sum_k \Big(\int_{m_k^j}^{m_{k+1}^j} w\Big)^{r/q} \Big(\int_{s(m_{k+1}^j)}^{h(m_k^j)} v^{1-p'}\Big)^{r/p'} \le C. \tag{3.1}$$

Now, let us see that $\psi_2 \in L^r(w)$; the proof of $\psi_1 \in L^r(w)$ is very similar. We have

$$\begin{aligned}
\int_a^b [\psi_2(x)]^r w(x)\,dx &= \sum_{j,k} \int_{m_k^j}^{m_{k+1}^j} [\psi_2(x)]^r w(x)\,dx \\
&\le C \sum_{j,k} \int_{m_k^j}^{m_{k+1}^j} \Big(\int_x^{m_{k+1}^j} w\Big)^{r/p} \Big(\int_{s(m_{k+1}^j)}^{h(m_k^j)} v^{1-p'}\Big)^{r/p'} w(x)\,dx \\
&\quad + C \sum_{j,k} \int_{m_k^j}^{m_{k+1}^j} \Big(\int_x^{m_{k+1}^j} w\Big)^{r/p} \Big(\int_{h(m_k^j)}^{h(x)} v^{1-p'}\Big)^{r/p'} w(x)\,dx \\
&= \mathrm{I} + \mathrm{II}\,.
\end{aligned}$$

Performing the integral in I and using (3.1), we see that $I$ is finite. In order to estimate II, we shall use a suitable partition of each interval $(m_k^j, m_{k+1}^j)$ (this idea is taken from [6, Lemma 1]). Since we wish to estimate II, it is clear that we only have to consider intervals $(m_k^j, m_{k+1}^j)$ such that

$$\int_{m_k^j}^{m_{k+1}^j} w > 0 \quad \text{and} \quad \int_{h(m_k^j)}^{h(m_{k+1}^j)} v^{1-p'} > 0.$$

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By using approximation arguments we may assume that

$$\alpha_{j,k} = \int_{m_k^j}^{m_{k+1}^j} w < \infty \quad \text{and} \quad \beta_{j,k} = \int_{h(m_k^j)}^{h(m_{k+1}^j)} v^{1-p'} < \infty.$$

On one hand, we define an increasing sequence $\{x_i\}$ such that $x_0 = m_k^j$ and

$$\int_{x_i}^{m_{k+1}^j} w = 2^{-i}\alpha_{j,k}.$$

On the other hand, we define a decreasing sequence $\{u'_s\}$ such that $u'_0 = m_{k+1}^j$ and

$$\int_{h(m_k^j)}^{h(u'_s)} v^{1-p'} = 2^{-s}\beta_{j,k}.$$

Notice that both sequences depend on $j, k$. Now, we shall select a finite subsequence $\{u_n\}$ of $\{u'_s\}$ by the following principle: if $[u'_{s+1}, u'_s) \cap \{x_i\} = \varnothing$, delete the element $u'_{s+1}$ from the sequence $\{u'_s\}$. Denote the subsequence by $\{u_n\}$. Observe that if

$$N = \min\{s : m_k^j < u'_s < x_1\},$$

then $u'_N$ is the last term that we can choose. Let $M$ be such that $u_{M-1} = u'_N$ and define $u_M = m_k^j$. Then $u_M = m_k^j < u_{M-1} < \cdots < u_0 = m_{k+1}^j$ is a partition of the interval $[m_k^j, m_{k+1}^j]$. Let $\mathcal{I}(n) = \{i : u_{n+1} \le x_{i+1} < u_n\}$. Now, we write

$$\begin{aligned} \mathrm{II}_{j,k} &= \int_{m_k^j}^{m_{k+1}^j} \Big(\int_x^{m_{k+1}^j} w\Big)^{r/p} \Big(\int_{h(m_k^j)}^{h(x)} v^{1-p'}\Big)^{r/p'} w(x)\,dx \\ &= \sum_{n=0}^{M-2} \sum_{i\in\mathcal{I}(n)} \int_{x_i}^{x_{i+1}} \Big(\int_x^{m_{k+1}^j} w\Big)^{r/p} \Big(\int_{h(m_k^j)}^{h(x)} v^{1-p'}\Big)^{r/p'} w(x)\,dx. \end{aligned}$$

For fixed $n \le M-2$ there exists $s$ such that $u'_{s+1} = u_{n+1}$. If $u_{n+1} \le x_{i+1} < u_n$, then it follows by the definition of the subsequence that $x_{i+1} < u'_s$ and $u_{n+2} \le u'_{s+2}$. Furthermore, by the definition of the sequence $\{u'_s\}$ we get

$$\int_{h(m_k^j)}^{h(x_{i+1})} v^{1-p'} \le \int_{h(m_k^j)}^{h(u'_s)} v^{1-p'} = 4\int_{h(u'_{s+2})}^{h(u'_{s+1})} v^{1-p'} \le 4\int_{h(u_{n+2})}^{h(u_{n+1})} v^{1-p'}.$$

From these inequalities we get that

$$\mathrm{II}_{j,k} \le 4^{r/p'} \sum_{n=0}^{M-2} \Big(\int_{h(u_{n+2})}^{h(u_{n+1})} v^{1-p'}\Big)^{r/p'} \sum_{i\in\mathcal{I}(n)} \int_{x_i}^{x_{i+1}} \Big(\int_x^{m_{k+1}^j} w\Big)^{r/p} w(x)\,dx.$$

By the definition of the sequence $\{x_i\}$, we get that

$$\mathrm{II}_{j,k} \le C \sum_{n=0}^{M-2} \Big( \int_{h(u_{n+2})}^{h(u_{n+1})} v^{1-p'} \Big)^{r/p'} \sum_{i \in \mathfrak{I}(n)} \Big( \int_{x_{i+1}}^{x_{i+2}} w \Big)^{r/q}.$$

Since

$$\sum_{i \in \mathfrak{I}(n)} \Big( \int_{x_{i+1}}^{x_{i+2}} w \Big)^{r/q} \le \Big( \sum_{i \in \mathfrak{I}(n)} \int_{x_{i+1}}^{x_{i+2}} w \Big)^{r/q}$$

and $\bigcup_{i \in \mathfrak{I}(n)} (x_{i+1}, x_{i+2}) \subset (u_{n+1}, u_{n-1})$, where $u_{-1} = u_0$, we get that

$$\mathrm{II}_{j,k} \le C \sum_{n=0}^{M-2} \Big( \int_{h(u_{n+2})}^{h(u_{n+1})} v^{1-p'} \Big)^{r/p'} \Big( \int_{u_{n+1}}^{u_{n-1}} w \Big)^{r/q}.$$

Now, since $s(u_{n-1}) \le s(m^j_{k+1}) \le h(m^j_k) \le h(u_{n+2})$ for all $0 \le n \le M-2$, we get that

$$\begin{aligned} \mathrm{II}_{j,k} \le& C \sum_{n=0}^{M-2} \Big( \int_{s(u_{n-1})}^{h(u_{n+1})} v^{1-p'} \Big)^{r/p'} \Big( \int_{u_{n+1}}^{u_{n-1}} w \Big)^{r/q} \\ =& C \Big( \sum_{n=2m} (\,\cdot\,) + \sum_{n=2m+1} (\,\cdot\,) \Big). \end{aligned}$$

Now, summing up in $j, k$ and using (ii) we see that II is finite.

(iii) $\Rightarrow$ (iv). It is an easy consequence of Lemmas 2.1, 2.3, 2.4 and 2.5.

(iv) $\Rightarrow$ (i). Taking into account the decomposition given in Lemma 1.4 we get that

$$\begin{aligned} \int_{a_j}^{b_j} [Tf]^q w &= \sum_k \int_{m^j_k}^{m^j_{k+1}} [Tf]^q w \\ &\le C \sum_k \int_{m^j_k}^{m^j_{k+1}} \Big[ \int_{s(x)}^{s(m^j_{k+1})} f \Big]^q w + C \sum_k \int_{m^j_k}^{m^j_{k+1}} \Big[ \int_{h(m^j_k)}^{h(x)} f \Big]^q w \\ &\quad + C \sum_k \Big( \int_{s(m^j_{k+1})}^{h(m^j_k)} f \Big)^q \Big( \int_{m^j_k}^{m^j_{k+1}} w \Big) \;=\; \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

In order to estimate I, we fix $k$ and select an increasing sequence $\{x_i\}_i$, $x_i \in [m^j_k, m^j_{k+1}]$ such that $x_0 = m^j_k$ and

$$\int_{s(x_i)}^{s(m^j_{k+1})} f = \int_{s(x_{i-1})}^{s(x_i)} f.$$

Then by the properties of the sequence $\{x_i\}_i$ and applying Hölder's inequality, we get

$$\begin{aligned}\int_{m_k^j}^{m_{k+1}^j}\Big[\int_{s(x)}^{s(m_{k+1}^j)} f\Big]^q w &= \sum_i \int_{x_i}^{x_{i+1}}\Big[\int_{s(x)}^{s(m_{k+1}^j)} f\Big]^q w\\
&\le \sum_i \Big(\int_{s(x_i)}^{s(m_{k+1}^j)} f\Big)^q\Big(\int_{x_i}^{x_{i+1}} w\Big)\\
&\le 4^q \sum_i \Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f\Big)^q\Big(\int_{x_i}^{x_{i+1}} w\Big)\\
&\le 4^q \sum_i \Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{q/p}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} v^{1-p'}\Big)^{q/p'}\Big(\int_{x_i}^{x_{i+1}} w\Big).\end{aligned}$$

Then, since $s(x_{i+2}) \le s(m_{k+1}^j) \le h(m_k^j) \le h(x_i)$, applying the Hölder inequality we get that

$$\begin{aligned}\mathrm{I} &\le C\sum_{k,i}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{q/p}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} v^{1-p'}\Big)^{q/p'}\Big(\int_{x_i}^{x_{i+1}} w\Big)\\
&\le C\Big(\sum_{k,i}\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{q/p}\Big[\sum_{k,i}\Big(\int_{s(x_{i+1})}^{h(x_i)} v^{1-p'}\Big)^{r/p'}\Big(\int_{x_i}^{x_{i+1}} w\Big)^{r/q}\Big]^{q/r}\\
&\le C\Big(\int_{s(a_j)}^{s(b_j)} f^p v\Big)^{q/p}\Big[\sum_{k,i}\Big(\int_{s(x_{i+1})}^{h(x_i)} v^{1-p'}\Big)^{r/p'}\Big(\int_{x_i}^{x_{i+1}} w\Big)^{r/q}\Big]^{q/r}.\end{aligned}$$

By applying the identity

$$\Big(\int_a^b f\Big)^t = t\int_a^b\Big(\int_a^x f\Big)^{t-1} f(x)\,dx, \quad \text{if } t \ge 1,$$

we get that

$$\begin{aligned}&\Big(\int_{x_i}^{x_{i+1}} w\Big)^{r/q}\Big(\int_{s(x_{i+1})}^{h(x_i)} v^{1-p'}\Big)^{r/p'}\\
&\qquad = r/q\Big(\int_{x_i}^{x_{i+1}}\Big(\int_{x_i}^{x} w\Big)^{r/p} w(x)\,dx\Big)\Big(\int_{s(x_{i+1})}^{h(x_i)} v^{1-p'}\Big)^{r/p'}\\
&\qquad \le r/q\int_{x_i}^{x_{i+1}}\Big(\int_{x_i}^{x} w\Big)^{r/p}\Big(\int_{s(x)}^{h(x_i)} v^{1-p'}\Big)^{r/p'} w(x)\,dx\\
&\qquad \le r/q\int_{x_i}^{x_{i+1}} \Phi^r(x) w(x)\,dx.\end{aligned}$$

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Then

$$
\begin{aligned}
\mathrm{I} &\le C\Big(\int_{s(a_j)}^{s(b_j)} f^p v\Big)^{q/p}\Big[\sum_k\sum_i\int_{x_i}^{x_{i+1}}\Phi^r(x)w(x)\,dx\Big]^{q/r}\\
&\le \Big(\int_{s(a_j)}^{h(b_j)} f^p v\Big)^{q/p}\Big(\int_{a_j}^{b_j}\Phi^r(x)w(x)\,dx\Big)^{q/r}.
\end{aligned}
$$

In a similar way, we can prove that

$$
\mathrm{II} \le C\Big(\int_{s(a_j)}^{h(b_j)} f^p v\Big)^{q/p}\Big(\int_{a_j}^{b_j}\Phi^r(x)w(x)\,dx\Big)^{q/r}.
$$

On the other hand, using Hölder's inequality we get that

$$
\begin{aligned}
\mathrm{III} &\le C\Big(\sum_k\int_{s(m^j_{k+1})}^{h(m^j_k)} f^p v\Big)^{q/p}\Big[\sum_k\Big(\int_{s(m^j_{k+1})}^{h(m^j_k)} v^{1-p'}\Big)^{r/p'}\Big(\int_{m^j_k}^{m^j_{k+1}} w\Big)^{r/q}\Big]^{q/r}\\
&\le C\Big(\int_{s(a_j)}^{h(b_j)} f^p v\Big)^{q/p}\Big(\int_{a_j}^{b_j}\Phi^r(x)w(x)\,dx\Big)^{q/r}.
\end{aligned}
$$

Now, putting together the estimates of I, II and III, summing up in $j$, and applying Hölder's inequality, we get that

$$
\int_a^b [Tf]^q w \le C\Big(\sum_j\int_{s(a_j)}^{h(b_j)} f^p v\Big)^{q/p}\Big(\int_a^b \Phi^r w\Big)^{q/r}.
$$

Finally, taking into account that the intervals $(s(a_j), h(b_j))$ are disjoint (see Lemma 1.4(i)) we have that

$$
\int_a^b [Tf]^q w \le C\Big(\int_{s(a)}^{h(b)} f^p v\Big)^{q/p}\Big(\int_a^b \Phi^r w\Big)^{q/r},
$$

and we are done. ■

## 4 Proof of Corollaries 1.6 and 1.7

**Proof of Corollary 1.6** Observe that the equivalence (i) ⇔ (iii) is given directly by Theorem 1.5. Since $\Psi \le \Phi$ then (iii) ⇒ (ii). To prove the converse, we only have to realize that $\{m_k\}_{k=0}^1$, with $m_0 = 0$ and $m_1 = \infty$, is a sequence satisfying the conditions in Lemma 1.4 for the functions $s(x) = 0$ and $h(x) = x$. The associated functions $\psi_1$ and $\psi_2$ in Theorem 1.5 are 0 and $\Psi$, respectively. Therefore, (ii) in Corollary 1.6 means that (iii) in Theorem 1.5 holds. Then (iv) (in Theorem 1.5) holds, but that is the same as (iii) in Corollary 1.6 and we are done. ■

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**Proof of Corollary 1.7** We first observe that (iv) in Corollary 1.7 is exactly the same as (iv) in Theorem 1.5 for $s(x) = Ax$, $h(x) = Bx$ and $g(x) = 1$. So we already have that (i) $\Leftrightarrow$ (iv). In what follows, we shall prove (iv) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(iv) $\Rightarrow$ (ii). By Fubini's Theorem

$$\begin{aligned} K_1 &= \Big( \int_0^\infty w(x) \int_x^{\frac{B}{A}x} \frac{1}{t} \Big( \int_{At}^{Bx} v^{1-p'} \Big)^{r/p'} \Big( \int_x^t w \Big)^{r/p} dt\,dx \Big)^{1/r} \\ &\le \Big( \int_0^\infty w(x) \int_x^{\frac{B}{A}x} \frac{1}{t} \Phi^r(x)\,dt\,dx \Big)^{1/r} \\ &= (\log(B/A))^{1/r} \Big( \int_0^\infty \Phi^r(x) w(x)\,dx(x) \Big)^{1/r}. \end{aligned}$$

The same estimate holds for $K_2$. Therefore (iv) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Clearly

$$K_1^r = \sum_{k=-\infty}^{\infty} \int_{\lambda^k}^{\lambda^{k+1}} \frac{1}{t} \int_{t/\lambda}^{t} \Big( \int_{At}^{Bx} v^{1-p'} \Big)^{r/p'} \Big( \int_x^t w \Big)^{r/p} w(x)\,dxdt.$$

Changing the variable ($t = \lambda^{k+1}s$),

$$K_1^r = \int_{1/\lambda}^{1} \sum_{k=-\infty}^{\infty} \frac{1}{s} \int_{\lambda^k s}^{\lambda^{k+1}s} \Big( \int_{A\lambda^{k+1}s}^{Bx} v^{1-p'} \Big)^{r/p'} \Big( \int_x^{\lambda^{k+1}s} w \Big)^{r/p} w(x)\,dxds.$$

Since $K_1^r < \infty$, we deduce that for almost every $s \in (1/\lambda, 1)$

$$\sum_{k=-\infty}^{\infty} \int_{\lambda^k s}^{\lambda^{k+1}s} \Big( \int_{A\lambda^{k+1}s}^{Bx} v^{1-p'} \Big)^{r/p'} \Big( \int_x^{\lambda^{k+1}s} w \Big)^{r/p} w(x)\,dx < \infty. \tag{4.1}$$

Analogously, from $K_2^r < \infty$ we obtain that for almost every $s \in (1/\lambda, 1)$

$$\sum_{k=-\infty}^{\infty} \int_{\lambda^k s}^{\lambda^{k+1}s} \Big( \int_{Ax}^{B\lambda^k s} v^{1-p'} \Big)^{r/p'} \Big( \int_{\lambda^k s}^{x} w \Big)^{r/p} w(x)\,dx < \infty. \tag{4.2}$$

Consequently, there exists $s \in (1/\lambda, 1)$ such that (4.1) and (4.2) hold simultaneously, *i.e.,* (iii) holds.

(iii) $\Rightarrow$ (iv). Given $t > 0$, the sequence $m_k = \lambda^k t$ satisfies the conditions in Lemma 1.4 for the functions $s(x) = Ax$, $h(x) = Bx$ and the unique connected component of $\Omega = \{x : s(x) < h(x)\} = (0, \infty)$. Then (iii) is nothing but condition (iii) in Theorem 1.5. It follows that (iv) (in Theorem 1.5) holds, but as we have already said, that is exactly the same as (iv) in Corollary 1.7. ■

## 5 Proof of Theorem 1.9

(ii) $\Rightarrow$ (i). We shall prove the weighted weak type inequality for nonnegative functions such that $\int_{s(a)}^{h(b)} f^p v = 1$. The general case follows easily. Observe that if $U = \{x \in (a,b) : Tf(x) > \lambda, \Phi(x) \le \lambda^{q/r}\}$, then

$$w(\{x \in (a,b) : Tf(x) > \lambda\} \le w(\{x \in (a,b) : \Phi(x) > \lambda^{q/r}\}) + w(U) \le \frac{\|\Phi\|_{r,\infty,w}^r}{\lambda^q} + w(U).$$

Therefore the implication will be proved if we establish that $w(U) \le \frac{C}{\lambda^q}$. Let $\Omega$, $(a_j, b_j)$ and $\{m_k^j\}$ be as in Lemma 1.4. Then for fixed $j$,

$$w(U \cap (a_j, b_j)) = \sum_k w(U \cap (m_k^j, m_{k+1}^j)). \tag{5.1}$$

If $x \in (m_k^j, m_{k+1}^j)$, since $s(m_{k+1}^j) \le h(m_k^j)$, we get that

$$Tf(x) = g(x) \int_{s(x)}^{s(m_{k+1}^j)} f + g(x) \int_{s(m_{k+1}^j)}^{h(m_k^j)} f + g(x) \int_{h(m_k^j)}^{h(x)} f.$$

It is clear that

$$\begin{aligned} w(U \cap (m_k^j, m_{k+1}^j)) &\le w\Big( \Big\{ x \in (m_k^j, m_{k+1}^j) : g(x) \int_{s(x)}^{s(m_{k+1}^j)} f > \lambda/3, \Phi(x) \le \lambda^{q/r} \Big\} \Big) \\ &\quad + w\Big( \Big\{ x \in (m_k^j, m_{k+1}^j) : g(x) \int_{s(m_{k+1}^j)}^{h(m_k^j)} f > \lambda/3, \Phi(x) \le \lambda^{q/r} \Big\} \Big) \\ &\quad + w\Big( \Big\{ x \in (m_k^j, m_{k+1}^j) : g(x) \int_{h(m_k^j)}^{h(x)} f > \lambda/3, \Phi(x) \le \lambda^{q/r} \Big\} \Big) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

To estimate II, we write $E = \{x \in (m_k^j, m_{k+1}^j) : g(x) \int_{s(m_{k+1}^j)}^{h(m_k^j)} f > \lambda/3, \Phi(x) \le \lambda^{q/r}\}$, $\alpha = \inf E$, $\beta = \sup E$, and we choose a sequence $\{z_l\}_l \downarrow \alpha$ such that $z_l \in E$. We have for every $x \in E$ that $\lambda/3 < g(x) \int_{s(m_{k+1}^j)}^{h(m_k)} f$ and therefore by the monotonicity of $g$

$$\lambda/3 \le \inf_{z \in (z_l, \beta)} g(z) \int_{s(m_{k+1}^j)}^{h(m_k^j)} f.$$

Applying the Hölder inequality and multiplying by $\left( \int_{z_l}^{\beta} w \right)^{1/p}$ we have

$$\lambda \Big( \int_{z_l}^{\beta} w \Big)^{1/p} \le \inf_{z \in (z_l, \beta)} g(z) \Big( \int_{z_l}^{\beta} w \Big)^{1/p} \Big( \int_{s(m_{k+1}^j)}^{h(m_k^j)} v^{1-p'} \Big)^{1/p'} \Big( \int_{s(m_{k+1}^j)}^{h(m_k^j)} f^p v \Big)^{1/p}.$$

Now, since $s$ and $h$ are increasing functions, the last term is dominated by

$$\inf_{z\in(z_l,\beta)} g(z)\Big(\int_{z_l}^{\beta} w\Big)^{1/p}\Big(\int_{s(\beta)}^{h(z_l)} v^{1-p'}\Big)^{1/p'}\Big(\int_{s(m^j_{k+1})}^{h(m^j_k)} f^p v\Big)^{1/p}$$

$$\leq \Phi(z_l)\Big(\int_{s(m^j_{k+1})}^{h(m^j_k)} f^p v\Big)^{1/p} \leq \lambda^{q/r}\Big(\int_{s(m^j_{k+1})}^{h(m^j_k)} f^p v\Big)^{1/p}.$$

Therefore,

$$\mathrm{II} = \int_E w \leq \lim_{z_l\to\alpha}\int_{z_l}^{\beta} w \leq \frac{1}{\lambda^q}\int_{s(m^j_{k+1})}^{h(m^j_k)} f^p v.$$

In order to estimate $I$, we select an increasing sequence $\{x_i\}_i$, $x_i \in (m^j_k, m^j_{k+1})$, such that $x_0 = m^j_k$ and

$$\int_{s(x_i)}^{s(m^j_{k+1})} f = \int_{s(x_{i-1})}^{s(x_i)} f.$$

Let $E_i = \{x \in (x_i, x_{i+1}) : g(x)\int_{s(x)}^{s(m^j_{k+1})} f > \lambda/3, \Phi(x) \leq \lambda^{q/r}\}$, $\alpha_i = \inf E_i$ and $\beta_i = \sup E_i$. Let us choose a sequence $\{z_l\}_l \uparrow \beta_i$. Using the monotonicity of $g$ and the property of the sequence $\{x_i\}_i$ we have

$$\lambda/3 \leq 4 \inf_{z\in(\alpha_i,z_l)} g(z)\int_{s(x_{i+1})}^{s(x_{i+2})} f.$$

Applying the Hölder inequality and multiplying by $\big(\int_{\alpha_i}^{z_l} w\big)^{1/p}$ we have

$$\lambda/3\Big(\int_{\alpha_i}^{z_l} w\Big)^{1/p} \leq 4\inf_{z\in(\alpha_i,z_l)} g(z)\Big(\int_{\alpha_i}^{z_l} w\Big)^{1/p}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} v^{1-p'}\Big)^{1/p'}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{1/p}.$$

Now since $s$ and $h$ are increasing functions and $s(m^j_{k+1}) \leq h(m^j_k)$, the last term is dominated by

$$4\inf_{z\in(\alpha_i,z_l)} g(z)\Big(\int_{\alpha_i}^{z_l} w\Big)^{1/p}\Big(\int_{s(z_l)}^{s(m^j_{k+1})} v^{1-p'}\Big)^{1/p'}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{1/p}$$

$$\leq 4\inf_{z\in(\alpha_i,z_l)} g(z)\Big(\int_{\alpha_i}^{z_l} w\Big)^{1/p}\Big(\int_{s(z_l)}^{h(\alpha_i)} v^{1-p'}\Big)^{1/p'}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{1/p}$$

$$\leq \Phi(z_l)\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{1/p} \leq \lambda^{q/r}\Big(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\Big)^{1/p}.$$

Therefore,

$$\int_{E_i} w \leq \lim_{z_l\to\beta_i}\int_{\alpha_i}^{z_l} w \leq \frac{C}{\lambda^q}\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v.$$

Now summing up in $i$, we obtain

$$\mathrm{I} \leq \frac{C}{\lambda^q} \int_{s(m_k^j)}^{s(m_{k+1}^j)} f^p v.$$

In a similar way it is proved that

$$\mathrm{III} \leq \frac{C}{\lambda^q} \int_{h(m_k^j)}^{h(m_{k+1}^j)} f^p v.$$

Putting together the estimates for I, II and III, we have

$$w(U \cap (m_k^j, m_{k+1}^j)) \leq \frac{C}{\lambda^q} \int_{s(m_k^j)}^{h(m_{k+1}^j)} f^p v.$$

Summing in $k$ in the above inequality and putting together with (5.1), we get that

$$w(U \cap (a_j, b_j)) \leq \frac{C}{\lambda^q} \int_{s(a_j)}^{h(b_j)} f^p v.$$

Keeping in mind Lemma 1.4 and summing up in $j$, we obtain the desired inequality. Therefore (ii) $\Rightarrow$ (i) is completely proved.

(i) $\Rightarrow$ (ii). We have to prove that

$$\sup_{\lambda>0} \lambda \Big( \int_{\{x \in (a,b) : \Phi(x) > \lambda\}} w \Big)^{1/r} < \infty.$$

Let $\lambda > 0$ and $S_\lambda = \{x \in (a,b) : \Phi(x) > \lambda\}$. For every $z \in S_\lambda$ there exist $a_z$ and $b_z$ with $a < a_z < z < b_z < b$ such that $s(b_z) < h(a_z)$ and

$$\lambda < \inf_{y \in (a_z, b_z)} g(y) \Big( \int_{a_z}^{b_z} w \Big)^{1/p} \Big( \int_{s(b_z)}^{h(a_z)} v^{1-p'} \Big)^{1/p'}.$$

Let $K \subset S_\lambda$ be a compact set. Then there exist $(a_{z_1}, b_{z_1}), \dots, (a_{z_k}, b_{z_k})$ which cover $K$. We may assume without loss of generality that $\sum_{j=1}^k \chi_{(a_{z_j}, b_{z_j})} \leq 2\chi_{\cup_{j=1}^k (a_{z_j}, b_{z_j})}$. Let $f\colon (s(a), h(b)) \to \mathbb{R}$ defined by

$$f(x) = \Big( \sum_{j=1}^k \Big( \frac{1}{\inf_{y \in (a_{z_j}, b_{z_j})} g(y) \int_{s(b_{z_j})}^{h(a_{z_j})} v^{1-p'}} \Big)^p v^{-p'}(x) \chi_{(s(b_{z_j}), h(a_{z_j}))}(x) \Big)^{1/p}.$$

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If $z \in (a_{z_j}, b_{z_j})$, then we have $Tf(z) = g(z)\int_{s(z)}^{h(z)} f \geq 1$. Therefore, $\bigcup_{j=1}^{k}(a_{z_j}, b_{z_j}) \subset \{x \in (a,b) : Tf(x) \geq 1\}$. Applying the weighted weak type inequality we obtain

$$\begin{aligned}\int_{\cup_{j=1}^{k}(a_{z_j},b_{z_j})} w &\leq C\Big(\sum_{j=1}^{k}\Big(\frac{1}{\inf_{y\in(a_{z_j},b_{z_j})} g(y)\int_{s(b_{z_j})}^{h(a_{z_j})} v^{1-p'}}\Big)^{p}\int_{s(b_{z_j})}^{h(a_{z_j})} v^{1-p'}\Big)^{q/p}\\ &= C\Big(\sum_{j=1}^{k}\frac{1}{(\inf_{y\in(a_{z_j},b_{z_j})} g(y))^{p}(\int_{s(b_{z_j})}^{h(a_{z_j})} v^{1-p'})^{p-1}}\Big)^{q/p}\\ &\leq \frac{C}{\lambda^{q}}\Big(\sum_{j=1}^{k}\int_{a_{z_j}}^{b_{z_j}} w\Big)^{q/p}\\ &\leq \frac{C}{\lambda^{q}}\Big(\int_{\cup_{j=1}^{k}(a_{z_j},b_{z_j})} w\Big)^{q/p}.\end{aligned}$$

The last inequality implies that $\lambda\big(\int_K w\big)^{1/r} \leq C$ for any compact set $K \subset S_\lambda$ and we are done.

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*IMAL-CONICET,*
*Güemes 3450*
*(3000) Santa Fe,*
*Argentina*
*e-mail: bernard@ceride.gov.ar*

*Análisis Matemático*
*Facultad de Ciencias*
*Universidad de Málaga*
*29071 Málaga*
*Spain*
*e-mail: martin_reyes@uma.es*
*ortega@anamat.cie.uma.es*