For example, in the case of the parabola we have

$$3y_2y_4 - 5y_3^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

whence by differentiation $\quad 3y_2y_5 - 7y_3y_4 = 0.$

This may be written $\quad 9y_2^2y_5 - 21y_2y_3y_4 = 0$;

or $\quad 9y_2^2y_5 - 45y_2y_3y_4 + 24y_2y_3y_4 = 0$;

that is using (18)

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0;$$

which is the general differential equation to a conic section.

Note on the Integration of $x^m(a+bx^n)^p dx$.

By Thomas Muir, LL.D.

The integration of differentials of the form $x^m(a+bx^n)^p dx$ seems to me to be susceptible of a more methodical mode of treatment than that commonly employed. In the ordinary way of presenting the matter there is little choice left to the student, when such an integration is required of him, between a haphazard, tentative process, and the consultation of a text-book, in which lists of "formulae of reduction" are given.

In beginning the subject with a learner, I should first state that the integration can be made dependent on any one of six different integrals, viz :—

$$(1) \int x^{m-n}(a+bx^n)^p dx,$$

$$(2) \int x^{m+n}(a+bx^n)^p dx,$$

$$(3) \int x^m(a+bx^n)^{p-1} dx,$$

$$(4) \int x^m(a+bx^n)^{p+1} dx,$$

$$(5) \int x^{m-n}(a+bx^n)^{p+1} dx,$$

$$(6) \int x^{m+n}(a+bx^n)^{p-1} dx;$$

that is to say, the integral can be expressed in terms of a like integral in which the index of the monomial factor is greater or less by n; in terms of a like integral in which the index of the binomial factor is greater or less by 1; in terms of a like integral in which the index of the monomial factor is less by n, and that of the binomial factor greater by 1; and in terms of a like integral in which the index of the monomial factor is greater by n, and that of the binomial factor less by 1.

Then I should proceed to show as follows how these transformations could be effected, asking the learner to note particularly that in order to minimize memory work, I had set myself the problem of obtaining them all *by one and the same process.*

(1) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m-n}(a+bx^n)^p dx$.

Take the integral $\int x^{m-n}(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\begin{aligned}\int x^{m-n}(a+bx^n)^{p+1}dx &= \frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1} - \int \frac{x^{m-n+1}}{m-n+1}(p+1)(a+bx^n)^p nbx^{n-1}dx,\\ &= \frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1} - \frac{(p+1)nb}{m-n+1}\int x^m(a+bx^n)^p dx. \qquad (\alpha)\end{aligned}$$

Again the fact that $(a+bx^n)^{p+1} = (a+bx^n)^p(a+bx^n)$ gives

$$\int x^{m-n}(a+bx^n)^{p+1}dx = a\int x^{m-n}(a+bx^n)^p dx + b\int x^m(a+bx^n)^p dx. \qquad (\beta)$$

Eliminating $\int x^{m-n}(a+bx^n)^{p+1}dx$ between (α) and (β) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m-n}(a+bx^n)^p dx$ as was required.

(2) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m+n}(a+bx^n)^p dx$.

Take the integral $\int x^m(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^{p+1}dx = \frac{x^{m+1}}{m+1}(a+bx^n)^{p+1} - \int \frac{x^{m+1}}{m+1}(p+1)(a+bx^n)^p nbx^{n-1}dx, \qquad (\alpha)$$

Again

$$\int x^m(a+bx^n)^{p+1}dx = a\int x^m(a+bx^n)^p dx + b\int x^{m+n}(a+bx^n)^p dx. \qquad (\beta)$$

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Eliminating $\int x^m(a+bx^n)^{p+1}dx$ between (a) and (β) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m+n}(a+bx^n)^p dx$ as was required.

(3) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p-1}dx$.

Take the integral $\int x^m(a+bx^n)^p dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}}{m+1}(a+bx^n)^p - \int \frac{x^{m+1}}{m+1}p(a+bx^n)^{p-1}nbx^{n-1}dx,$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^p - \frac{pnb}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx. \qquad (a)$$

Again

$$\int x^m(a+bx^n)^p dx = a\int x^m(a+bx^n)^{p-1}dx + b\int x^{m+n}(a+bx^n)^{p-1}dx. \qquad (\beta)$$

Eliminating $\int x^{m+n}(a+bx^n)^{p-1}dx$ between (a) and (β) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p-1}dx$ as was required.

(4) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p+1}dx$.

Take the integral $\int x^m(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^{p+1}dx$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^{p+1} - \int \frac{x^{m+1}}{m+1}(p+1)(a+bx^n)^p nbx^{n-1}dx,$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^{p+1} - \frac{(p+1)nb}{m+1}\int x^{m+n}(a+bx^n)^p dx, \qquad (a)$$

Again

$$\int x^m(a+bx^n)^{p+1}dx = a\int x^m(a+bx^n)^p dx + b\int x^{m+n}(a+bx^n)^p dx. \qquad (\beta)$$

Eliminating $\int x^{m+n}(a+bx^n)^p dx$ between (a) and (β) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p+1}dx$ as was required.

(5) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m-n}(a+bx^n)^{p+1}dx$.

Take the integral $\int x^{m-n}(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\int x^{m-n}(a+bx^n)^{p+1}dx$$

$$= \frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1} - \int \frac{x^{m-n+1}}{m-n+1}(p+1)(a+bx^n)^p nbx^{n-1}dx.$$

$$= \frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1} - \frac{(p+1)nb}{m-n+1}\int x^m(a+bx^n)^p dx.$$

Here it is not necessary to proceed further; we have already got what was required.

(6) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m+n}(a+bx^n)^{p-1}dx$.

Take the integral $\int x^m(a+bx^n)^p dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}}{m+1}(a+bx^n)^p - \int \frac{x^{m+1}}{m+1}\, p(a+bx^n)^{p-1}nbx^{n-1}dx$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^p - \frac{pnb}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx.$$

Here again we have got what was required by one operation only.

The transformations thus being effected attention could be drawn to the leading points of the process by which all the six results are obtained. These are

(*a*) The choice of an integral for operating on.
(*b*) Integrating "by parts."
(*c*) Partition of the integral into two.
(*d*) Elimination of an integral between the two equations thus got.

The only question remaining then to be answered would be as to how the integral we begin with is selected. The answer to this is that the monomial factor is always x^m unless we wish to make the required integral dependent on one where the monomial factor is x^{m-n}, in which case we start with x^{m-n}; and that the binomial factor is always $(a+bx^n)^{p+1}$ unless we wish to make the required integral dependent on one where the binomial factor is $(a+bx^n)^{p-1}$, in which case we start with $(a+bx^n)^p$. If the learner's memory were not very robust and his spirit did not contemn the aid of a rule in rhyme, I should prefer to dictate to him the answer in a more condensed and winning form, viz:—

Monomial's index leave unchanged,
Except to lessen when required;
Binomial's increase by one,
But change not if 'tis less desired.

This, it seems to me, is Todhunter's third chapter in a nutshell. I am inclined to view it as one of the few unobjectionable outcomes of the modern over-examination system, it having been devised in 1868 when preparing for an approaching hour of trial. Another such product of the same period but of a less artificial character was published in the *Journal of Education* for 1875 with the title "On Integration by Parts." Both of them I have found of considerable service in teaching.

Historical Note on the so-called Simson line.

By Thomas Muir, LL.D.

The theorem that the feet of the perpendiculars drawn to the sides of a triangle from any point in the circumference of the circumscribing circle are collinear is ascribed (Gerg. Ann. iv., p. 250, ca. 1814) by Servois, though not with confident knowledge, to Simson. Baltzer, who gives us this information, refers also to Gerg. Ann. xiv., p. 28, p. 280, and to Poncelet. Fuller details, showing how the question of authority has hitherto stood, will be found in an extract from a letter of Mr Mackay's in *Nature*, xxx., p. 635. Mr Mackay has further stated that he has not found the property mentioned in any of Simson's published works.

It seems, therefore, of some interest to point out that the theorem is enunciated and proved in a paper with the title "Mathematical Lucubrations," published in Leybourn's *Mathematical Repository*, old series, vol. II., p. 111. The author is Mr William Wallace, assistant mathematical master in Perth Academy, afterwards professor in the Royal Military Academy of Woolwich, and in the University of Edinburgh. The date of publication is 1798. No reference is made to Simson, the theorem apparently being given as new.
