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Some characterizations of ρ-Einstein solitons on Sasakian manifolds

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Abstract. The ρ-Einstein soliton is a self-similar solution of the Ricci–Bourguignon flow, which includes or relates to some famous geometric solitons, for example, the Ricci soliton and the Yamabe soliton, and so on. This paper deals with the study of ρ-Einstein solitons on Sasakian manifolds. First, we prove that if a Sasakian manifold M admits a nontrivial ρ-Einstein soliton (M, g, V, λ), then M is $\mathcal{D}$-homothetically fixed null η-Einstein and the soliton vector field V is Jacobi field along trajectories of the Reeb vector field ξ, nonstrict infinitesimal contact transformation and leaves φ invariant. Next, we find two sufficient conditions for a compact ρ-Einstein almost soliton to be trivial (Einstein) under the assumption that the soliton vector field is an infinitesimal contact transformation or is parallel to the Reeb vector field ξ.

1 Introduction

In recent years, the pioneering works of Hamilton [14] and Perelman [19] toward the solution of the Poincaré conjecture have produced a flourishing activity in the research of self-similar solutions, or solitons, of the Ricci flow. For more details on Ricci solitons, we refer to the reader to [1, 9, 11–13, 18, 21]. In general, Bourguignon [4] introduced a perturbed version of the Ricci flow on an n-dimensional Riemannian manifold (M, g), which satisfies the following evolution equation [4] considered a geometric flow of the following type:

$$\frac{\partial g}{\partial t} = -2\left(\mathrm{Ric_g} - \rho\, r\, g\right), \tag{1.1}$$

where $\mathrm{Ric_g}$ is the Ricci tensor, r is the scalar curvature of g and $\rho \in \mathbb{R}$. This flow is known as *the Ricci-Bourguignon flow (or shortly RB flow)* and the short time existence $\left(\text{for } \rho < \frac{1}{2(n-1)}\right)$ of this flow is provided in [7]. This family of geometric flows contains, as a special case, the Einstein flow $\left(\rho = \frac{1}{2}\right)$, the traceless Ricci flow $\left(\rho = \frac{1}{n}\right)$, the Schouten flow $\left(\rho = \frac{1}{2(n-1)}\right)$ and the Ricci flow $(\rho = 0)$, and so on, see [7, 15, 19]. On the other hand, by choosing $\rho \to -\infty$, the RB flow behaves like a Yamabe flow. Recently, Ho [15] studied this flow on locally homogeneous 3-manifolds. More examples of this flow were constructed on the product of an Anti de Sitter space with sphere in

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[2]. The self-similar solution associated to the flow (1.1) is described by the following definition:

Definition 1.1 An n-dimensional Riemannian manifold (M, g), $n \geq 3$, is said to be a *Ricci–Bouguignon soliton (or RB soliton or ρ-Einstein soliton)* if there is a smooth vector field V (called potential vector field) satisfying

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric}_g - \rho\, r\, g = \lambda\, g, \tag{1.2}$$

where $\rho, \lambda \in \mathbb{R}$ and $\mathcal{L}_V$ denotes the Lie-derivative in the direction of V.

It is a natural generalization of Einstein metrics and is a topic of current research in Riemannian geometry, see details [7, 10, 16, 15, 20]. It is denoted by (M, g, V, λ). A ρ-Einstein soliton is called *trivial* if V is a Killing vector field, i.e., $\mathcal{L}_V g = 0$. We say that a soliton is shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. For particular value of the parameter ρ, a ρ-Einstein soliton is called

(I) Einstein soliton if $\rho = \frac{1}{2}$,

(II) traceless Ricci soliton if $\rho = \frac{1}{n}$, and

(III) Schouten soliton if $\rho = \frac{1}{2(n-1)}$.

Recently, Dwivedi [10] introduced the notion of *Ricci–Bourguignon almost soliton (or RB almost soliton or ρ-Einstein almost soliton)* by allowing the soliton constant λ to be a smooth function. If ∇f is the gradient of a smooth function f on M, then the *Hessian of f* is defined by

$$\nabla^2 f(X_1, Y_1) = \mathrm{Hess}_f(X_1, Y_1) = g(\nabla_{X_1}\nabla f, Y_1)$$

for all vector fields X_1, Y_1 on M. In particular, if $V = \nabla f$ for some smooth function f, then $\mathcal{L}_{\nabla f} = 2\nabla^2 f = \mathrm{Hess_f}$ and the ρ-Einstein soliton and ρ-Einstein almost soliton is called the *gradient ρ-Einstein soliton* and *gradient ρ-Einstein almost soliton*, respectively, see details in [16, 20, 22].

In [7], Catino et al. studied the ρ-Einstein solitons where they obtained important rigidity results and proved that every compact gradient Einstein, Schouten, or traceless Ricci soliton is trivial. Recently, Dwivedi [10] studied ρ-Einstein almost solitons and presented some nontrivial examples. He found some integral formulas for compact gradient ρ-Einstein solitons and compact gradient ρ-Einstein almost solitons. Using the integral formulas, he found some sufficient conditions under which a gradient ρ-Einstein almost solitons are isometric to a unit sphere or Einstein (trivial). On the other hand, Ho [15] studied gradient shrinking ρ-Einstein solitons under the assumption of Bach flatness, and Huang [16] found some integral pinching rigidity results for compact gradient shrinking ρ-Einstein solitons. Recently, Shaikh et al. [20] found some geometric characterizations of gradient ρ-Einstein solitons.

In [12], Ghosh-Sharma proved that *if a Sasakian manifold of dimension* > 3 *admits a nontrivial Ricci soliton, then M is $\mathcal{D}$-homothetically fixed null η-Einstein, and the potential vector field V leaves the structure tensor φ invariant, and is an infinitesimal contact $\mathcal{D}$-homothetic transformation*. It is well known that the scalar curvature on a Sasakian manifold is not a constant, and therefore, ρ-Einstein solitons are an interesting generalization of Ricci solitons. In our first result, we generalize and improve the Theorem 1 of [12] for ρ-Einstein soliton. Now, we recall the following definitions.

A contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be *η-Einstein* if its Ricci curvature tensor has the form

$$\mathrm{Ric}_g = \alpha\, g + \beta\, \eta \otimes \eta,$$

where α, β are smooth functions on M. For a K-contact manifold of dimension > 3, α and β are constants, see Yano and Kon [24, p. 286]. Moreover, under a $\mathcal{D}$-homothetic deformation:

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta$$

for a positive real constant a, a K-contact η-Einstein manifold transforms to another K-contact η-Einstein manifold, in which $\bar{\alpha} = (\alpha + 2 - 2a)/a$ and $\bar{\beta} = 2n - \bar{\alpha}$. In particular, for $\alpha = -2$ remains fixed under a homothetic deformation, and as $\alpha + \beta = 2n$, β also remains fixed, and therefore, we define: a K-contact η-Einstein manifold with $\alpha = -2$ is said to be *$\mathcal{D}$-homothetically fixed*.

Let J denotes the restrictionof φ to the contact sub-bundle $\mathcal{D}(\eta = 0)$. Then for a Sasakian manifold, $(\mathcal{D}, J, d\eta)$ defines a Kähler metric on $\mathcal{D}$, with the transverse Kähler metric g^T related to the Sasakian metric g as $g = g^T + \eta \otimes \eta$. One can easily compute the relation between the transverse Ricci tensor ${\mathrm{Ric}_g}^T$ of g^T and the Ricci tensor Ric_g of g by

$${\mathrm{Ric}_g}^T(X_1, Y_1) = \mathrm{Ric}_g(X_1, Y_1) + 2\, g(X_1, Y_1), \quad X_1, Y_1 \in \mathcal{D}.$$

The Ricci form τ and transverse Ricci form τ^T are defined by

$$\tau(X_1, Y_1) = \mathrm{Ric}_g(X_1, \varphi Y_1), \qquad \tau^T(X_1, Y_1) = {\mathrm{Ric}_g}^T(X_1, \varphi Y_1), \qquad X_1, Y_1 \in \mathcal{D}.$$

The basic first Chern class $2\pi c_1^B$ of $\mathcal{D}$ is represented by τ^T. A Sasakian structure is said to be *null (transverse Calabi-Yau)* if $c_1^B = 0$. An η-Einstein Sasakian manifold with $\alpha = -2$ and $\beta = 2n + 2$ is known as null-Sasakian, which is characterized by $c_1^B = 0$. Further, an η-Einstein Sasakian manifold with $\alpha > -2$ is called a *positive-Sasakian* manifold. We refer the reader to [6] for details of its importance, examples and geometrical characteristics.

Theorem 1.1 *If a Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, $n > 1$, admits a nontrivial ρ-Einstein soliton (M, g, V, λ), then*

(i) *the soliton vector field V is a Jacobi field along trajectories of the Reeb vector field ξ,*
(ii) *M is $\mathcal{D}$-homothetically fixed null η-Einstein,*
(iii) *V is a non-strict infinitesimal contact transformation and is equal to $-\frac{1}{2}\varphi\nabla f + f\xi$ for a smooth function f on M such that $\xi(f) = -4(n+1)$, and*
(iv) *V leaves the structure tensor φ invariant.*

Remark 1.2 One may ask: *under what conditions the above result holds for ρ-Einstein almost soliton*? Following Theorem 3.1 of [11] and our Proposition 2.1, one can see that the above theorem also holds for ρ-Einstein almost soliton if V is a Jacobi field along trajectories of the Reeb vector field ξ.

In the geometry of a Ricci almost soliton, an important question is: *under what conditions a Ricci almost soliton is trivial (Einstein)?* Several results are proved in finding conditions under which a compact Ricci almost soliton is trivial, see details in [1, 9, 8, 11, 18]. Note that a ρ-Einstein almost soliton is a generalization of an Einstein manifold, Ricci soliton, Ricci almost soliton, as well as ρ-Einstein soliton; therefore, the following question arises:

Under what conditions a ρ-Einstein almost soliton on a K-contact manifold is trivial (Einstein)?

In [10], Dwivedi obtained some sufficient conditions under which a compact ρ-Einstein almost soliton is trivial (Einstein) on Riemannian manifold. Here, we find some sufficient conditions under which a ρ-Einstein almost soliton on K-contact manifold is trivial (Einstein). For this, in the next theorem, we consider the potential vector field as an infinitesimal contact transformation.

Theorem 1.3 *If a compact K-contact manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits a ρ-Einstein almost soliton (M, g, V, λ) with the potential vector field V is an infinitesimal contact transformation, then V is an infinitesimal automorphism and g is trivial $(Einstein)$ Sasakian and of constant scalar curvature $2n(2n+1)$. Moreover, the soliton is expanding, steady, or shrinking if $\rho > \frac{1}{2n+1}$, $\rho = \frac{1}{2n+1}$, or $\rho < \frac{1}{2n+1}$, respectively.*

Next, considering that the potential vector field V is parallel to the Reeb vector field ξ, we find one more sufficient condition for trivial ρ-Einstein almost soliton.

Theorem 1.4 *If a K-contact manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits a ρ-Einstein almost soliton (M, g, V, λ), whose potential vector field V is parallel to the Reeb vector field ξ, then V is a Killing vector field, g is trivial $(Einstein)$ and of constant scalar curvature $2n(2n+1)$. Moreover, the soliton is expanding, steady, or shrinking if $\rho > \frac{1}{2n+1}$, $\rho = \frac{1}{2n+1}$, or $\rho < \frac{1}{2n+1}$, respectively.*

Finally, applying Boyer and Galicki's result (see Theorem 11.1.7 of [5, p. 372]): *"any compact K-contact Einstein manifold is Sasakian"* in the previous result, we conclude the following.

Corollary 1.5 *If a compact K-contact manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits a ρ-Einstein almost soliton (M, g, V, λ), whose potential vector field V is parallel to the Reeb vector field ξ, then V is a Killing vector field, g is trivial $(Einstein)$ Sasakian and of constant scalar curvature $2n(2n+1)$. Moreover, the soliton is expanding, steady, or shrinking if $\rho > \frac{1}{2n+1}$, $\rho = \frac{1}{2n+1}$, or $\rho < \frac{1}{2n+1}$, respectively.*

Remark 1.6 We notice that Kumara et al. [17] studied K-contact manifold whose metric is a gradient Einstein-type. From their result (see Theorem 3.2 in [17]) it follows that *if a complete K-contact manifold admits a gradient ρ-Einstein almost soliton, then it is compact Einstein Sasakian and isometric to the unit sphere $\mathbb{S}^{2n+1}$.*

2 Background

In this section, we recall some basic facts about Sasakian geometry and fixing the notation which will be adopted in the rest of the paper. All manifolds are assumed to be smooth and connected.

Let ∇ be the *Levi-Civita connection* of a Riemannian manifold (M, g) and R the *Riemann curvature tensor* of g, given by

$$R(X_1, Y_1) = [\nabla_{X_1}, \nabla_{Y_1}] - \nabla_{[X_1,Y_1]}, \quad X_1, Y_1 \in \mathfrak{X}(M), \tag{2.1}$$

where $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M. We recall that *the Ricci operator* Q is a symmetric $(1,1)$-tensor field defined by

$$g(QX_1, Y_1) = \mathrm{Ric}_g(X_1, Y_1) = \mathrm{Tr}_g\{Z_1 \to R(Z_1, X_1)Y_1\}, \quad X_1, Y_1, Z_1 \in \mathfrak{X}(M),$$

and the scalar curvature of g is the smooth function defined by $r = \mathrm{Tr}_g Q$. The gradient of the scalar curvature r is given by

$$\frac{1}{2} g(X_1, \nabla r) = (\mathrm{div}Q)(X_1) = \sum_i g((\nabla_{E_i} Q)X_1, E_i), \quad X_1 \in \mathfrak{X}(M), \tag{2.2}$$

where $\{E_i\}$ is a local orthonormal frame on M.

A $(2n+1)$-dimensional manifold M is said to be a *contact manifold* if it admits a global 1-form η (called a *contact form*) such that $\eta \wedge (d\eta)^n \neq 0$. For such a structure, there exists a unique vector field ξ, called the *Reeb vector field* or *characteristic vector field*, satisfying $d\eta(\xi, \cdot) = 0$ and $\eta(\xi) = 1$. In addition, polarization of $d\eta$ on the *contact sub-bundle* $\mathcal{D}$ (defined by $\eta = 0$) gives a $(1,1)$-tensor field φ, and the Riemannian metric g satisfying

$$\varphi^2 = -id + \eta \otimes \xi, \quad \eta = g(\xi, \cdot), \tag{2.3}$$

$$d\eta(\cdot, \cdot) = g(\cdot, \varphi \cdot), \tag{2.4}$$

where $id : TM \to TM$ is the identity operator. The above structure (φ, ξ, η, g) is called a *contact metric structure*, $(M^{2n+1}, \varphi, \xi, \eta, g)$ a *contact metric manifold* and g an *associated metric*. It follows from (2.3) that

$$\eta \circ \varphi = 0, \quad \varphi(\xi) = 0, \quad \text{and} \quad \mathrm{rank}(\varphi) = 2n.$$

Moreover, a contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be *Sasakian* if the metric cone $(C(M), \bar{g}) = (M \times \mathbb{R}^+, r^2 g + dr^2)$ over M, is *Kähler*, or equivalently, a contact metric manifold is said to be *Sasakian manifold* (see [3, p. 86]) if

$$(\nabla_{X_1} \varphi)Y_1 = g(X_1, Y_1)\xi - \eta(Y_1)X_1, \quad X_1, Y_1 \in \mathfrak{X}(M). \tag{2.5}$$

The curvature tensor R of a Sasakian manifold has the following property:

$$R(X_1, Y_1)\xi = \eta(Y_1)X_1 - \eta(X_1)Y_1, \quad X_1, Y_1 \in \mathfrak{X}(M). \tag{2.6}$$

A vector field X_1 on a Riemannian manifold (M, g) is said to be *Killing* if $\mathcal{L}_{X_1} g = 0$. It is well known that if ξ is a Killing vector field, then M is said to be a *K-contact manifold*. A Sasakian manifold is a K-contact manifold, but the converse is true only

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in dimension 3 (e.g., [3, p. 87]). On a K-contact manifold, the following formulae are valid (see Blair [3, p. 113]):

$$\nabla_{X_1}\xi = -\varphi X_1, \quad X_1 \in \mathfrak{X}(M), \tag{2.7}$$

$$Q\xi = 2n\xi. \tag{2.8}$$

Further, using (2.7), we find the covariant derivative of (2.8) along an arbitrary vector field $X_1 \in \mathfrak{X}(M)$,

$$(\nabla_{X_1}Q)\xi = Q\varphi X_1 - 2n\,\varphi X_1. \tag{2.9}$$

According to Blair [3], a vector field V on a contact manifold (M,η) is said to be an *infinitesimal contact transformation (or a contact vector field)* if it preserves the contact form η, i.e., there exists a smooth function $f : M \to \mathbb{R}$ satisfying

$$\mathcal{L}_V\eta = f\,\eta, \tag{2.10}$$

and if $f = 0$, then the vector field V is said to be *strict*. A vector field V on a contact metric manifold is said to be *an infinitesimal automorphism* if it leaves φ, ξ, η and g invariant.

Proposition 2.1 *Let a K-contact (Sasakian) metric g admits an ρ-Einstein almost soliton. Then the following formula is valid:*

$$\begin{aligned}(\mathcal{L}_V\nabla)(X_1,\xi) + 2\,Q\varphi X_1 = 4n\,\varphi X_1 + X_1(\lambda+\rho\,r)\xi\\ + \xi(\lambda+\rho\,r)X_1 - \eta(X_1)\,\nabla(\lambda+\rho\,r), \quad X_1 \in \mathfrak{X}(M).\end{aligned} \tag{2.11}$$

Proof Since the metric g is parallel, taking the covariant derivative of (1.2) along an arbitrary $Z_1 \in \mathfrak{X}(M)$, we get

$$(\nabla_{Z_1}\mathcal{L}_V g)(X_1,Y_1) + 2\,(\nabla_{Z_1}\mathrm{Ric}_g)(X_1,Y_1) = 2\,Z_1(\lambda+\rho\,r)\,g(X_1,Y_1) \tag{2.12}$$

for all $X_1, Y_1 \in \mathfrak{X}(M)$. According to Yano (see [23, p. 23]), we write

$$\begin{aligned}&(\mathcal{L}_V\nabla_{Z_1}g - \nabla_{Z_1}\mathcal{L}_V g - \nabla_{[V,Z_1]}g)(X_1,Y_1)\\ &\quad = -g((\mathcal{L}_V\nabla)(Z_1,X_1),Y_1) - g((\mathcal{L}_V\nabla)(Z_1,Y_1),X_1)\end{aligned}$$

for all $X_1, Y_1, Z_1 \in \mathfrak{X}(M)$. Since a Riemannian metric g is parallel, inserting (2.12) into the preceding equality yields the following:

$$\begin{aligned}g((\mathcal{L}_V\nabla)(Z_1,X_1),Y_1) + g((\mathcal{L}_V\nabla)(Z_1,Y_1),X_1)\\ + 2\,(\nabla_{Z_1}\mathrm{Ric}_g)(X_1,Y_1) = 2\,Z_1(\lambda+\rho\,r)\,g(X_1,Y_1).\end{aligned}$$

In view of symmetry $(\mathcal{L}_V\nabla)(X_1,Y_1) = (\mathcal{L}_V\nabla)(Y_1,X_1)$ of a $(1,2)$-type tensor field $\mathcal{L}_V\nabla$, interchanging cyclically the roles of X_1, Y_1, Z_1 in the previous equality and by a direct calculation, we obtain

$$\begin{aligned}g((\mathcal{L}_V\nabla)(X_1,Y_1),Z_1) =\,& g((\nabla_{Z_1}Q)X_1,Y_1) - g((\nabla_{X_1}Q)Y_1,Z_1) - g((\nabla_{Y_1}Q)Z_1,X_1)\\ &+ X_1(\lambda+\rho\,r)\,g(Y_1,Z_1) + Y_1(\lambda+\rho\,r)\,g(Z_1,X_1)\\ &- Z_1(\lambda+\rho\,r)\,g(X_1,Y_1),\end{aligned} \tag{2.13}$$

by using the notion $(\nabla_{Z_1}\mathrm{Ric}_g)(X_1, Y_1) = g((\nabla_{Z_1} Q)X_1, Y_1)$. As ξ is killing, i.e., $\mathcal{L}_\xi \mathrm{Ric}_g = 0$, and therefore, $(\nabla_\xi Q)X_1 - \nabla_{QX_1}\xi + Q(\nabla_{X_1}\xi) = 0$ for $X_1 \in \mathfrak{X}(M)$. It follows from (2.7) that

$$(\nabla_\xi Q)X_1 = Q\varphi X_1 - \varphi Q X_1, \quad X_1 \in \mathfrak{X}(M). \tag{2.14}$$

In view of this, the self-adjoint property of the Ricci operator Q and (2.9), taking ξ instead of Y_1 in (2.13), one can deduce the required formula. ■

3 Proof of main results

Proof of Theorem 1.1 Since ξ is Killing on a Sasakian manifold, $\mathcal{L}_\xi \mathrm{Ric}_g = 0$, and therefore, $\xi(r) = 0$. So, in case of ρ-Einstein soliton, equation (2.11) reduces to

$$(\mathcal{L}_V\nabla)(X_1, \xi) + 2\,Q\varphi X_1 = 4n\,\varphi X_1 + \rho\left\{X_1(r)\xi - \eta(X_1)\,\nabla r\right\}, \quad X_1 \in \mathfrak{X}(M). \tag{3.1}$$

Since the Riemannian metric g is parallel and $\varphi(\xi) = 0$, from (2.7) and (3.1), we obtain

$$\begin{aligned}(\nabla_\xi\mathcal{L}_V\nabla)(X_1, \xi) &+ (\mathcal{L}_V\nabla)(\nabla_\xi X_1, \xi) + 2\left\{(\nabla_\xi Q)\varphi X_1 + Q(\nabla_\xi\varphi)X_1 + Q\varphi(\nabla_\xi X_1)\right\}\\ &= 4n\left\{(\nabla_\xi\varphi)X_1 + \varphi(\nabla_\xi X_1)\right\} + \rho\Big\{g(\nabla_\xi X_1, \nabla r)\xi + g(X_1, \nabla_\xi\nabla r)\xi\\ &\quad - \eta(\nabla_\xi X_1)\,\nabla r - \eta(X_1)\,\nabla_\xi\nabla r\Big\}.\end{aligned} \tag{3.2}$$

On Sasakian manifold, the Ricci operator Q commutes with the contact metric structure φ, see [3, p. 116], and therefore, (2.14) gives us $\nabla_\xi Q = 0$. Thus, (3.1) and $\nabla_\xi\varphi = 0$ (follows from (2.5)) transform (3.2) into the following:

$$(\nabla_\xi\mathcal{L}_V\nabla)(X_1, \xi) = \rho\left\{g(X_1, \nabla_\xi\nabla r)\xi - \eta(X_1)\,\nabla_\xi\nabla r\right\}, \quad X_1 \in \mathfrak{X}(M). \tag{3.3}$$

Now, setting $X_1 = \xi$ in (3.1) and using $\xi(r) = \varphi(\xi) = 0$, we get $(\mathcal{L}_V\nabla)(\xi, \xi) = -\rho\,\nabla r$, and therefore, by (2.7), one can find

$$(\nabla_{X_1}\mathcal{L}_V\nabla)(\xi, \xi) = 2\,(\mathcal{L}_V\nabla)(\varphi X_1, \xi) - \rho\nabla_{X_1}\nabla r, \quad X_1 \in \mathfrak{X}(M). \tag{3.4}$$

Next, by the following commutation formulas on a Riemannian manifold, see Yano [23, p. 23],

$$(\mathcal{L}_V R)(X_1, Y_1)Z_1 = (\nabla_{X_1}\mathcal{L}_V\nabla)(Y_1, Z_1) - (\nabla_{Y_1}\mathcal{L}_V\nabla)(X_1, Z_1), \tag{3.5}$$

by (3.1), (3.3), and (3.4), we obtain

$$\begin{aligned}(\mathcal{L}_V R)(X_1, \xi)\xi &= (\nabla_{X_1}\mathcal{L}_V\nabla)(\xi, \xi) - (\nabla_\xi\mathcal{L}_V\nabla)(X_1, \xi)\\ &= 2\,(\mathcal{L}_V\nabla)(\varphi X_1, \xi) - \rho\nabla_{X_1}\nabla r - \rho\left\{g(X_1, \nabla_\xi\nabla r)\xi - \eta(X_1)\nabla_\xi\nabla r\right\}\\ &= 2\left\{-2Q(-X_1 + \eta(X_1)\xi) + 4n(-X_1 + \eta(X_1)\xi) + \rho\,(\varphi X_1)(r)\,\xi\right\}\\ &\quad - \rho\nabla_{X_1}\nabla r - \rho\left\{g(X_1, \nabla_\xi\nabla r)\xi - \eta(X_1)\,\nabla_\xi\nabla r\right\}\\ &= 4\,QX_1 - 8n\,X_1 + 2\rho\,(\varphi X_1)(r)\,\xi - \rho\nabla_{X_1}\nabla r\\ &\quad - \rho\left\{g(X_1, \nabla_\xi\nabla r)\xi - \eta(X_1)\,\nabla_\xi\nabla r\right\},\end{aligned} \tag{3.6}$$

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by using (2.3) and (2.8). On the other hand, taking the Lie derivative of the equation $R(X_1,\xi)\xi = X_1 - \eta(X_1)\xi$ (follows from (2.6)) along V and using (2.6), we compute

$$\begin{aligned}(\mathcal{L}_V R)(X_1,\xi)\xi =& R(\mathcal{L}_V\xi, X_1)\xi - R(X_1,\xi)\mathcal{L}_V\xi - (\mathcal{L}_V g)(X_1,\xi)\xi\\ &- g(X_1,\mathcal{L}_V\xi)\xi - \eta(X_1)\,\mathcal{L}_V\xi\\ =& -2\,\eta(\mathcal{L}_V\xi)X_1 - (\mathcal{L}_V g)(X_1,\xi)\xi, \quad X_1\in\mathfrak{X}(M).\end{aligned}\tag{3.7}$$

Now, we deduce from (2.7) and $\xi(r)=0$ that $(\varphi X_1)(r) = g(\xi, \nabla_{X_1}\nabla r)$, and therefore, combining (3.6) and (3.7), we acquire

$$\begin{aligned}\rho\nabla_{X_1}\nabla r =& \rho\Big\{g(\xi,\nabla_{X_1}\nabla r)\xi + \eta(X_1)\,\nabla_\xi\nabla r\Big\}\\ &+ 4\,QX_1 - 8n\,X_1 + 2\,\eta(\mathcal{L}_V\xi)X_1 + (\mathcal{L}_V g)(X_1,\xi)\xi,\end{aligned}\tag{3.8}$$

by using the symmetric property of Hess. Next, using (2.8) in (1.2), we achieve

$$(\mathcal{L}_V g)(X_1,\xi) = 2(\lambda+\rho\,r-2n)\,\eta(X_1), \quad X_1\in\mathfrak{X}(M),\tag{3.9}$$

and applying this in the Lie derivative of $\eta(\xi)=1$ leads to $\eta(\mathcal{L}_V\xi) = -(\lambda+\rho\,r-2n)$. Further, making use of this and (3.8), the equality (3.9) transform into

$$\begin{aligned}\rho\nabla_{X_1}\nabla r =& \rho\Big\{g(\xi,\nabla_{X_1}\nabla r)\xi + \eta(X_1)\,\nabla_\xi\nabla r\Big\} + 4\,QX_1 - 2(\lambda+\rho\,r+2n)X_1\\ &+ 2(\lambda+\rho\,r-2n)\,\eta(X_1)\xi, \quad X_1\in\mathfrak{X}(M).\end{aligned}\tag{3.10}$$

Now, taking its covariant derivative along an arbitrary $Y_1\in\mathfrak{X}(M)$ and using (2.7), we obtain

$$\begin{aligned}\rho\nabla_{Y_1}\nabla_{X_1}\nabla r =& \rho\Big\{g(\xi,\nabla_{Y_1}\nabla_{X_1}\nabla r)\xi - g(\varphi Y_1,\nabla_{X_1}\nabla r)\xi - g(\xi,\nabla_{X_1}\nabla r)\varphi Y_1\\ &+ \eta(\nabla_{Y_1}X_1)\,\nabla_\xi\nabla r - g(X_1,\varphi Y_1)\nabla_\xi\nabla r + \eta(X_1)\,\nabla_{Y_1}\nabla_\xi\nabla r\Big\}\\ &+ 4\Big\{(\nabla_{Y_1}Q)X_1 + Q(\nabla_{Y_1}X_1)\Big\}\\ &- 2(\lambda+\rho\,r+2n)\nabla_{Y_1}X_1 - 2\rho\,\big\{X_1 - \eta(X_1)\xi\big\}\,Y_1(r)\\ &+ 2(\lambda+\rho\,r-2n)\,\Big\{\eta(\nabla_{Y_1}X_1)\xi + g(\varphi X_1, Y_1)\xi - \eta(X_1)\varphi Y_1\Big\},\end{aligned}\tag{3.11}$$

as the Rimannian metric g is parallel. Since Hess is symmetric and φ is anti-symmetric, equations (3.10), (3.11), and the curvature expression (2.1) gives us

$$\begin{aligned}\rho\,R(X_1,Y_1)\nabla r =& \rho\Big\{g(\varphi\,Y_1,\nabla_{X_1}\nabla r)\xi - g(\varphi X_1,\nabla_{Y_1}\nabla r)\xi + g(\xi, R(X_1,Y_1)\nabla r)\xi\\ &+ g(\xi,\nabla_{X_1}\nabla r)\,\varphi\,Y_1 - g(\xi,\nabla_{Y_1}\nabla r)\,\varphi X_1 - 2\,g(\varphi X_1, Y_1)\,\nabla_\xi\nabla r\\ &+ \eta(Y_1)\,\nabla_{X_1}\nabla_\xi\nabla r - \eta(X_1)\,\nabla_{Y_1}\nabla_\xi\nabla r\Big\} + 4\Big\{(\nabla_{X_1}Q)Y_1 - (\nabla_{Y_1}Q)X_1\Big\}\\ &- 2\rho\,\Big\{X_1(r)\,(Y_1 - \eta(Y_1)\xi) - Y_1(r)\,(X_1 - \eta(X_1)\xi)\Big\}\\ &- 2(\lambda+\rho\,r-2n)\,\Big\{2\,g(\varphi X_1, Y_1)\xi + \eta(Y_1)\,\varphi X_1 - \eta(X_1)\,\varphi Y_1\Big\}.\end{aligned}\tag{3.12}$$

Now, consider a local orthonormal frame $\{E_i\}_{1\le i\le 2n+1}$ on M. By curvature properties and (2.6), one can compute the following formulae:

$$\sum_{i=1}^{2n+1} g(\varphi Y_1, \nabla_{E_i}\nabla r)\, g(\xi, E_i) = g(\xi, \nabla_{\varphi Y_1}\nabla r),$$
$$\sum_{i=1}^{2n+1} g(\xi, \nabla_{E_i}\nabla r)\, g(\varphi Y_1, E_i) = g(\varphi Y_1, \nabla_{\xi}\nabla r),$$
$$\sum_{i=1}^{2n+1} g(\xi, R(E_i, Y_1)\nabla r)\, g(\xi, E_i) = Y_1(r) - \eta(\nabla r)\,\eta(Y_1)$$

for all $Y_1 \in \mathfrak{X}(M)$. Next, contracting (3.12) over X_1 and using the antisymmetry of φ, the symmetry of Hess, $\mathrm{Tr}_{\mathrm{g}}\varphi = 0 = \varphi(\xi)$ and the above formulae, we obtain

$$\begin{aligned}\rho\,\mathrm{Ric}_{\mathrm{g}}(Y_1, \nabla r) =& \rho\Big\{4\, g(\varphi Y_1, \nabla_\xi \nabla r) + (4n-1)\, Y_1(r)\\ &+ \eta(Y_1)\,\mathrm{div}(\nabla_\xi\nabla r) - g(\xi, \nabla_{Y_1}\nabla_\xi\nabla r)\Big\} - 2\, Y_1(r).\end{aligned} \tag{3.13}$$

Further, replacing $X_1 = \varphi X_1$ and $Y_1 = \varphi Y_1$ in (3.12) and by using the formula for Sasakian manifold, see Blair [3, p. 137]:

$$\begin{aligned}&R(\varphi X_1, \varphi Y_1)Z_1\\ &\quad = R(X_1, Y_1)Z_1 + g(X_1, Z_1)Y_1 - g(Y_1, Z_1)X_1 - g(\varphi X_1, Z_1)\varphi Y_1 + g(\varphi Y_1, Z_1)\varphi X_1,\end{aligned}$$

$\varphi(\xi) = 0 = \eta\circ\varphi$, symmetry of Hess, antisymmetry of φ and (2.3), we obtain

$$\begin{aligned}\rho\Big\{R(X_1, Y_1)\nabla r + X_1(r)Y_1 - Y_1(r)X_1\Big\} =& \rho\Big\{g(X_1, \nabla_{\varphi Y_1}\nabla r)\xi - g(Y_1, \nabla_{\varphi X_1}\nabla r)\xi\\ &+ 2\,\eta(Y_1)\, g(\xi, \nabla_{\varphi X_1}\nabla r)\xi - 2\,\eta(X_1)\, g(\xi, \nabla_{\varphi Y_1}\nabla r)\xi + g(\varphi Y_1, \nabla_\xi\nabla r)X_1\\ &- g(\varphi X_1, \nabla_\xi\nabla r)Y_1 + 2\, g(X_1, \varphi Y_1)\,\nabla_\xi\nabla r\Big\}\\ &+ 4\Big\{(\nabla_{\varphi X_1}Q)(\varphi Y_1) - (\nabla_{\varphi Y_1}Q)(\varphi X_1)\Big\}\\ &- \rho\Big\{(\varphi X_1)(r)\,\varphi Y_1 - (\varphi Y_1)(r)\,\varphi X_1\Big\} - 4(\lambda + \rho\, r - 2n)\, g(\varphi X_1, Y_1)\xi.\end{aligned} \tag{3.14}$$

Covariant derivative of $Q\varphi = \varphi Q$ and formula (2.2) provide the following formulae, see Lemma 5.1 in [13]:

$$\sum_{i=1}^{2n+1} g\big((\nabla_{Y_1}Q)\,\varphi E_i, E_i\big) = 0, \qquad \sum_{i=1}^{2n+1} g\big((\nabla_{\varphi E_i}Q)\,\varphi Y_1, E_i\big) = -\frac{1}{2}\, Y_1(r)$$

for all $Y_1 \in \mathfrak{X}(M)$. Using $\varphi(\xi) = 0$, one can easily compute that

$$\sum_{i=1}^{2n+1} g(Y_1, \nabla_{\varphi E_i}\nabla r)\, g(\xi, E_i) = -g(\xi, \varphi(\nabla_{Y_1}\nabla r)) = 0, \qquad Y_1 \in \mathfrak{X}(M).$$

Now, contracting (3.14) over X_1 and using the preceding formulae, we compute

$$\begin{aligned}\rho\Big\{\mathrm{Ric}_{\mathrm{g}}(Y_1, \nabla r) - 2n\, Y_1(r)\Big\} =& \rho\Big\{2(n+1)\, g(\varphi Y_1, \nabla_\xi\nabla r) - g(\xi, \nabla_{\varphi Y_1}\nabla r)\Big\} - 2\, Y_1(r)\\ &+ \rho\,\big\{Y_1(r) - \xi(r)\,\eta(Y_1)\big\}.\end{aligned} \tag{3.15}$$

Since $\xi(r) = 0$ and $\nabla_\xi \xi = 0$, and therefore, we have $g(\xi, \nabla_\xi \nabla r) = \xi(\xi(r)) = 0$. Differentiating it along $Y_1 \in \mathfrak{X}(M)$ and recalling (2.7), we get

$$g(\xi, \nabla_{Y_1} \nabla_\xi \nabla r) = g(\varphi Y_1, \nabla_\xi \nabla r), \quad Y_1 \in \mathfrak{X}(M).$$

In [12], Ghosh-Sharma studied the case $\rho = 0$. Here we consider ρ is nonzero. Then combining (3.13) and (3.15) with the last equality, we reach

$$\eta(Y_1)\,\mathrm{div}(\nabla_\xi \nabla r) = 2(n-1)\left\{g(\varphi Y_1, \nabla_\xi \nabla r) - Y_1(r)\right\}, \quad Y_1 \in \mathfrak{X}(M). \tag{3.16}$$

Next, replacing Y_1 by φY_1 in (3.16) and using (2.3) and $\varphi(\xi) = 0$, we acquire

$$(n-1)\left\{g(Y_1, \nabla_\xi \nabla r) - \eta(Y_1)\,g(\xi, \nabla_\xi \nabla r) + (\varphi Y_1)(r)\right\} = 0 \tag{3.17}$$

and, in view of (2.3) and $\xi(r) = 0$, it is easily seen that $g(Y_1, \nabla_\xi \nabla r) = g(\xi, \nabla_{Y_1} \nabla r) = (\varphi Y_1)(r)$. Thus, the equality (3.17) becomes $(n-1)\,(\varphi Y_1)(r) = 0$, as $g(\xi, \nabla_\xi \nabla r) = 0$; and therefore, $\varphi \nabla r = 0$ for $n > 1$. Operating this by φ and applying (2.3) yields that $\nabla r = \xi(r)\xi = 0$; hence, r is constant on M^{2n+1} for $n > 1$. Thus, equation (3.1) and $\varphi(\xi) = 0$ gives us $(\mathcal{L}_V \nabla)(\xi, \xi) = 0$. Applying this in the well known formula:

$$\nabla_{X_1} \nabla_{Y_1} V - \nabla_{\nabla_{X_1} Y_1} V - R(X_1, V) Y_1 = (\mathcal{L}_V \nabla)(X_1, Y_1),$$

see [23, p. 23], we acquire $R(\xi, V)\xi = \nabla_\xi \nabla_\xi V$, i.e., V is Jacobi along the geodesics determined by ξ, which proves part (i).

Next, we prove part (ii). Since r is constant, equation (3.10) becomes

$$\mathrm{Ric}_g(X_1, Y_1) = \frac{1}{2}\left\{(\lambda + \rho r + 2n)\,g(X_1, Y_1) - (\lambda + \rho r - 2n)\,\eta(X_1)\eta(Y_1)\right\}, \tag{3.18}$$

and therefore, the scalar curvature is given by

$$r = n(\lambda + \rho r + 2n + 2), \tag{3.19}$$

which is constant, and the squared norm of the Ricci operator is given by

$$||Q||^2 = \frac{n}{2}\left\{(\lambda + \rho r + 2n)^2 + 8n\right\}. \tag{3.20}$$

Now, recalling the following integrability formula on $(2n+1)$-dimensional contact metric manifold, see equation (8) of [21]:

$$\mathcal{L}_V r = 2\,||Q||^2 + \Delta r - 2(\lambda + \rho r)r - 4n\,\Delta(\lambda + \rho r), \tag{3.21}$$

where we have used for convenience $\Delta r = -\mathrm{div}(\nabla r)$. As λ and r are constants, taking into account (3.19) as well as (3.20), the previous formula becomes

$$\begin{aligned} 0 &= 2\,||Q||^2 - 2(\lambda + \rho r)r \\ &= -n(\lambda + \rho r - 2n)(\lambda + \rho r + 2n + 4). \end{aligned} \tag{3.22}$$

If $\lambda + \rho r - 2n = 0$, then according to (3.18), (M, g) is an Einstein manifold, contradicting our hypothesis; and therefore, (3.22) implies that $\lambda + \rho r + 2n + 4 = 0$, which reduces (3.18) to the form

$$\mathrm{Ric}_g(X_1, Y_1) = -2\left\{g(X_1, Y_1) - (n+1)\,\eta(X_1)\eta(Y_1)\right\}. \tag{3.23}$$

Hence, (M, g) is $\mathcal{D}$-homothetically fixed null η-Einstein manifold, which proves part (ii).

To prove part (iii). Now, using (3.23) in the soliton equation (1.2), we get

$$\frac{1}{2}(\mathcal{L}_V g)(X_1, Y_1) = -2(n+1)\Big\{g(X_1, Y_1) + \eta(X_1)\eta(Y_1)\Big\}, \quad X_1, Y_1 \in \mathfrak{X}(M). \tag{3.24}$$

In view of (2.7) and (3.23), we deduce

$$(\nabla_{Z_1}\mathrm{Ric}_g)(X_1, Y_1) = 2(n+1)\{g(\varphi X_1, Z_1)\,\eta(Y_1) + \eta(X_1)\,g(\varphi Y_1, Z_1)\}, \tag{3.25}$$

by using the parallelism of the Riemannian metric g. Repeated application of (3.25) and the antisymmetry of φ in (2.13) gives us

$$(\mathcal{L}_V\nabla)(X_1, Y_1) = 4(n+1)\{\eta(Y_1)\,\varphi X_1 + \eta(X_1)\,\varphi Y_1\}, \tag{3.26}$$

and its covariant derivative along $Z_1 \in \mathfrak{X}(M)$ yields that

$$\begin{aligned}(\nabla_{Z_1}\mathcal{L}_V\nabla)(X_1, Y_1) = 4(n+1)\Big\{&g(\varphi Y_1, Z_1)\,\varphi X_1 + g(\varphi X_1, Z_1)\,\varphi Y_1\\ &+ \eta(Y_1)\,g(X_1, Z_1)\xi + \eta(X_1)\,g(Y_1, Z_1)\xi - 2\,\eta(X_1)\eta(Y_1)Z_1\Big\},\end{aligned} \tag{3.27}$$

by using (2.7). In view of this and the communication formula (3.5), we obtain

$$\begin{aligned}(\mathcal{L}_V R)(Z_1, X_1)Y_1 =& (\nabla_{Z_1}\mathcal{L}_V\nabla)(X_1, Y_1) - (\nabla_{X_1}\mathcal{L}_V\nabla)(Z_1, Y_1)\\ =& 4(n+1)\Big\{g(\varphi Y_1, Z_1)\,\varphi X_1 + 2\,g(\varphi X_1, Z_1)\,\varphi Y_1\\ &- g(\varphi Y_1, X_1)\,\varphi Z_1 + \eta(X_1)\,g(Y_1, Z_1)\xi - \eta(Z_1)\,g(X_1, Y_1)\xi\\ &- 2\,\eta(X_1)\eta(Y_1)Z_1 + 2\,\eta(Z_1)\eta(Y_1)X_1\Big\},\end{aligned} \tag{3.28}$$

by using the antisymmetry of φ. Next, contracting it over Z_1 and using $\mathrm{Tr}_g\varphi = 0$ and (2.3), we obtain

$$(\mathcal{L}_V\mathrm{Ric}_g)(X_1, Y_1) = 8(n+1)\Big\{g(X_1, Y_1) - 2(n+1)\,\eta(X_1)\eta(Y_1)\Big\}. \tag{3.29}$$

On the other hand, taking into account (3.23) as well as (3.24), it suffices to show that

$$\begin{aligned}(\mathcal{L}_V\mathrm{Ric}_g)(X_1, Y_1) =& 8(n+1)\Big\{g(X_1, Y_1) + \eta(X_1)\eta(Y_1)\Big\}\\ &+ 2(n+1)\Big\{\eta(Y_1)\,(\mathcal{L}_V\eta)(X_1) + \eta(X_1)\,(\mathcal{L}_V\eta)(Y_1)\Big\}.\end{aligned} \tag{3.30}$$

In addition, it follows from $\eta(\xi) = 1$ that $(\mathcal{L}_V\eta)(\xi) = -\eta(\mathcal{L}_V\xi) = (\lambda + \rho r - 2n) = -4(n+1)$. Thus, comparing (3.29) with (3.30) and then replacing ξ instead of Y_1, we have

$$(\mathcal{L}_V\eta)(X_1) = -4(n+1)\,\eta(X_1), \quad X_1 \in \mathfrak{X}(M), \tag{3.31}$$

and therefore, V is a nonstrict infinitesimal contact transformation and is equal to $-\frac{1}{2}\varphi\nabla f + f\xi$ for a smooth function f on M such that $\xi(f) = -4(n+1)$, see [3, p. 72]. This proves part (iii).

Furthermore, taking into account (3.24) as well as (3.31), it is easy to check that $\mathcal{L}_V\xi = 4(n+1)$. Since the exterior derivative d commutes with the Lie derivative, i.e.,

$d \circ \mathcal{L}_V = \mathcal{L}_V \circ d$, applying d to (3.31), we acquire

(3.32)
$$(\mathcal{L}_V d\eta)(X_1, Y_1) = d(\mathcal{L}_V \eta)(X_1, Y_1) = -4(n+1)\, g(X_1, \varphi Y_1), \quad X_1, Y_1 \in \mathfrak{X}(M).$$

Therefore, applying this and (3.24) in the Lie-derivative of (2.4), we conclude that V leaves the structural tensor φ invariant, which finishes the proof.

Proof of Theorem 1.3 Operating (2.10) by d and since the exterior derivative d commutes with the Lie-derivative, i.e., $d \circ \mathcal{L}_V = \mathcal{L}_V \circ d$, we obtain

$$(\mathcal{L}_V d\eta)(X_1, Y_1) = \frac{1}{2}\Big\{X_1(f)\,\eta(Y_1) - Y_1(f)\,\eta(X_1)\Big\} + f\, d\eta(X_1, Y_1) \tag{3.33}$$

for any $X_1, Y_1 \in \mathfrak{X}(M)$. On the other hand, Lie-derivative of (2.4) along V and making use of (1.2) yields that

$$\begin{aligned}(\mathcal{L}_V d\eta)(X_1, Y_1) &= (\mathcal{L}_V g)(X_1, \varphi Y_1) + g(X_1, (\mathcal{L}_V \varphi) Y_1)\\ &= -2\mathrm{Ric}_g(X_1, \varphi Y_1) + 2(\lambda + \rho\, r)\, g(X_1, \varphi Y_1) + g(X_1, (\mathcal{L}_V \varphi) Y_1).\end{aligned}$$

By virtue of this and (2.4), we deduce from (3.33) the relation

$$(\mathcal{L}_V \varphi)(X_1) = 2\, Q\varphi X_1 + (f - 2\lambda - 2\rho\, r)\varphi X_1 + \frac{1}{2}\Big\{\eta(X_1)\nabla f - X_1(f)\xi\Big\} \tag{3.34}$$

for any $X_1 \in \mathfrak{X}(M)$. Next, taking into account (3.37) and (2.10) in the Lie-derivative of $\eta(X_1) = g(X_1, \xi)$, we get

$$\mathcal{L}_V \xi = (f - 2\lambda - 2\rho\, r + 4n)\xi. \tag{3.35}$$

Further, taking Lie-derivative of $\varphi(\xi) = 0$ along V and using (3.35), $\varphi(\xi) = 0$, we have $(\mathcal{L}_V \varphi)\xi = 0$; thus, (3.34) gives us $\nabla f = \xi(f)\xi$, or, in terms of the exterior derivation,

$$df = \xi(f)\,\eta. \tag{3.36}$$

Now taking its exterior derivative and using Poincaré lemma ($d^2 = 0$), and then applying the wedge product with η, we obtain $\xi(f)\,\eta \wedge d\eta = 0$; thus, $\xi(f) = 0$, as $\eta \wedge d\eta$ vanishes nowhere on a contact manifold M. It follows from (3.36) that $df = 0$; and hence, f is a constant on M. Further, using (2.8) in the soliton equation (1.2), we acquire

$$(\mathcal{L}_V g)(X_1, \xi) = 2\,(\lambda + \rho\, r - 2n)\,\eta(X_1), \quad X_1 \in \mathfrak{X}(M). \tag{3.37}$$

At this point, taking Lie-derivative of $g(\xi, \xi) = 1$ and using (3.37), (3.35) yields $\lambda + \rho\, r = f + 2n$, a constant, and therefore, g is a Ricci soliton.

Let ω be the volume form of M, i.e., $\omega = \eta \wedge (d\eta)^n \neq 0$. Then taking Lie-derivative of $\omega = \eta \wedge (d\eta)^n$ along V and using $\mathcal{L}_V\, \omega = (\mathrm{div} V)\omega$ and (3.33), we obtain $\mathrm{div} V = (n+1)f$. Integrating it over compact M and applying the divergence theorem, we get $f = 0$ (as f is constant), and therefore, $\mathrm{div} V = 0$ and $\lambda + \rho\, r = 2n$. Hence (2.6) and (3.35) prove that V leaves η invariant and ξ invariant, respectively. Using $\mathrm{div} V = 0$ and $\lambda + \rho\, r = 2n$ in the trace of (1.2), we have $r = (2n+1)(\lambda + \rho\, r) = 2n(2n+1)$.

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Therefore, using all these consequences in the integrability condition (3.12), we deduce

$$||Q||^2 = (\lambda + \rho\, r) r = 4n^2(2n+1).$$

This is equivalent to $||Q - 2n\, I||^2 = 0$, i.e., $\mathrm{Ric}_g = 2n\, g$. Hence (M, g) is Einstein with Einstein constant $2n$; thus, (3.34) and (1.2) imply that V leaves φ and g invariant, respectively. Since M is compact, applying Boyer and Galicki's result (see Theorem 11.1.7 of [5, p. 372]), we can conclude that M is Sasakian. Moreover, relations $\lambda + \rho\, r = 2n$ and $r = 2n(2n+1)$ gives $\lambda = 2n(1 - \rho(2n+1))$, which states that the soliton is expanding, steady, or shrinking if $\rho > \frac{1}{2n+1}$, $\rho = \frac{1}{2n+1}$, or $\rho < \frac{1}{2n+1}$, respectively. This completes the proof.

Proof of Theorem 1.4 By conditions, $V = \sigma\, \xi$ for a nonzero smooth function σ on M. Using its covariant derivative, the antisymmetry of φ and (2.7), we find

$$(\mathcal{L}_V g)(X_1, Y_1) = g(\nabla_{X_1} V, Y_1) + g(\nabla_{Y_1} V, X_1) = X_1(\sigma)\, \eta(Y_1) + Y_1(\sigma)\, \eta(X_1). \tag{3.38}$$

In view of this, equation (1.2) becomes

$$X_1(\sigma)\eta(Y_1) + Y_1(\sigma)\eta(X_1) + 2\, \mathrm{Ric}_g(X_1, Y_1) = 2(\lambda + \rho\, r)\, g(X_1, Y_1). \tag{3.39}$$

At this point, replacing Y_1 by ξ in (3.39) and using (2.8), we achieve

$$X_1(\sigma) = \Big\{2(\lambda + \rho\, r - 2n) - \xi(\sigma)\Big\}\, \eta(X_1), \quad X_1 \in \mathfrak{X}(M). \tag{3.40}$$

Again, substituting ξ for both X_1 and Y_1 in (3.39) and using (2.8), we get $\xi(\sigma) = \lambda + \rho\, r - 2n$, and therefore, equation (3.40) follows that $\nabla\sigma = \xi(\sigma)\xi$. Taking its covariant derivative along $Y_1 \in \mathfrak{X}(M)$ and using (2.7), we find

$$g(\nabla_{Y_1}\nabla\sigma, X_1) = Y_1(\xi(\sigma))\, \eta(X_1) + \xi(\sigma)\, g(\varphi X_1, Y_1). \tag{3.41}$$

By symmetry of Hess_σ and antisymmetry of φ, (3.41) yields that

$$2\, \xi(\sigma)\, g(\varphi X_1, Y_1) = X_1(\xi(\sigma))\, \eta(Y_1) - Y_1(\xi(\sigma))\, \eta(X_1).$$

Choosing $X_1, Y_1 \perp \xi$ implies that $\xi(\sigma) = 0$ on M, as $d\eta$ is non-zero on M; consequently, $\nabla\sigma = 0$ on M. Therefore, we conclude that σ is constant on M. It follows from (3.38) and (3.40) that V is a Killing vector field and $\lambda + \rho\, r = 2n$, respectively; hence, rest of the proof follows from (3.39). ∎

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