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ON THE NORM CONTINUITY OF $\mathscr{S}'$-VALUED GAUSSIAN PROCESSES

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Summary

Let $\mathscr{S}$ be the Schwartz space of all rapidly decreasing functions on $\boldsymbol{R}^n$, $\mathscr{S}'$ be the topological dual space of $\mathscr{S}$ and for each positive integer p, $\mathscr{S}'_p$ be the space of all elements of $\mathscr{S}'$ which are continuous in the p-th norm defining the nuclear Fréchet topology of $\mathscr{S}$. The main purpose of the present paper is to show that if $\{X_t, t \in [0, +\infty)\}$ is an $\mathscr{S}'$-valued Gaussian process and for any fixed $\varphi \in \mathscr{S}$ the real Gaussian process $\{X_t(\varphi), t \in [0, +\infty)\}$ has a continuous version, then for any fixed $T > 0$ there is a positive integer p such that $\{X_t, t \in [0, T]\}$ has a version which is continuous in the norm topology of $\mathscr{S}'_p$.

§1. Introduction

Let E be a locally convex topological vector space, E' be the topological dual space of E and denote by $C(E', E)$ the smallest σ-algebra of subsets of E' that makes all functions $\{\langle x, \xi\rangle : \xi \in E\}$ measurable, where $\langle x, \xi\rangle$ is the canonical bilinear form on $E' \times E$. An E'-valued stochastic process is a collection $X = \{X_t, t \in [0, +\infty)\}$ of measurable maps X_t from a complete probability space $(\Omega, \mathscr{B}, P)$ into the measurable space $(E', C(E', E))$. Throughout this paper R_+, T_+ and N denote the half line $[0, +\infty)$, the closed interval $[0, T]$ and the set of all positive integers.

X is said to be *Gaussian* if the family of real random variables $\{\langle X_t, \xi\rangle : t \in R_+, \xi \in E\}$ forms a Gaussian system.

We shall study below sample path continuity of E'-valued Gaussian processes in case where E is a nuclear Fréchet space or a countable strict inductive limit of nuclear Fréchet spaces. In the following definitions we assume that E is one of such spaces. Then the Borel field of E' coincides with $C(E', E)$. If X is Gaussian the probability law μ_t of X_t which is

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defined by $\mu_t(A) = P(X_t^{-1}(A))$, $A \in C(E', E)$ is a Gaussian measure on E' with mean $m_t(\xi)$ and variance $v_t(\xi)$ and then there always exists m_t in E' such that $\langle m_t, \xi \rangle = m_t(\xi)$ for every ξ in E. Two E'-valued processes $\{X_t, t \in I\}$ and $\{Y_t, t \in I\}$ on the same probability space $(\Omega, \mathscr{B}, P)$ is said to be *versions* of each other if $P(\omega: X_t = Y_t) = 1$ for any $t \in I$, where I is a subset of R_+. If we change "E'-valued" for "real valued" in the above sentence, that is the definition of versions for real processes. X is said to be *quasi weakly continuous* if for any fixed $\xi \in E$ there is a P-null set N_ξ such that $\langle X_t(\omega), \xi \rangle$ is a continuous real function of t for each $\omega \in \Omega \backslash N_\xi$. X is said to be *weakly continuous* if there is a P-null set Λ such that for each $\omega \in \Omega \backslash \Lambda$, $\langle X_t(\omega), \xi \rangle$ is a continuous real function of t for any $\xi \in E$. X is said to be *continuous* if X is continuous in the strong topology of E' almost surely. X is said to be *additive* if $X_0 = 0$ almost surely and if for every $n \in N$ and $t_0 < t_1 < \cdots < t_n$, $X_{t_i} - X_{t_{i-1}}$, $i = 1, 2, \cdots, n$, are independent E'-valued random variables.

Let E be a nuclear Fréchet space, $\|\cdot\|_1 \leqq \|\cdot\|_2 \leqq \cdots \leqq \|\cdot\|_p \leqq \cdots$ be an increasing sequence of Hilbertian semi-norms defining the topology of E, E_p be the completion of E by $\|\cdot\|_p$ and $\|\cdot\|_{-p}$ be the norm of E'_p. Then we have $E = \bigcap_{p=1}^{+\infty} E_p$ and $E' = \bigcup_{p=1}^{+\infty} E'_p$.

The foundation of this paper is in the proof of the following theorem which will be given in Section 2.

THEOREM 1. *Let E be a nuclear Fréchet space and $X = \{X_t, t \in R_+\}$ be an E'-valued quasi weakly continuous Gaussian process. Then for any fixed $T > 0$ there is $p = p_T \in N$ such that $\{X_t, t \in T_+\}$ is $\|\cdot\|_{-p}$-continuous almost surely.*

Using the idea of the proof of Theorem 1 it will be shown that if X is an E'-valued Gaussian process and for any fixed ξ in E the real process $\{\langle X_t, \xi \rangle, t \in R_+\}$ has a continuous version, then X has a quasi weakly continuous version. Hence in such a case, by Theorem 1, $\{X_t, t \in T_+\}$ has a $\|\cdot\|_{-p}$-continuous version. (Theorem 2).

Appealing to Theorem 2 we can extend the Fernique's result about sample path continuity of real Gaussian processes to E'-valued Gaussian processes. (About Fernique's result, see R. M. Dudley [1], Theorem 7.1). In Section 2 we will also give necessary and sufficient conditions for the norm continuity in the case of the whole time interval R_+ and give examples which show that the conditions are not trivial. Section 3 is devoted to the norm continuity in the case where E is a countable strict inductive limit of nuclear Fréchet spaces.

The present paper was motivated by the following proposition proved by K. Itô. (see Theorem 4.1 of [5]).

PROPOSITION 1. *If X is an $\mathscr{S}'$-valued Gaussian additive process where for any φ in $\mathscr{S}$, $m_t(\varphi)$ and $v_t(\varphi)$ are continuous real functions of t, then for any fixed $T > 0$ there is $p = p_T \in N$ such that $\{X_t, t \in T_+\}$ has a version which is continuous in the norm topology of $\mathscr{S}'_p$.*

§ 2. Nuclear Fréchet space

Throughout this section we assume that E is a nuclear Fréchet space. We shall begin with proving Theorem 1.

Proof of Theorem 1. For any ξ in E put $X(\xi) = \sup_{t \in T_+} |\langle X_t, \xi \rangle|$. Since X is quasi weakly continuous, $X(\xi)$ is $\mathscr{B}$-measurable and $P(\omega\colon X(\xi) < +\infty) = 1$ so that

$$V_T(\xi) = \boldsymbol{E}[X(\xi)^2] < +\infty ,$$

where $\boldsymbol{E}$ denotes the mathematical expectation. (see [*]: H. J. Landau and L. A. Shepp [6], Theorem 5 and X. Fernique [2], Theorem 1.3.2.).

Then we obtain the following lemma.

LEMMA 1. *Let X be an E'-valued quasi weakly continuous Gaussian process. Then for any fixed $T > 0$ there exist $q = q_T \in N$ and a constant $L = L_T > 0$ such that*

$$V_T(\xi) \leqq L \|\xi\|_q^2 .$$

Proof. Since for each $t \in T_+$, $\langle X_t(\omega), \xi \rangle$ is a continuous function of ξ, $X(\xi)(\omega) = \sup_{t \in T_+} |\langle X_t(\omega), \xi \rangle|$ is a lower semi-continuous function of ξ. Hence $V_T(\xi)$ is also a lower semi-continuous function of ξ because if ξ_n converges to ξ in E then we have

$$\begin{aligned} \liminf_{n \to +\infty} V_T(\xi_n) &\geqq \boldsymbol{E}\left[\liminf_{n \to +\infty} X(\xi_n)^2\right] \\ &\geqq \boldsymbol{E}[X(\xi)^2] \\ &= V_T(\xi) \end{aligned}$$

by the Fatou's lemma. Obviously $V_T(\xi)$ is a symmetric and convex function of ξ and satisfies $V_T(a\xi) = a^2 V_T(\xi)$ for all $a \geqq 0$. Since E is a complete metrizable space, by the Baire's category theorem, (see p. 62 of [4]), there exist $q \in N$ and a constant L satisfying the desired inequality, which proves the lemma.

Since E is nuclear, there is an integer $\gamma > q$ such that E_γ is nuclearly imbedded into E_q. Namely, if $\{\eta_j\}$ is a C.O.N.S.[1] (complete orthonormal system) in E_γ, then it holds that

$$\sum_{j=1}^{+\infty} \|\eta_j\|_q^2 < +\infty .$$

Of course $\sup_{t \in T_+} \|X_t\|_{-\gamma}^2$ is $\mathscr{B}$-measurable.

Using Lemma 1 and the Sazonov-Minlos' theorem in [3], we have

$$\begin{aligned} E\Big[\sup_{t \in T_+} \|X_t\|_{-\gamma}^2\Big] &= E\Big[\sup_{t \in T_+} \sum_{j=1}^{+\infty} \langle X_t, \eta_j \rangle^2\Big] \\ &\leqq \sum_{j=1}^{+\infty} E\Big[\Big(\sup_{t \in T_+} |\langle X_t, \eta_j \rangle|\Big)^2\Big] \\ &= \sum_{j=1}^{+\infty} V_T(\eta_j) \\ &\leqq L \sum_{j=1}^{+\infty} \|\eta_j\|_q^2 < +\infty . \end{aligned} \tag{2.1}$$

Thus we have $P(\omega\colon \sup_{t \in T_+} \|X_t\|_{-\gamma}^2 < +\infty) = 1$. This implies that there exists a P-null set Ω_1 such that for $\omega \in \Omega \backslash \Omega_1$,

$$\sup_{t \in T_+} \|X_t(\omega)\|_{-\gamma}^2 < +\infty .$$

Again by the nuclearity of E, there is an integer $p > \gamma$ such that E_p is nuclearly imbedded into E_γ. Let $\{\zeta_j\}$ be a C.O.N.S. in E_p. Put $\Omega_2 = \bigcup_{j=1}^{+\infty} N_{\zeta_j}$ and $\Omega_3 = \Omega \backslash (\Omega_1 \cup \Omega_2)$ and so $P(\Omega_3) = 1$. Furthermore for $\omega \in \Omega_3$ there is a finite real number $N(\omega)$ such that

$$\sup_{t \in T_+} \|X_t(\omega)\|_{-\gamma}^2 \leqq N(\omega) .$$

Then for $\omega \in \Omega_3$ and for $t, s \in T_+$, we get the following estimate:

$$\langle X_t(\omega) - X_s(\omega), \zeta_j \rangle^2 \leqq 4N(\omega) \|\zeta_j\|_\gamma^2 .$$

Therefore by the Lebesgue's convergence theorem, for the above ω and for $t, s \in T_+$ we have

$$\lim_{t \to s} \|X_t(\omega) - X_s(\omega)\|_{-p}^2 = \lim_{t \to s} \sum_{j=1}^{+\infty} \langle X_t(\omega) - X_s(\omega), \zeta_j \rangle^2$$

1) We always choose a C.O.N.S. from E.

$$= \sum_{j=1}^{+\infty} \lim_{t\to s} \langle X_t(\omega) - X_s(\omega), \zeta_j \rangle^2$$
$$= 0 .$$

This completes the proof of Theorem 1.

Remark. Theorem 1 implies the following statements are equivalent.
(1) X is quasi weakly continuous.
(2) X is weakly continuous.
(3) X is continuous.

THEOREM 2. *Let E be a nuclear Fréchet space and $X = \{X_t, t \in R_+\}$ be an E'-valued Gaussian Process and for any fixed ξ in E the real Gaussian process $\{\langle X_t, \xi\rangle, t \in R_+\}$ have a continuous version. Then for any fixed $T > 0$ there is $p = p_T \in N$ such that $\{X_t, t \in T_+\}$ has a $\|\cdot\|_{-p}$-continuous version.*

Proof. Since for any fixed ξ in E, $\{\langle X_t, \xi\rangle, t \in R_+\}$ has a continuous version, we denote it by $\hat{X}_t(\xi)$. Put $\hat{X}(\xi) = \sup_{t\in T_+} |\hat{X}_t(\xi)|$ and $X_Q(\xi) = \sup_{t\in Q} |\langle X_t, \xi\rangle|$, where Q is a set of all rational numbers in T_+. Then we have

$$\hat{X}(\xi) = \sup_{t\in Q} |\hat{X}_t(\xi)|$$
$$= X_Q(\xi) < +\infty$$

almost surely, so that

$$V_Q(\xi) = \boldsymbol{E}[X_Q(\xi)^2] = \boldsymbol{E}[\hat{X}(\xi)^2] < +\infty$$

(see [*]).

By the proof of Lemma 1 we have that there exist $q = q_Q \in N$ and a constant $L_Q > 0$ such that

$$V_Q(\xi) \leqq L_Q \|\xi\|_q^2 . \tag{2.2}$$

We assume $\gamma, \{\eta_j\}$ are the same notations as in the proof of Theorem 1. By (2.1) we have

$$P\left(\sum_{j=1}^{+\infty} \sup_{t\in T_+} (\hat{X}_t(\eta_j))^2 < +\infty\right) = 1 ,$$

so that there exists a P-null set Ω_4 such that for $\omega \in \Omega\backslash\Omega_4$ there is a finite real number $M(\omega)$ satisfying

$$\sum_{j=1}^{+\infty} \sup_{t\in T_+} (\hat{X}_t(\eta_j)(\omega))^2 \leqq M(\omega) .$$

Any ξ in E has the following unique expansion as an element of E_r:

$$\xi = \sum_{j=1}^{+\infty} C_j(\xi)\eta_j .$$

So we can define for $t \in T_+$,

$$\tilde{X}_t(\xi)(\omega) = \begin{cases} \sum_{j=1}^{+\infty} C_j(\xi)\hat{X}_t(\eta_j)(\omega) & \text{if } \omega \in \Omega \backslash \Omega_4 , \\ 0 & \text{if } \omega \in \Omega_4 . \end{cases}$$

Then for $\omega \in \Omega\backslash\Omega_4$ and for $t, s \in T_+$, we get the following estimate:

$$|\hat{X}_t(\eta_j)(\omega) - \hat{X}_s(\eta_j)(\omega)|^2 \leqq 4 \sup_{t \in T_+} (\hat{X}_t(\eta_j)(\omega))^2 .$$

Therefore by the Lebesgue's convergence theorem, $\tilde{X}_t(\xi)(\omega)$ is a continuous real function of t on T_+ for almost all ω. Furthermore for $\omega \in \Omega\backslash\Omega_4$ and for $t \in T_+$, we have

$$|\tilde{X}_t(\xi)(\omega)|^2 \leqq \left(\sum_{j=1}^{+\infty} C_j(\xi)^2\right)\left(\sum_{j=1}^{+\infty} (\hat{X}_t(\eta_j)(\omega))^2\right)$$
$$\leqq M(\omega)\, \|\xi\|_r^2 .$$

Hence there exists an element $\tilde{x}_t(\omega)$ in E'_r such that

$$\tilde{X}_t(\xi)(\omega) = \langle \tilde{x}_t(\omega), \xi \rangle .$$

Define

$$\tilde{X}_t(\omega) = \begin{cases} \tilde{x}_t(\omega) & \text{if } \omega \in \Omega\backslash\Omega_4 , \\ 0 & \text{if } \omega \in \Omega_4 , \end{cases}$$

so that by (2.2) and the Sazonov-Minlos' theorem in [3], $\{\tilde{X}_t, t \in T_+\}$ is a version of $\{X_t, t \in T_+\}$. Since Theorem 1 guarantees that there exists $p = p_T \in N$ such that $\{\tilde{X}_t, t \in T_+\}$ is $\|\cdot\|_{-p}$-continuous almost surely, the proof is completed.

The following theorem is immediate from Theorem 2.

THEOREM 3. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued Gaussian process and for any fixed ξ in E the real Gaussian process $\{\langle X_t, \xi\rangle, t \in R_+\}$ have a continuous version. Then X has a continuous version.*

EXAMPLE 1. Let X be an E'-valued Gaussian process. According to Fernique's condition, we consider the following inequality:

$$E[\langle X_t - X_s, \xi\rangle^2] \leqq \phi_\xi^2(|t - s|) \tag{2.3}$$

for any $t, s \in R_+$ and ξ in E, where $\phi_\xi(u)$ is a non-negative function which is monotone increasing on $0 < u < \alpha_\xi$ and satisfies

$$\int_{M_\xi}^{+\infty} \phi_\xi(e^{-x^2})dx < +\infty \qquad \text{for some } M_\xi < +\infty .$$

Under the condition (2.3), by Theorem 3, X has an E'-valued continuous version. This is an extension of Fernique's result to E'-valued processes. (see R. M. Dudley [1] and X. Fernique [2]).

We have the following theorem for the whole time interval R_+. Denote by $\mathscr{F}_+$ the set of all positive locally bounded functions on R_+.

THEOREM 4. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued quasi weakly continuous Gaussian process. Then there exists $p \in N$ such that X is $\|\cdot\|_{-p}$-continuous almost surely if and only if there is $f(t) \in \mathscr{F}_+$ such that*

$$\sup_{T \in R_+} \frac{V_T(\xi)}{f(T)} < +\infty$$

for any ξ in E.

Proof. Since $V_T(\xi)$ is a lower semi-continuous function of ξ as we have proved, $\sup_{T \in R_+} V_T(\xi)/f(T)$ is also a lower semi-continuous function of ξ. To prove the sufficiency it suffices only to repeat word by word the proof of Theorem 1. The necessity can be shown as follows. By the hypothesis of $\|\cdot\|_{-p}$-continuity, we have $P(\omega\colon \sup_{t \in T_+} \|X_t\|_{-p}^2 < +\infty) = 1$ for any fixed $T > 0$, so that $\boldsymbol{E}[\sup_{t \in T_+} \|X_t\|_{-p}^2] < +\infty$. (see [*]). Put $f(T) = \boldsymbol{E}[\sup_{t \in T_+} \|X_t\|_{-p}^2]$, then $f(t) \in \mathscr{F}_+$ and satisfies the desired inequality.

Moreover if X is additive, the condition is given in terms of mean and variance of X.

COROLLARY 1. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued quasi weakly continuous additive (necessarily Gaussian) process. Then there exists $p \in N$ such that X is $\|\cdot\|_{-p}$-continuous almost surely if and only if there is $g(t) \in \mathscr{F}_+$ such that*

$$\sup_{t \in R_+} \frac{m_t^2(\xi) + v_t(\xi)}{g(t)} < +\infty$$

for any ξ in E.

The above corollary is proved by combining Theorem 4 with the following theorem.

THEOREM 5. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued Gaussian additive process. Then there exists $p \in N$ such that X has a $\|\cdot\|_{-p}$-continuous version if and only if there is $h(t) \in \mathscr{F}_+$ such that*

$$\sup_{t \in R_+} \frac{m_t^2(\xi) + v_t(\xi)}{h(t)} < +\infty ,$$

and $m_t(\xi)$ and $v_t(\xi)$ are continuous real functions of t for any ξ in E.

Proof. We first prove the sufficiency. By the Baire's category theorem there exist $q \in N$ and a constant $D > 0$ such that

$$\max\{m_t^2(\xi), v_t(\xi)\} \leqq Dh(t)\|\xi\|_q^2 .$$

Hence m_t belongs to $E'_{r'}$ $(r > q)$, for every $t \in R_+$. From the nuclearity of E there is an integer $p > q$ such that E_p is nuclearly imbedded into E_q. For any fixed $T > 0$, we have

$$\langle m_t - m_s, \zeta_j\rangle^2 \leqq 4\Big(\sup_{t \in T_+} h(t)\Big)D\|\zeta_j\|_q^2$$

for $t, s \in T_+$. Therefore by the Lebesgue's convergence theorem, m_t is $\|\cdot\|_{-p}$-continuous.

Put $Y_t = X_t - m_t$. Then it can be shown that Y_t has a $\|\cdot\|_{-p}$-continuous version by following the same argument as in the proof of Theorem 4.1 of [5].

Set up $h(t) = \sup_{\substack{\|\xi\|_p \leq 1 \\ \xi \in E}} \{m_t^2(\xi) + v_t(\xi)\}$, then it can be shown in a way similar to the proof of the necessity of Theorem 4 that $h(t)$ satisfies the desired properties, which proves the necessity.

The following Example 2 does not satisfy the condition of Theorem 4 and Example 3 does not satisfy the condition of Corollary 1, consequently that of Theorem 5.

EXAMPLE 2. Let $\{x_j\}$ be a sequence of points of $\mathscr{S}'$ whose element x_j belongs to $\mathscr{S}'_j \backslash \mathscr{S}'_{j-1}$ if $j \geqq 2$ and x_1 belongs to $\mathscr{S}'_1$. Set up

$$y(t) = \begin{cases} t(1-t)x_1 & \text{if } 0 \leqq t \leqq 1 , \\ (t-1)(2-t)x_2 & \text{if } 1 \leqq t \leqq 2 , \\ \quad\vdots & \\ (t-(n-1))(n-t)x_n & \text{if } n-1 \leqq t \leqq n , \\ \quad\vdots & \end{cases}$$

Define $X_t = B(t)y(t)$, where $B(t)$ is a one dimensional Brownian motion.

Then X is an $\mathscr{S}'$-valued continuous Gaussian process but it does not stay $\mathscr{S}'_p$ for the whole time interval R_+.

EXAMPLE 3. Let $\{x_j\}$ be the same sequence as in Example 2. Define

$$X_t = \begin{cases} B_1(t)x_1 & \text{if } 0 \leqq t \leqq 1\,, \\ B_1(t)x_1 + B_2(t-1)x_2 & \text{if } 1 \leqq t \leqq 2\,, \\ \quad\vdots & \\ \sum_{j=1}^{n} B_j(t-(j-1))x_j & \text{if } n-1 \leqq t \leqq n\,, \\ \quad\vdots & \end{cases}$$

where $\{B_j(t)\}$ is a sequence of mutually independent one dimensional Brownian motions such that $B_j(0) = 0$ almost surely, $j = 1, 2, \cdots$. Then X is an $\mathscr{S}'$-valued continuous additive process but is on the same situation as above.

§ 3. Countable strict inductive limit of nuclear Fréchet spaces

Throughout this section we assume that E is a countable strict inductive limit of an increasing sequence of nuclear Fréchet spaces $\{F_n, n \in N\}$.

Let $X = \{X_t, t \in R_+\}$ be an E'-valued stochastic process and I be a subset of R_+. Then a Hilbert space H with norm $\|\cdot\|_H$ satisfying the following properties (a), (b), (c) is called a *common Hilbertian support over I.*

(a) H is a $C(E', E)$-measurable linear subspace of E'.

(b) $\mu_t(H) = 1$ for every $t \in I$.

(c) The injection from H into E' equipped with the strong topology is continuous.

We will begin with an extension of Theorem 1.

THEOREM 6. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued quasi weakly continuous Gaussian process. Then for any fixed $T > 0$ there exists a common separable Hilbertian support H over T_+ such that $\{X_t, t \in T_+\}$ is $\|\cdot\|_H$-continuous almost surely, so that X is continuous and simultaneously weakly continuous.*

Proof. Let $\|\cdot\|_{n,1} \leqq \|\cdot\|_{n,2} \leqq \cdots \leqq \|\cdot\|_{n,p} \leqq \cdots$ be an increasing sequence of Hilbertian semi-norms defining the topology of F_n. Let $F_{n,p}$ be the completion of F_n by $\|\cdot\|_{n,p}$ and $\|\cdot\|_{n,-p}$ be the norm of $F'_{n,p}$. Then for any fixed $n \in N$ Theorem 1 shows that for any fixed $T > 0$ there is $p_n =$

$p_T^n \in N$ such that $\{X_t, t \in T_+\}$ is $\|\cdot\|_{n,-p_n}$-continuous almost surely. We consider $Z = \bigcap_{n=1}^{+\infty} F'_{n,p_n}$ which is metrized by

$$\rho(x) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{\|x\|_{n,-p_n}}{1 + \|x\|_{n,-p_n}} .$$

Since F'_{n,p_n} is a separable Hilbert space, Z is a separable Fréchet space. In a way similar to that of J. Kuelbs [7], (see H. Sato [8]), we can choose an increasing sequence $\{G_j\}$ of bounded, closed and absolutely convex subsets of Z satisfying

$$jG_j \subset G_{j+1} \quad \text{and} \quad \lim_{j \to +\infty} P(\omega: X_t \in G_j, t \in T_+) = 1 .$$

Define an inner product on $H_0 = \bigcup_{j=1}^{+\infty} G_j$ by

$$\|x\|_H^2 = (x, x)_H = \sum_{j=1}^{+\infty} \frac{1}{2^j a_j} \|x\|_{j,-p_j}^2 ,$$

where $a_j = \sup_{x \in G_j} (1 + \|x\|_{j,-p_j}^2)$, then the completion of H_0 by $\|\cdot\|_H$ is the desired Hilbertian support. This completes the proof.

By Theorem 2, similarly we have

THEOREM 7. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued Gaussian process and for any fixed ξ in E, the real Gaussian process $\{\langle X_t, \xi\rangle, t \in R_+\}$ have a continuous version. Then for any fixed $T > 0$ there exists a common separable Hilbertian support H over T_+ such that $\{X_t, t \in T_+\}$ has a $\|\cdot\|_H$-continuous version.*

Appealing Theorem 7, we have an extension of Proposition 1.

COROLLARY 2. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued Gaussian additive process where for any ξ in E, $m_t(\xi)$ and $v_t(\xi)$ are continuous real functions of t. Then for any fixed $T > 0$ there exists a common separable Hilbertian support H over T_+ such that $\{X_t, t \in T_+\}$ has a $\|\cdot\|_H$-continuous version.*

As a corollary of Theorem 7, we have

COROLLARY 3. *Under the same assumption as in Theorem 7, X has a continuous version.*

Appealing to Corollary 3, Fernique's result can be extended to E'-valued processes.

For the whole time interval R_+, we have the following theorems.

THEOREM 8. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued quasi weakly continuous Gaussian process. Then there exists a common separable Hilbertian support H over R_+ such that X is $\|\cdot\|_H$-continuous almost surely if and only if there is $f(t) \in \mathscr{F}_+$ such that*

$$\sup_{\tau \in R^+} \frac{V_\tau(\xi)}{f(\tau)} < +\infty$$

for any ξ in E.

Combining Theorem 8 with Theorem 9 yields

COROLLARY 4. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued quasi weakly continuous additive process. Then there exists a common separable Hilbertian support H over R_+ such that X is $\|\cdot\|_H$-continuous almost surely if and only if there is $g(t) \in \mathscr{F}_+$ such that*

$$\sup_{t \in R_+} \frac{m_t^2(\xi) + v_t(\xi)}{g(t)} < +\infty$$

for any ξ in E.

THEOREM 9. *Let $X = \{X_t, t \in R_+\}$ be an E'-valued Gaussian additive process. Then there exists a common separable Hilbertian support H over R_+ such that X has a $\|\cdot\|_H$-continuous version if and only if there is $h(t) \in \mathscr{F}_+$ such that*

$$\sup_{t \in R_+} \frac{m_t^2(\xi) + v_t(\xi)}{h(t)} < +\infty ,$$

and $m_t(\xi)$ and $v_t(\xi)$ are continuous real functions of t for any ξ in E.

Proof. First we prove the sufficiency of Theorem 8. From Theorem 4, we can choose a sequence $\{G_{mj}: m \in N, j \in N\}$ of bounded, closed and absolutely convex subsets of Z satisfying

$$jG_{mj} \subset G_{mj+1} , \qquad G_{mj} \subset G_{m+1j}$$

and

$$\lim_{j \to +\infty} P(\omega: X_t \in G_{mj}, t \in [0, m]) = 1 .$$

If we take $H_0 = \bigcup_{m=1}^{+\infty} G_{mm}$, the rest of the proof is similar to that of Theorem 6.

By Theorem 5, the sufficiency of Theorem 9 can be proved similarly.

The necessities of Theorems 8 and 9 are due to quite same reasons in Theorems 4 and 5 by virtue of the following Remark, which completes the proof.

Remark. If X is an E'-valued additive process and there exists a common separable Hilbertian support H over I such that $\{X_t, t \in I\}$ is $\|\cdot\|_H$-continuous almost surely, then $\{X_t, t \in I\}$ is an H-valued additive process. If X is an E'-valued Gaussian process and $\{X_t, t \in I\}$ satisfies the same assumption as above, then $\{X_t, t \in I\}$ is an H-valued Gaussian process because the range of the adjoint of the continuous injection from H into E' is dense in H'.

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References

[1] Dudley, R. M., The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, Journal of Funct. Anal., **1**, No. **3** (1967), 290–330.
[2] Fernique, X., Régularité des trajectoires des fonctions aléatoires gaussiennes. Ecole d'été de Calcul des Probabilités, St. Flour. Lect. Notes in Math., **480** (1974), 1–96.
[3] Gelfand, I. M. and Vilenkin, N. Ya., Generalized functions, **4**, Academic Press, (1964).
[4] Horváth, J., Topological vector spaces and distributions, **1**, Addison-Wesley (1966).
[5] Itô, K., Continuous additive $\mathscr{S}'$-processes, Lect. Notes in Control and Information Sciences, **25** (1980), 143–151.
[6] Landau, H. J. and Shepp, L. A., On the supremum of a Gaussian process, Sankhya Ser. A. **32** (1970), 369–378.
[7] Kuelbs, J., Some results for probability measures on linear topological vector spaces with an application to Strassen's Log Log Law, Journal of Funct. Anal., **14** (1973), 28–43.
[8] Sato, H., Banach support of a probability measure in a locally convex space., Lect. Notes in Math., **526** (1976), 221–226.

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