17

Conformal invariance in four-dimensional field theories and in QCD

Conformal symmetry was shown to be extremely powerful in two-dimensional field theories and obviously also in string theories. This is due to the fact that the conformal algebra is infinite dimensional and hence supplies a set of infinitely many conserved charges. In four dimensions the conformal algebra is finite and therefore less powerful.[1] The purpose of this chapter is to explore the use in 4d conformal field theories of notions and tools that we encountered in two-dimensional CFT, such as primary fields, conformal operator expansion, conformal anomalies and Ward identities.

Free massless theories are obviously scale and conformal invariant. However, field theories that describe the elementary particles of nature and their interactions, are interacting field theories. The question is thus, to what extent can one apply the techniques of CFT to those theories and in particular QCD in four dimensions? QCD with massless quarks is a prototype model of theories which are classically conformal invariant. In fact even for theories with masses and other dimension-full parameters, in certain cases these can be neglected in the high energy and high momentum transfer regime of the theory. However, even in the massless case and with only dimensionless couplings, it is easy to realize that the quantum picture lacks conformal invariance. This follows from the fact that one has to introduce dimension-full parameters such as UV cutoff, which turns into a scale where the coupling is defined, after renormalization. Thus there is an anomaly in the conformal symmetry in the sense that it is a symmetry of the classical system but not of the quantum one.

We will investigate in this chapter conformal symmetry in four-dimensional field theories and in particular its applications in the context of QCD in four dimensions. We start with the description of conformal transformations, their corresponding generators and the $SO(2,4)$ conformal algebra. We then analyze the Noether currents that follow from the conformal transformation and their conservation laws. Next we present the $SL(2,\mathcal{R})$ collinear subgroup associated with light-cone conformal transformations. In a similar manner to the treatment of two-dimensional conformal symmetry, we define primary and descendant operators of the collinear group. We then define and study the conformal operator product expansion (COPE). We proceed and describe the conformal Ward identity, the Callan–Symanzik equation. We then make use of the conformal toolbox

[1] The full conformal algebra in four dimensions was introduced in [156]

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in the study of four dimensional QCD which is conformal only at the classical level. We analyze the non-local operators built from a quark and an anti-quark and expand them in terms of Gegenbauer polynomials. We use the COPE to write down the operator product of two electromagnetic currents. Finally we determine, in the limit of large momentum exchange, the pion distribution amplitude.

Conformal invariance in four dimensions was described in several review papers and books. The original studies on conformal symmetry are summarized in [207] and [66]. A modern review that we follow in this chapter is [43].

17.1 Conformal symmetry algebra in four dimensions

In general in d space-time dimensions the conformal group is the subgroup of coordinate transformations that leaves the metric invariant up to a scale, namely,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{17.1}$$

It is obvious from (2.2) that the 2d conformal transformations (2.1) indeed produce such a variation of the metric. An important property of conformal transformations in any dimension is that they preserve the angle $\frac{\vec{A}\cdot\vec{B}}{\sqrt{A^2B^2}}$ between two vectors $\vec{A}$ and $\vec{B}$.

Starting from flat space-time, the general infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$ induces a change of the metric,

$$\mathrm{d}s^2 \to \mathrm{d}s^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{17.2}$$

so that the condition for conformal transformations reads,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}, \tag{17.3}$$

where $g_{\mu\nu}$ is $\eta_{\mu\nu}$ or $\delta_{\mu\nu}$ for a Minkowskian signature or Euclidean signature, respectively, and the factor of $\frac{2}{d}$ is fixed by tracing both sides of the equation with $g^{\mu\nu}$. To check what are all the possible solutions for ϵ_μ we differentiate (2.4) twice to find that,

$$[(d-2)\partial_\mu \partial_\nu + g_{\mu\nu}\partial_\alpha \partial^\alpha]\partial_\beta \epsilon^\beta = 0. \tag{17.4}$$

This equation together with (2.4) implies that the third derivatives of ϵ^μ vanish, which means that ϵ^μ can be of order $0, 1, 2$ in x^ν. Obviously the parameters associated with the Poincare group, since they are an isometry and hence do not change the metric, obey (17.1). These transformation parameters together with additional infinitesimal parameters which are linear and quadratic in x^μ are summarized as follows:

$$\begin{aligned}
\epsilon^\mu &= \epsilon_0^\mu && \textit{space-time translations}\\
\epsilon^\mu &= \epsilon_0^{\mu\nu} x_\nu && \textit{Lorentz transformations}\\
\epsilon^\mu &= \epsilon_0 x^\mu && \textit{scale transformations}\\
\epsilon^\mu &= \tilde{\epsilon}_0^\mu x^2 - 2x^\mu \tilde{\epsilon}_0^\nu x_\nu && \textit{special conformal transformations}
\end{aligned} \tag{17.5}$$

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where ϵ_0^μ, $\epsilon_0^{\mu\nu}$, ϵ_0, $\tilde{\epsilon}_0^\mu$ are vector, antisymmetric tensor, scalar and vector infinitesimal constants, respectively. The corresponding finite transformations are,

$$\begin{aligned}
x^\mu &\to x^{\mu\prime} = x^\mu + a^\mu \\
x^\mu &\to x^{\mu\prime} = a^\mu_\nu x^\nu \\
x^\mu &\to x^{\mu\prime} = a x^\mu \\
x^\mu &\to x^{\mu\prime} = \frac{x^\mu + \tilde{a}^\mu x^2}{1 + 2\tilde{a}^\mu x_\mu + \tilde{a}^2 x^2},
\end{aligned} \tag{17.6}$$

where the various forms of a are the finite parameters of transformation that correspond to the infinitesimal ones above. The last transformation is referred to as the special conformal transformation. It can in fact be decomposed into an inversion transformation $x^\mu \to -\frac{x^\mu}{x^2}$, a space-time shift transformation and another inversion. The sum of all these transformations is $d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+1)(d+2)}{2}$, which is the dimensions of $SO(2, d)$, the algebra of the conformal group in Minkowski space-time.

Let us analyze now the generators of the conformal transformations and show that indeed they obey the $SO(2, d)$ algebra. The generators are,

$$\begin{aligned}
P_\mu &= -i\partial_\mu && \textit{space-time translations} \\
L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && \textit{Lorentz transformations} \\
D &= -ix^\mu\partial_\mu && \textit{scale transformation} \\
K_\mu &= -i\left[x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu\right] && \textit{special conformal transformations}
\end{aligned} \tag{17.7}$$

Using these expressions for the generators it is a straightforward exercise to realize that they obey the following algebra,

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\mu, L_{\nu\rho}] &= i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= -i(\eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\nu\rho}L_{\mu\sigma}) \\
[D, P_\mu] &= -iP_\mu \qquad [D, K_\mu] = iK_\mu \\
[P_\mu, K_\nu] &= 2i(L_{\mu\nu} - \eta_{\mu\nu}D) \\
[D, L_{\mu\nu}] &= 0 \\
[K_\mu, K_\nu] &= 0,
\end{aligned} \tag{17.8}$$

which is indeed the $SO(2, 4)$ algebra. The first three lines constitute the Poincare algebra in four dimensions. It is well known that (17.8) is not the most general form of the $SO(2, 4)$ algebra. One can further generalize the construction of the generators by modifying $L_{\mu\nu}$ in the following way,

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}, \tag{17.9}$$

where $\Sigma_{\mu\nu}$ does not act on the space-time points and obeys,

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = -i(\eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\mu\sigma}\Sigma_{\nu\rho} + \eta_{\nu\sigma}\Sigma_{\mu\rho} - \eta_{\nu\rho}\Sigma_{\mu\sigma}). \tag{17.10}$$

Shortly the role of these generators in the conformal transformation of fields will be discussed.

17.2 Conformal invariance of fields, Noether currents and conservation laws

So far we have discussed the conformal transformations as they act on the points in space-time. Now we would like to consider field theories in four dimensions that are classically invariant under the group of conformal transformations. The fields associated with conformal invariant theories may be scalar fields, spinors, vectors or tensors. We denote such a field by $\Phi(x)$.

The transformation of the field under the conformal transformations,

$$\delta\Phi(x) = \delta x^\mu \partial_\mu \Phi(x) + \delta_I \Phi(x), \tag{17.11}$$

is composed of two parts, the one due to that of the space-time point with δx^μ given in (17.5) and an "internal transformation" $\delta_I \Phi(x)$, which vanishes for space-time translations, while for Lorentz transformations, dilatations and special conformal transformations, takes the form,

$$\begin{aligned}
&\delta_I \Phi(x) = -e_{\mu\nu} \Sigma^{\mu\nu} \Phi && \textit{Lorentz transformations} \\
&\delta_I \Phi(x) = e\, l \Phi && \textit{scale translations} \\
&\delta_I \Phi(x) = 2\tilde{e}_\mu (l x^\mu - x_\nu \Sigma_{\mu\nu}) \Phi && \textit{special conformal transformations}
\end{aligned} \tag{17.12}$$

where l is the conformal dimension of the field and the internal Lorentz generators $\Sigma\mu\nu$ are given by,

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \;\; - \textit{Dirac spinors} \qquad [\Sigma^{\mu\nu}]^\beta_\alpha = \eta^{\mu\beta}\delta^\nu_\alpha - \eta^{\nu\beta}\delta^\mu_\alpha \;\; - \textit{gauge fields} \tag{17.13}$$

Recall that all the parameters of transformations are global, namely space-time independent. To determine the Noether currents associated with the various symmetry transformations, one elevates the transformations into local ones and reads the currents from the variation of the action,

$$\delta S = \int \mathrm{d}^4 x J^a_\mu \partial^\mu e_a \tag{17.14}$$

where e^a is any of the parameters of transformations given in (17.5). The outcome of the Noether procedure are the following conserved currents,

$$\begin{aligned}
J^{(P)}{}^\mu_\nu \equiv T^\mu_\nu &= \Pi^\mu \partial_\nu \Phi - \delta^\mu_\nu \mathcal{L} \\
J^{(M)}{}^\mu_{\nu\rho} &= x_\nu T^\mu_\rho - x_\rho T^\mu_\nu - \Pi_\mu \Sigma_{\nu\rho} \Phi \\
J^{(D)}{}^\mu \equiv D^\mu &= x_\nu T^{\mu\nu} + l \Pi^\mu \Phi \\
J^{(K)}{}^\mu_\nu &= (2x_\rho x_\nu - \eta_{\nu\rho} x^2) T^{\mu\rho} + 2x_\rho \Pi^\mu (l\delta^\rho_\nu - \Sigma^\rho_\nu)\Phi,
\end{aligned} \tag{17.15}$$

where $\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}$. In fact the variation of the action with respect to dilatations may lead in addition to the divergence of $J^{(D)^\mu}$, to another total derivative term Δ_D. However for Lagrangians that are polynomials in the fields and their derivatives, this term vanishes. For the special conformal transformations the additional term is defined by

$$\delta_K S = \int \mathrm{d}^4 x e_\mu(x) \left[-\partial_\nu K^{\mu\nu} + 2x^\mu \Delta_D + \Delta_k^\mu\right], \tag{17.16}$$

with $\Delta_k^\mu = 2\Pi_\nu \Phi(l g^{\mu\nu} + \Sigma^{\mu\nu})\Phi$. For invariance we need $\Delta_D = 0$ and $\Delta_k^\mu = 2\partial_\nu \sigma^{\mu\nu}$. For $l = 1$ and $l = 3/2$, $\sigma_{\mu\nu}$ vanishes, so in these cases $J^{(K)}{}^\mu_\nu$ are really the generators of conformal transformations.

An interesting observation is that all the Noether currents associated with the full conformal group can be expressed in terms of a modified energy-momentum tensor. First note that the energy-momentum tensor defined above in not necessarily symmetric. In fact from the conservation of $J^{(M)}{}^\mu_{\nu\rho}$ the antisymmetric part of $T_{\mu\nu}$ can be determined since,

$$\partial_\mu J^{(M)}{}^\mu_{\nu\rho} = T_{\rho\nu} - T_{\nu\rho} - \partial_\mu(\Pi_\mu \Sigma_{\mu\nu} \Phi) = 0. \tag{17.17}$$

Using this result it is now easy to define a modified conserved symmetric energy-momentum tensor,

$$T^{(S)}_{\mu\nu} = T_{\mu\nu} + \frac{1}{2}\partial^\rho(\Pi_\rho \Sigma_{\mu\nu}\Phi - \Pi_\mu \Sigma_{\rho\nu}\Phi - \Pi_\nu \Sigma_{\rho\mu}\Phi). \tag{17.18}$$

The current associated with the Lorentz transformations can be expressed in terms of $T^{(S)}_{\mu\nu}$ as,

$$J^{(M)}{}^\mu_{\nu\rho} = x_\nu T^{(S)}{}^\mu_\rho - x_\rho T^{(S)}{}^\mu_\nu. \tag{17.19}$$

One can further modify the symmetric energy-momentum tensor to render it also traceless,

$$T^{(TL)}_{\mu\nu} = T^{(S)}_{\mu\nu} + \frac{1}{2}\partial^\rho \partial^\sigma X_{\rho\sigma\mu\nu}, \tag{17.20}$$

where $X_{\rho\sigma\mu\nu}$ is defined such that the energy-momentum tensor is traceless and conserved and $\eta^{\mu\nu}\partial^\rho\partial^\sigma X_{\rho\sigma\mu\nu} = 2\partial^\rho\partial^\nu \sigma_{\rho\nu} = -2\eta^{\mu\nu} T^{(S)}_{\mu\nu}$. In terms of this traceless energy-momentum tensor the dilatation current and the current associated with the special conformal transformation are given by,

$$J^{(D)^\mu} = x_\nu T^{(TL)^{\mu\nu}} \qquad J^{(K)}{}^\mu_\nu = (2x_\nu x_\rho - x^2 \eta_{\rho\nu}) T^{(TL)^{\mu\rho}}. \tag{17.21}$$

It is thus clear that $J^{(D)^\mu}$ and $J^{(K)}{}^\mu_\nu$ are conserved only provided that $T^{(TL)^{\mu\nu}}$ is conserved and traceless.

Note that in the latter form, scale invariance, namely a traceless energy-momentum tensor, implies also conformal invariance.

17.3 Collinear and transverse conformal transformations of fields

Recall that in 2d conformal field theories it is very useful to employ light-cone coordinates (in Minkowski space-time) or holomorphic and anti-holomorphic coordinates (in complex Euclidean space-time). We want to argue now that it is also quite useful to use light-cone coordinates, when analyzing four-dimensional conformal field theories.

Consider the two light-like vectors n_+^μ and n_-^μ,

$$n_+^\mu n_{+\,\mu} = n_-^\mu n_{-\,\mu} = 0 \qquad n_+^\mu n_{-\,\mu} = 1. \tag{17.22}$$

We can then decompose any Lorentz four-vector A_μ as follows,

$$A^\mu = A_- n_+^\mu + A_+ n_-^\mu + A_T^\mu, \tag{17.23}$$

where,

$$A_+ \equiv A_\mu n_+^\mu \quad A_- \equiv A_\mu n_-^\mu, \tag{17.24}$$

and the transverse part of the four-vector A_T^μ is defined using the transverse part of the metric defined as,

$$\eta_{\mu\nu}^T = \eta_{\mu\nu} - n_-^\mu n_+^\nu - n_+^\mu n_-^\nu \quad A_T^\mu \equiv \eta_{\mu\nu}^T A_\nu. \tag{17.25}$$

Using this decomposition we find that $A_\mu A^\mu = 2A_+A_- - A_T^2$.

We can now also consider transformations associated with a subgroup of the full conformal group. In particular consider the special transformations associated with a light-like parameter $\tilde{a}^\mu = \tilde{a} n_-^\mu$. The transformation of x_- takes the form

$$x_- \to x'_- = \frac{x_-}{1 + 2\tilde{a}x_-}. \tag{17.26}$$

Combining this transformation with the translation along the x_- direction $x_- \to x_- + a_-$ and scaling $x_- \to a x_-$ these transformations constitute a subgroup of the full conformal group, the *collinear subgroup* which is an $SL(2,R)$.[2] To verify this group structure we define the following generators,

$$\begin{aligned} L_+ &= -iP_+ & L_- &= \frac{i}{2}K_- \\ L_0 &= \frac{i}{2}(D + M_{-+}) & E &= \frac{i}{2}(D - M_{-+}). \end{aligned} \tag{17.27}$$

The generators $L_\pm$ and L_0 obey the algebra,

$$[L_0, L_\pm] = \pm L_\pm \qquad [L_-, L_+] = -2L_0, \tag{17.28}$$

which is indeed the $SL(2,R) \sim SO(2,1)$ algebra; E commutes with them. It is, actually, the L_0 of the other $SL(2,R)$, the one in the x_+ direction.

[2] The use of the $SL(2,R)$ group in applications of conformal symmetry to QCD was introduced in [150] and [83]

The $SL(2,R)$ collinear subgroup is particularly useful for collinear fields, where for instance $\Phi(x)$ takes the form $\Phi(\alpha) \equiv \Phi(\alpha n_+^\mu)$, with α a real number and n_+^μ the light-cone direction defined above. In particular, as will be shown below, this will apply to parton description of quarks. The field $\Phi(\alpha)$ is taken to be an eigenstate of the spin operator Σ_{+-},

$$\Sigma_{+-}\Phi(\alpha) = s\Phi(\alpha), \tag{17.29}$$

so that s is the spin projection to the n_+ direction. The collinear subgroup of the conformal group now acts on the coordinate α as an $SL(2,R)$ transformation,

$$\alpha \to \alpha' = \frac{a\alpha + b}{c\alpha + d}, \tag{17.30}$$

where a, b, c, d are real numbers with $ad - bc = 1$, and correspondingly the field $\Phi(\alpha)$ transforms as,

$$\Phi(\alpha) \to (c\alpha + d)^{-2j}\,\Phi\left(\frac{a\alpha + b}{c\alpha + d}\right), \tag{17.31}$$

with,

$$j = \frac{1}{2}(l + s). \tag{17.32}$$

Thus $\Phi(\alpha)$ is a representation of $SL(2,R)$ or an $SL(2,R)$ form of degree j.

The generators of the $SL(2,R)$ group and E act on the collinear field as,

$$\begin{aligned}
[L_+, \Phi(\alpha)] &= -\partial_\alpha \Phi(\alpha) \\
[L_0, \Phi(\alpha)] &= (\alpha\partial_\alpha + j)\Phi(\alpha) \\
[L_-, \Phi(\alpha)] &= (\alpha^2\partial_\alpha + 2j\alpha)\Phi(\alpha) \\
[E, \Phi(\alpha)] &= \frac{1}{2}(l - s)\Phi(\alpha)
\end{aligned} \tag{17.33}$$

where $t = (l - s)$ is referred to as the *collinear twist*. In addition $\Phi(\alpha)$ is an eigenstate of the Casimir operator with,

$$\sum_{i=0,1,2} [L_i, [L_i, \Phi(\alpha)]] = j(j-1)\Phi(\alpha). \tag{17.34}$$

Another subgroup of the conformal group is the transverse subgroup $SL(2,C)$ acting on the transverse coordinates $x_T^\mu = (0, x_1, x_2, 0)$ or in complex coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, with fields $\Phi(z, \bar{z})$. This is in fact identical to the $SL(2,C)$ discussed in Chapter 3 where the conformal symmetry of two-dimensional field theories is discussed. In terms of the conformal generators (17.7) the generators of the $SL(2,C)$ are the $P_T^\mu, M_T^{\mu\nu}, D, K_T^\mu$. The coordinate z transforms under the $SL(2,C)$ transformation,

$$z \to z' = \frac{az + b}{cz + d}, \tag{17.35}$$

which implies the following transformation of $\Phi(z,\bar{z})$,

$$\Phi(z,\bar{z}) \to (cz+d)^{-2h}\,\Phi\left(\frac{az+b}{cz+d},\bar{z}\right), \tag{17.36}$$

where $h = \frac{1}{2}(l+\lambda)$ with λ is the helicity defined by $\Sigma_{z\bar{z}}\Phi = \lambda\Phi$. Similarly for the transformation of $\bar{z}$ and the corresponding transformation of $\Phi(z,\bar{z})$, with $\bar{h} = \frac{1}{2}(l-\lambda)$.

17.4 Collinear primary fields and descendants

In two-dimensional conformal field theories fields were classified into primary fields and descendant ones. The classification was based on their conformal transformations. Correspondingly the states were put into Verma modules each containing a highest weight state and its descendants. Recall the definition of the former,

$$L_0\,[\phi(0)|0>] = h\,[\phi(0)|0>] \qquad L_n\,[\phi(0)|0>] = 0, \quad n>0. \tag{17.37}$$

In a similar manner the primary operator and the highest weight state of the four-dimensional collinear group are defined [171], [42] as,

$$\begin{aligned} [L_0,\Phi(0)] &= j\Phi(0) \qquad [L_-\phi(0)] = 0 \qquad \Rightarrow \\ L_0[\Phi(0)|0>] &= j[\Phi(0)|0>] \qquad L_-[\Phi(0)|0>] = 0, \end{aligned} \tag{17.38}$$

$\Phi(0)$ is by definition collinear since it is defined at the origin of the light-cone direction. The fact that in the 2d case the conformal algebra is infinite while in 4d it is finite is manifested by the fact that in the former case there is an infinite set of annihilation operators $L_n,\ n>0$ that annihilate the highest weight state, whereas in the latter case it is the single operator L_-.

The descendant fields and correspondingly the descendant states are obtained by repeatedly applying the creation operators, which are L_{-n} in 2d while in 4d it is the operator L_+. So in 4d,

$$\mathcal{O}_n = [L_+,\ldots[L_+,[L_+,\Phi(0)]]] = (-\partial_+)^n\,\Phi(\alpha)|_{\alpha=0}. \tag{17.39}$$

Note the difference in notation, as it is L_- in 2d while it is L_+ in 4d, both raising.

The descendant operators obey the following commutation relations,

$$[L_0,\mathcal{O}_n] = (j+n)\mathcal{O}_n \qquad [L_+,\mathcal{O}_n] = \mathcal{O}_{n+1} \qquad [L_-,\mathcal{O}_n] = -n(n+2j-1)\mathcal{O}_{n-1}. \tag{17.40}$$

In two-dimensions we discussed the Verma module that includes a highest weight state and all its descendants, and similarly, in four dimensions we consider the so-called conformal tower which also includes the highest weight state and all its descendants. Recall however that there is an essential difference between the two cases due to the fact that in the 2d the algebra is infinite dimensional whereas in 4d it is finite dimensional. In particular the notion of null vectors that played an important role in the 2d case does not exist in four-dimensional CFTs.

We can associate the complete sets of $\Phi(\alpha)$ and $\mathcal{O}_n$ by the following Taylor expansion,

$$\Phi(\alpha) = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \mathcal{O}_n. \tag{17.41}$$

An interesting and useful map relates the descendant operators and polynomials. Consider for instance the descendent operator defined in (17.39), which can be re-expressed as,

$$\mathcal{O}_n = \mathcal{P}_n(\partial_\alpha)\Phi(\alpha)|_{\alpha=0} \qquad \mathcal{P}_n(u) = (-u)^n. \tag{17.42}$$

It is straightforward to realize that in terms of these polynomials the operation of $L_0, L_\pm$ takes the form,

$$\begin{aligned} L_+ &\to \tilde{L}_- = -u \\ L_- &\to \tilde{L}_+ = (u\partial_u^2 + 2j\partial_u) \\ L_0 &\to \tilde{L}_0 = (u\partial_u + j). \end{aligned} \tag{17.43}$$

This correspondence can be viewed as first mapping,

$$\partial_\alpha \to u \qquad \alpha \to \partial_u, \tag{17.44}$$

then interchanging the + and − components, and finally some "normal ordering" of taking the derivatives with respect to u to the right of the factors of u.

The representation in terms of polynomials is referred to as the 'adjoint representation'. Note that since in the original algebra L_- includes a term proportional to α^2, the new algebra includes a second-derivative term ∂_u^2 in $\tilde{L}_+$. This can be avoided by introducing a different argument of the polynomials defined as,

$$\frac{u^n}{\Gamma(n+2j)} \to \kappa^n, \tag{17.45}$$

so that the action $L_0, L_\pm$ on $\tilde{\mathcal{P}}(\kappa)$ is the same as (17.33) with $\alpha \to \kappa$ and the interchange of L_- and L_+.

We now discuss composite operators built from two "elementary" operators of the form,

$$O(\alpha_1, \alpha_2) = \Phi_{j_1}(\alpha_1)\Phi_{j_2}(\alpha_2), \tag{17.46}$$

with $\alpha_1 \neq \alpha_2$. The operator product expansion with $|\alpha_1 - \alpha_2| \to 0$ is expressed in terms of the composite operators,

$$\mathcal{O}_n(0) = \mathcal{P}_n(\partial_1, \partial_2)\Phi_{j_1}(\alpha_1)\Phi_{j_2}(\alpha_2)|_{\alpha_1=\alpha_2=0}, \tag{17.47}$$

where $\mathcal{P}_n(\partial_1, \partial_2)$ is a homogeneous polynomial of degree n. It can be shown that the complete set of local operators with which one can perform the conformal

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operator expansion (COPE) takes the form,

$$\mathcal{O}_n^{j_1,j_2}(x) = \partial_+^n \left[\Phi_{j_1}(x) P_n^{(2j_1-1,2j_2-1)} \left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+} \right) \Phi_{j_2}(x) \right], \tag{17.48}$$

$P_n^{(a,b)}(x)$ are the Jacobi polynomials, and we are back to x space here.[3] One can further generalize this construction to a product of three or more operators.

17.5 Conformal operator product expansion

The conformal operator expansion in two dimensions, discussed in Section 3.7.2, was shown to be a very powerful tool in determining correlation functions. Obviously, we anticipate that in four dimensions the COPE will be less powerful. We discuss now the general structure of the COPE.[4] Consider the OPE of two local conformal operators $A(x)B(0)$ with twists and spin projection along the $+$ direction $(t_A, s_A), (t_B, s_B)$, respectively. We perform an expansion for fixed x_- and $x_+, x_T \to 0$, namely $x^2 \to 0$. We want to expand the product in terms of a complete set $O_{n,n+k}^{j_1,j_2}$ and to leading order in the twist. Such an expansion takes the form,

$$A(x)B(0) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_{n,k} \left(\frac{1}{x^2} \right)^{1/2(t_A+t_B-t_n)} x_-^{n+k+\Delta} \mathcal{O}_{n,n+k}^{j_1,j_2}(0) + \ldots, \tag{17.49}$$

where $\Delta = s_1 + s_2 - s_A - s_B$, $\mathcal{O}_{n,n+k} = (-\partial_+)^k \mathcal{O}_n$, s_1 and s_2 are the spin projections of the constituent fields in the local operators $\mathcal{O}_{n,n+k}^{j_1,j_2}$, $t_n = l_n - n - s_1 - s_2 = l_1 + l_2 - s_1 - s_2$ the twist of the operator $\mathcal{O}_n$, actually independent of n, and the dots refer to higher twist contributions. We want to check to what extent conformal invariance enables us to determine the coefficients $C_{n,k}$. For this purpose we act on the OPE with L_- as,

$$[L_-, A(x)B(0)] = \left(x_-(2j_A + x \cdot \partial_x)A(x) - \frac{1}{2}x^2 \bar{n} \cdot \partial_x A(x) \right) B(0) + \ldots \tag{17.50}$$

Inserting (17.49) and taking into account that,

$$[L_-, \mathcal{O}_{n,n+k}^{j_1,j_2}(0)] = -k(k + 2j_n - 1)\mathcal{O}_{n,n+k-1}^{j_1,j_2}, \tag{17.51}$$

[3] The Jacobi polynomial is given by,

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)}{n!\Gamma(a+b+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(a+b+m+n+1)}{\Gamma(a+m+1)} \left(\frac{z-1}{2} \right)^m .$$

[4] COPE in four dimensions was introduced in [90] and used in QCD in [49], [50], [51].

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with $j_n = j_1 + j_2 + n$, we find the following recursion relation for the coefficients $C_{n,k}$,

$$C_{n,k+1} = -\frac{j_A - j_B + j_n + k}{(k+1)(k+2j_n)} C_{n,k}, \tag{17.52}$$

which is solved by,

$$C_{n,k} = (-1)^k \frac{1}{k!} \frac{\Gamma(j_A - j_B + j_n + k)}{\Gamma(j_A - j_B + j_n)} \frac{\Gamma(2j_n)}{\Gamma(2j_n + k)} C_n, \tag{17.53}$$

where $C_n = C_{n,0}$. Plugging this into (17.49) we get the following form for the OPE,

$$\begin{aligned} A(x)B(0) = \sum_{n=0}^{\infty} C_n \left(\frac{1}{x^2}\right)^{1/2(t_A + t_B - t_n)} & \frac{x_-^{n+s_1+s_2-s_A-s_B}}{B(j_A - j_B + j_n, j_B - j_A + j_n)} \\ \times \int_0^1 du u^{(j_A - j_B + j_n - 1)} (1-u)^{(j_B - j_A + j_n - 1)} & \mathcal{O}_n^{j_1, j_2}(ux_-), \end{aligned} \tag{17.54}$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

17.6 Conformal Ward identities

The application of conformal invariance in determining correlation functions was discussed in Section 2.8 for the case of two-dimensional conformal field theories. It includes both the use of the Ward identities associated with the global $SL(2,C)$ transformation[5] and the full holomorphic conformal transformation. Here in discussing four-dimensional field theories we will encounter two major differences:

(i) Due to the fact that the conformal symmetry group is finite dimensional, there are Ward identities only associated with global transformations.
(ii) When discussing theories with conformal anomalies, there will be modifications of the Ward identities.

Let us start by reminding ourselves of the concept of Ward identities and in particular the conformal ones. Associated with any infinitesimal transformation of a given field $\phi(x) \to \phi(x) + \delta\phi(x)$, the action that describes the system is transformed into,

$$S \to S + \delta S = S + \int \mathrm{d}^4 x \Delta(\phi, \partial_\mu \phi). \tag{17.55}$$

Associated with this transformation there is a current J_μ such that $\partial^\mu J_\mu = \Delta$. Obviously where $\Delta = 0$ (or a total derivative) the transformation is a symmetry and the corresponding Noether current is conserved. Associated with such a

[5] Conformal Ward identities which were studied in [168] are identical to the Callan–Symanzik equation [55] and [204].

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transformation of the field there is a constraint on correlation functions of this field. This constraint which is referred to as a *Ward identity* takes the form,

$$\begin{aligned}&\partial_{y^\mu} <TJ_\mu(y)\phi(x_1)...\phi(x_N)> = <T\Delta(y)\phi(x_1)...\phi(x_N)>\\&-i\delta^4(x_1-y) <T\delta\phi(x_1)...\phi(x_N)> ...\\&-i\delta^4(x_i-y) <T\phi(x_1)...\delta\phi(x_i)...\phi(x_N)> ...\\&-i\delta^4(x_N-y) <T\phi(x_1)...\delta\phi(x_N)> .\end{aligned} \tag{17.56}$$

This relation can be derived straightforwardly using the path integral formulation of correlation functions. The Ward identity takes a simpler form when integrating over y^μ,

$$\begin{aligned}&<T\delta\phi(x_1)...\phi(x_N)> +...+ <T\phi(x_1)...\delta\phi(x_i)...\phi(x_N)> +...\\&\quad <T\phi(x_1)...\delta\phi(x_N)> + <Ti\delta S\phi(x_1)...\phi(x_N)> = 0.\end{aligned} \tag{17.57}$$

In particular in analogy with (2.56) the Ward identities associated with dilation and special conformal transformation take the form,

$$\begin{aligned}&\sum_i^N (l_\phi + x_i\partial_i) <T\phi(x_1)...\phi(x_N)> = -i\int \mathrm{d}^4x <T\Delta_D(x)\phi(x_1)...\phi(x_N)>\\&\sum_i^N (2x_i^\mu(l_\phi + x_i\partial_i) - 2\Sigma^\mu_\nu x_i^\nu - x_i^2\partial_i^\mu) <T\phi(x_1)...\phi(x_N)>\\&= -i\int d^4x 2x^\mu <T\Delta_D(x)\phi(x_1)...\phi(x_N)>,\end{aligned} \tag{17.58}$$

where l_ϕ is the canonical dimension, namely that of the free field. Similarly to the way we extracted information about the structure of correlators in 2d CFT in Section 2.9, we can now constrain the form of correlators in 4d. The Ward identities associated with the Poincare transformations imply that any correlation function is in fact not a general function of the N coordinates x_i^μ, but only of the invariants $x_{ij}^2 \equiv (x_i - x_j)^2$.

To understand the implication of the dilatation transformation on the correlation function let us first study the theory at its fixed point, namely at a coupling $g^* = g(\mu^*)$ such that $\beta(g^*) = 0$. Recall that the β function is defined as,

$$\beta(g(\mu)) = \mu\frac{\partial}{\partial_\mu} g(\mu), \tag{17.59}$$

and hence the vanishing β function implies a fixed point of the coupling constant g. This will be further discussed below for the case of 4d QCD. In this case the dilatation Ward identity takes the form of that of a free theory, like the one given in (17.58), apart from the change of scaling dimension,

$$\sum_i^N (l_\Phi + \gamma(g^*) + x_i\partial_i) <T\phi(x_1)...\phi(x_N)> = 0, \tag{17.60}$$

where $\gamma(g^*)$ is the anomalous dimension of the filed Φ. As a consequence of this form of the Ward identity, the two-point function of two scalar fields at the fixed point has the form,

$$<\phi(x_1)\phi(x_2)> = N_2(g^*)(\mu^*)^{-2\gamma(g^*)}\left[\frac{1}{(x_1-x_2)^2}\right]^{l_\phi+\gamma(g^*)} . \tag{17.61}$$

For particles with spin s and the same projection on the light-cone,

$$<\phi(x_1)\phi(x_2)> = N_2(g^*)(\mu^*)^{-2\gamma(g^*)}\left[\frac{1}{(x_1-x_2)^2}\right]^{l_\phi+\gamma(g^*)}\left(\frac{(x_1-x_2)_+}{(x_1-x_2)_-}\right)^s , \tag{17.62}$$

where it is assumed that $(x_1-x_2)_T = 0$. At the fixed point, namely $\beta(g^*) = 0$, the Ward identity associated with the special conformal transformation takes the form of that of a free theory with l_ϕ again shifted by the anomalous dimension $l_\phi \to l_\phi + \gamma(g^*)$.

In two dimensions (see Section 2.8) it was found that the three-point function of primary fields is fully determined by the $SL(2, C)$ symmetry and any four-point function of primary fields is determined up to a function of the cross ratio (or anharmonic ratio) $\frac{z_{12}z_{34}}{z_{13}z_{24}}$ and its complex conjugate coordinate. Based on the Poincare, dilation and special conformal transformation in four dimensions, the three-point function is determined here too. For instance, the three-point function of a scalar field is,

$$<\phi(x_1)\phi(x_2)\phi(x_3)>=N_3(g^*)(\mu^*)^{-3\gamma(g^*)}\left[\frac{1}{(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2}\right]^{[l_\phi+\gamma(g^*)]/2} , \tag{17.63}$$

and any correlator of $n > 3$ operators depends only on the ratios $\frac{x_{ij}x_{kl}}{x_{il}x_{jl}}$.

It is worth noting that the Ward identity associated with the dilation (the first equation of (17.58)) is in fact the same as the Callan–Symanzik renormalization group equation. First note that based on dimensional counting and Lorentz invariance the dependence of the N point function on the scale μ takes the form,

$$<T\Phi(x_1)...\Phi(x_N)> = \mu^{N l_\Phi} G(x_{ik}^2\mu^2; g(\mu)), \tag{17.64}$$

which means that the following relation holds,

$$\sum_{i=1}^{N}(l_\Phi + x_i\partial_i) <T\Phi(x_1)...\Phi(x_N)> = \mu\frac{\partial}{\partial\mu} <T\Phi(x_1)...\Phi(x_N)> . \tag{17.65}$$

It is easy to realize that the right-hand side of the conformal Ward identity can be rewritten in the form,

$$i\int d^4x <T\Delta_D(x)\Phi(x_1)...\Phi(x_N)> = -M\frac{\partial}{\partial M} <T\Phi(x_1)...\Phi(x_N)>, \tag{17.66}$$

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as follows from,

$$\Delta_D(x) = -M\frac{\partial}{\partial M}\mathcal{L}_{\text{eff}}\,, \tag{17.67}$$

for the cases with no explicit dimension-full parameters. We will show this explicitly for the effective theory of 4d QCD below. On the other hand the dependence of the correlator on M follows from the dependence of the field renormalization factor and the dependence of the coupling constant so that,

$$-M\frac{\partial}{\partial M} <T\Phi(x_1)...\Phi(x_N)> = \left[\beta(g)\frac{\partial}{\partial g} + \sum_{i=1}^{N}\gamma_{\Phi_i}\right] <T\Phi(x_1)...\Phi(x_N)> . \tag{17.68}$$

Combining this together with (17.58) and (17.60) we get the Callan–Symanzik equation,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \sum_{i=1}^{N}\gamma_{\Phi_i}\right] <T\Phi(x_1)...\Phi(x_N)> = 0. \tag{17.69}$$

17.7 Conformal invariance and QCD_4

So far we have discussed the implications of conformal invariance in general, and in particular the invariance properties under the $SL(2,R)$ collinear group and conformal Ward identities. We are now in the position to examine the application of conformal symmetry to four-dimensional QCD. Recall that the action of four-dimensional $SU(N)$ gauge theory with massless quarks takes the form,

$$\mathcal{L}_{QCD4} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + i\bar{\psi}\not{D}\,\psi, \tag{17.70}$$

where $D_\mu = \partial_\mu - igt^a A^a_\mu$ is the covariant derivative, and t^a as usual are the $N\times N$ matrices in the fundamental representation of the $SU(N)$ algebra. It is straightforward to check that the corresponding classical action is invariant under the full set of fifteen transformations associated with the $SO(2,4)$ symmetry group. In particular it is invariant under the scale transformation given by,

$$x_\mu \to \lambda x_\mu \qquad A_\mu(x) \to \lambda A_\mu(\lambda x) \qquad \psi(x) \to \lambda^{3/2}\psi(\lambda x). \tag{17.71}$$

The invariance under these transformations manifests itself in the form of conservation of the corresponding Noether current,

$$D_\mu = x_\nu T^{(TL)\mu\nu} = x_\nu\left[F^{\mu\rho\,a}F^{\nu\,a}_{\rho} + \frac{i}{2}\bar{\psi}(\overleftrightarrow{D})^{(\mu}\gamma^{\nu)}\psi\right], \quad \partial^\mu D_\mu = 0, \tag{17.72}$$

where $(\overleftrightarrow{D}) \equiv \overrightarrow{D} - \overleftarrow{D}$. The classical invariance is not maintained quantum mechanically. This situation of having classical conformal symmetry but not a corresponding quantum mechanical one is referred to as the *conformal*

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anomaly.[6] In string theory the two-dimensional conformal symmetry is local. Having an anomaly in a local symmetry renders the theory into an inconsistent one. This implies that (at least in flat space-time) the theory will be defined in a critical dimension where the conformal anomaly vanishes. In the four-dimensional field theories discussed here, like QCD_4, conformal invariance is a global symmetry and the theory is consistent even when having an anomaly. There are several ways to show that the quantum theory is not scale and hence also not conformal invariant. One may say that the anomaly follows from the fact that the theory has infinities that are cured just by the introduction of a renormalization procedure. The latter involves the introduction of a cutoff scale. Once a scale is introduced the theory is not any more scale invariant.

To see it more explicitly let us consider the low energy effective action of massless QCD_4. We expand the gluons and quarks in terms of modes and distinguish the low energy (momentum) modes and the high energy modes. Next we integrate the high energy modes to derive the one loop low energy effective action. It takes the form,[7]

$$S_{\rm LE} = -\frac{1}{4}\int \mathrm{d}^4x \left[\left(\frac{1}{g_0^2} - \frac{\beta_0}{16\pi^2}\ln\left(\frac{M^2}{\mu^2}\right)\right) F^a_{\mu\nu}F^{\mu\nu\, a} + ...\right]_{\rm low}, \tag{17.73}$$

where M is the UV cutoff, and $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f$ is the coefficient of the one loop beta function. It is easy to check that this one loop renormalized action *is not invariant* under the scale transformations of (17.71). The variation of the action under those transformation reads,

$$\delta S = -\frac{1}{32\pi^2}\beta_0 \ln\lambda \int \mathrm{d}^4x \left[\frac{1}{g_0^2}F^a_{\mu\nu}F^{\mu\nu\, a} + ...\right]_{\rm low}. \tag{17.74}$$

Thus the quantum mechanically (unlike the classical case) dilatation Noether current is not conserved,

$$\partial_\mu D^\mu \equiv \Delta_D = -\frac{1}{32\pi^2}\left[\beta(g)F^a_{\mu\nu}F^{\mu\nu a}\right]_{\rm low}, \tag{17.75}$$

and in deriving the right-hand side of the equation we have used the equations of motion.

The effective action admits also an anomaly with respect to the special conformal transformations.

In (17.46) we discussed the general structure of non-local operators of four-dimensional conformal field theory. In QCD in many cases we encounter a non-local operator built from a quark and an anti-quark at light-like separation, with

[6] The conformal anomaly was introduced in [169] and [6].

[7] The explicit calculation is a one loop perturbative calculation. Since we do not deal with perturbative methods in this book, we do not present here the derivation and refer the reader to references that deal with perturbation theory in QCD_4.

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a line integral connecting them,

$$\mathcal{Q}_\mu(\alpha_1,\alpha_2) = \bar\psi(\alpha_1)\gamma_\mu P \mathrm{e}^{ig\int_{\alpha_2}^{\alpha_1} \mathrm{d}t A_+(t)}\psi(\alpha_2) \tag{17.76}$$

where P stands for path ordering. The path integral factor will be denoted $[\alpha_1,\alpha_2]$. In performing the short distance expansion we need now to identify the corresponding conformal operators. To relate the operator ψ to a primary operator we first have to make a spin projection in the following way,

$$\psi_+ = \Gamma_+\psi \qquad \psi_- = \Gamma_-\psi \qquad \psi = \psi_+ + \psi_-, \tag{17.77}$$

where,

$$\Gamma_+ = \frac{1}{2}\gamma_-\gamma_+, \qquad \Gamma_- = \frac{1}{2}\gamma_+\gamma_-, \qquad \Gamma_- + \Gamma_+ = 1. \tag{17.78}$$

The spin projected parts are,

$$\psi_+(s=+1/2,\ j=1,\ t=1) \qquad \psi_-(s=-1/2,\ j=1/2,\ t=2). \tag{17.79}$$

With this identification we define the quark anti-quark operators:

$$\begin{aligned}
&twist-2: \mathcal{Q}_+ = \bar\psi_+\gamma_+\psi_+ \equiv \mathcal{Q}^{(1,1)}\\
&twist-3: \mathcal{Q}_T = \bar\psi_+\gamma_T\psi_- + \bar\psi_-\gamma_T\psi_+ \equiv \mathcal{Q}^{(1,1/2)} + \mathcal{Q}^{(1/2,1)}\\
&twist-4: \mathcal{Q}_- = \bar\psi_-\gamma_-\psi_- \equiv \mathcal{Q}^{(1/2,1/2)}.
\end{aligned} \tag{17.80}$$

The corresponding local conformal operators are,

$$\begin{aligned}
Q_n^{1,1}(\alpha) &= (i\partial_+)^n\left[\bar\psi(\alpha)\gamma_+ C_n^{3/2}\left(\overleftrightarrow{D}_+/d_+\right)\psi(\alpha)\right],\\
Q_n^{1,1/2}(\alpha) &= (i\partial_+)^n\left[\bar\psi(\alpha)\gamma_+\gamma_T\gamma_- P_n^{1,0}\left(\overleftrightarrow{D}_+/d_+\right)\psi(\alpha)\right],\\
Q_n^{1/2,1/2}(\alpha) &= (i\partial_+)^n\left[\bar\psi(\alpha)\gamma_- C_n^{1/2}\left(\overleftrightarrow{D}_+/d_+\right)\psi(\alpha)\right],
\end{aligned} \tag{17.81}$$

where

$$\overleftrightarrow{D}_+ = \overrightarrow{D}_+ - \overleftarrow{D}_+ \qquad d_+ = \overrightarrow{D}_+ + \overleftarrow{D}_+, \tag{17.82}$$

and where the Jacobi polynomials with two identical indices were replaced by the Gegenbauer polynomials $P^{(1,1)} \sim C_n^{3/2}$ and $P^{(0,0)} \sim C_n^{1/2}$.

A similar analysis can be carried out for the gluons. The various components of the gluon field have the following properties,

$$\begin{aligned}
&F_{+T}(s=+1,\ j=3/2,\ t=1) \qquad F_{TT}, F_{+-}(s=0,\ j=1,\ t=2)\\
&F_{-T}(s=-1,\ j=1/2,\ t=3).
\end{aligned} \tag{17.83}$$

Local operators built from two-gluon fields with leading twist are,

$$\mathcal{G}_n^{3/2,3/2}(\alpha) = (i\partial_+)^n\left[F_{+T}(\alpha)C_n^{5/2}\left(\overleftrightarrow{D}_+/d_+\right)F_{+T}(\alpha)\right]. \tag{17.84}$$

Another application of conformal invariance to QCD is the determination of the OPE of two electromagnetic currents $j_\mu^{\mathrm{EM}} = \sum_i e_i\bar\psi_i\gamma_\mu\psi_i$ where the e_i are the

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charges of the u, d and s quarks. At the tree level only the transverse components are of interest. The latter have spin $s_j = 0$ and twist $t_j = 3$. The quark operators $\mathcal{Q}_n^{1,1}$ are the relevant basis for the expansion, with conformal spin $j_n = (l_n + 1 + n)/2 = n + 2$ and $t_n = (l_n - 1 - n)/2 = 2$. As $\Delta = 1$ we find

$$J^T(x)J^T(0) \sim \sum_{n=0}^{\infty} C_n \left(\frac{1}{x^2}\right)^{(6-t_n)/2} (-ix_-)^{n+1} \frac{\Gamma(2j_n)}{\Gamma(j_n)\Gamma(j_n)} \int_0^1 \mathrm{d}u[u(1-u)]^{j_n-1} \mathcal{Q}_n^{1,1}(ux_-). \tag{17.85}$$

The coefficients C_n can be extracted from deep inelastic scattering via the following matrix element of forward scattering,

$$<P|J^T(x)J^T(0)|P> \sim \sum_{n=0}^{\infty} C_n \left(\frac{1}{x^2}\right)^{(6-t_n)/2} (-ix_-)^{n+1} <P|\mathcal{Q}_n^{1,1}(0)|P> . \tag{17.86}$$

Another application of the COPE is the determination of the short-distance expansion of the operator $\mathcal{Q}_+$ (17.76). This case is characterized by $s_A = s_B = s_1 = s_2 = \frac{1}{2}$ so that $\Delta = 0$ and $l_A = l_B = l_1 = l_2 = \frac{3}{2}$ and we find,

$$\mathcal{Q}_+(\alpha_1, \alpha_2) \sim \sum_{n=0}^{\infty} \tilde{C}_n (-i)^n (\alpha_1 - \alpha_2)^n \int_0^1 \mathrm{d}u u^{n+1}(1-u)^{n+1} \mathcal{Q}_n^{1,1}(u\alpha_1 + (1-u)\alpha_2), \tag{17.87}$$

where $\tilde{C}_n = \frac{C_n \Gamma(n+2)^2}{\Gamma(2n+4)}$ which can be determined again from forward matrix elements and are found to be $\tilde{C}_n = \frac{2(2n+3)}{(n+1)!}$.

Conformal invariance can be used at short distances to give predictions for the quark distribution amplitudes for flavor non-singlet mesons, namely the wave functions which control the behavior of the exclusive mesons processes at large momentum transfer. Here we discuss as an example the pion distribution amplitude in the leading twist order.

The basic ingredient in computing exclusive reactions including a large momentum transfer to a pion is the matrix element of a quark anti-quark between the vacuum and a one pion state. By using the light-cone gauge $A_+ = 0$ the Wilson line (17.76) is set to unity. We choose a frame where $p_\mu = p_+ n_{-\mu}$ and $x^\mu = x_- n_+^\mu + x_T^\mu,\ x_+ = 0$ so that $x^2 = x_T^2$. The matrix element can then be written as,

$$\begin{aligned} &<0|\bar{d}(0)[0, \infty n]\gamma_+\gamma_5[\infty n + x, x]u(x)|\pi^+(p)> = \\ &\qquad if_\pi p_+ \int_0^1 \mathrm{d}y \mathrm{e}^{-iy(p\cdot x)} f(y, \ln x^2) + O(x^2). \end{aligned} \tag{17.88}$$

This matrix element is the probability amplitude to find the pion in the valence state consisting of a u-quark carrying a momentum y and an anti-d quark of momentum $\bar{y} = 1 - y$ and have a transverse separation x_T. This amplitude is intimately related to the pion electromagnetic form factor for large momentum transfer Q^2 and small separation distance of the order $x_T \sim 1/Q^2$. To approach

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this limit, one defines the pion distribution amplitude taken at exactly light-like separation where $x_T = 0$. This amplitude reads,

$$<0|\bar{d}(0)[0,\alpha]\gamma_+\gamma_5 u(\alpha)|\pi^+(p)> = if_\pi p_+ \int_0^1 \mathrm{d}y \mathrm{e}^{-iy(\alpha p_+)}\phi_\pi(y,\mu). \qquad (17.89)$$

The distribution amplitude $\phi_\pi(y,\mu)$ is scale and scheme dependent. In fact the small transverse distance behavior of the valence component of the pion wave function is traded for the scale dependence of the distribution amplitude.

It can be shown that the evolution equation of $\phi_\pi(y,\mu)$ is given by,

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\phi_\pi(y,\mu) = \int_0^1 \mathrm{d}\tilde{y} V(y,\tilde{y},\alpha_s(\mu)\phi_\pi(y,\mu), \qquad (17.90)$$

where to leading order in α_s the integral kernel is given by,

$$V_0(y,\tilde{y}) = C_\mathrm{F}\left[\frac{1-y}{1-\tilde{y}}\left(1+\frac{1}{y-\tilde{y}}\right)\theta(y-\tilde{y}) + \frac{\tilde{y}}{y}\left(1+\frac{1}{y-\tilde{y}}\right)\theta(y-\tilde{y})\right]_+, \qquad (17.91)$$

where $]_+$ stands for,

$$[V(\tilde{y},y)]_+ = V(\tilde{y},y) - \delta(y-\tilde{y})\int_0^1 \mathrm{d}t V(t,\tilde{y}). \qquad (17.92)$$

Instead of solving this evolution equation one can alternatively proceed by expanding both sides of (17.90) in powers of α. In this way moments of the distribution amplitude are related to matrix elements of renormalized local operators in the following form,

$$<0|\bar{d}(0)\gamma_+\gamma_5(i\overleftrightarrow{D}_+)^n u(0)|\pi^+(p)> = if_\pi(p_+)^{n+1}\int_0^1 \mathrm{d}y(2y-1)^n\phi_\pi(y,\mu). \qquad (17.93)$$

This is similar to the leading twist operators that enter the OPE for the unpolarized deep inelastic scattering apart from the flavor, the additional γ_5 factor and the fact that now one has to take into account mixing with operators that contain total derivatives of the form,

$$\mathcal{O}_{n-k,k} = (i\partial_+)^k \bar{d}(0)\gamma_+\gamma_5(i\overleftrightarrow{D}_+)^{n-k}u(0). \qquad (17.94)$$

The mixing matrix is in fact triangular since operators with fewer total derivatives can only mix with operators with more total derivatives but not the other way around. The components of the matrix on the diagonal are true anomalous dimensions, which are identical to those of inelastic scattering,

$$\gamma_n^{(0)} = C_\mathrm{F}\left(1 - \frac{2}{(n+1)(n+2)} + 4\sum_{m=2}^{n+1}\frac{1}{m}\right), \qquad (17.95)$$

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where,

$$<P|\mathcal{O}_{n,0}(\mu)|P> = <P|\mathcal{O}_{n,0}(\mu_0)|P> \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_n^{(0)}}{\beta_0}} \qquad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f. \tag{17.96}$$

Conformal invariance is useful in finding the eigenvectors of the mixing matrix since conformal operators with different conformal spins cannot mix under renormalization to leading order. This happens since to leading order the renormalization is determined by counter terms of the tree level which is conformal invariant. Thus the mixing eigenvector operators are $Q^{1,1}(x)$ defined in (17.80) with the right flavor and γ matrices structure,

$$Q_n^{1,1}(x) = (i\partial_+)^n \left[\bar{d}(x)\gamma_+\gamma_5 C_n^{3/2}\left(\overleftrightarrow{D}_+/d_+\right)u(x)\right]. \tag{17.97}$$

Note that because of their flavor content these operators cannot mix with operators made out of gluons and they also cannot mix with operators with more fields since they have higher twist. Thus the operators (17.97) are the only relevant ones and they must be multiplicatively renormalized. Comparing (17.93) with (17.97) one concludes that the Gegenbauer moments of the pion distribution amplitudes are given in terms of reduced matrix elements of conformal operators,

$$if_\pi p_+^{n+1}\int_0^1 \mathrm{d}y C_n^{3/2}(2y-1)\phi_\pi(y,\mu) = <0|Q_n^{1,1}(0)|\pi^+(p)> . \tag{17.98}$$

As was mentioned above these operators are renormalized by a multiplication and the corresponding anomalous dimension is given by (17.95). Thus the final picture is that the distribution amplitude $\phi_\pi(u,\mu)$ can be expanded in a series of Gegenbauer polynomials,

$$\begin{aligned}
\phi_\pi(u,\mu) &= 6u(1-u)\sum_{n=0}^{\infty}\phi_n(\mu)C_n^{3/2}(2u-1)\\
\phi_n(\mu) &= (if_\pi p_+^{n+1})^{-1}\frac{2(2n+3)}{3(n+1)(n+2)} <0|Q_n^{1,1}(0)|\pi^+(p)>\\
\phi_n(\mu) &= \phi_n(\mu_0)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_n^{(0)}}{\beta_0}} .
\end{aligned} \tag{17.99}$$

This example demonstrates the application of conformal invariance to solve the problem of operator mixing. There are other applications of conformal symmetry to four-dimensional QCD. We refer the interested reader to [43]. The predictions based on conformal symmetry beyond one loop, for the sector with $\beta = 0$ [51], turned out to be in contradiction with explicit calculations [79]. This paradox was resolved in [166].

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