J. Austral Math. Soc. (*Series A*) **29** (1980), 177–205

MEAN VALUE THEOREMS FOR MULTIPLICATIVE FUNCTIONS BOUNDED IN MEAN α-POWER, $\alpha > 1$

P. D. T. A. ELLIOTT

(Received April 27; revised July 23, 1979)

Communicated by A. J. van der Poorten

Abstract

On analogy with functions of Lebesgue class L^α on the real line the author considers those multiplicative arithmetic functions which are bounded in mean α-power, $\alpha > 1$. Necessary and sufficient conditions are obtained in order that they should have a mean-value, zero or non-zero. An application is made to Ramanujan's τ-function.

1980 Mathematics subject classification (*Amer. Math. Soc.*): 10 H 25, 10 K 20.

1

An arithmetic function $g(n)$, possibly complex-valued, is said to be *multiplicative* if it satisfies the relation $g(ab) = g(a)g(b)$ whenever a and b are coprime positive integers.

The study of the average behaviour of arithmetic functions goes back at least as far as Dirichlet (1849), and, according to him, as far as Gauss (1801, 1870). Dirichlet proved that the multiplicative function $d(n)$, which counts the number of divisors of the integer n, has the average estimate

$$\sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}), \quad x \geqslant 2.$$

For a discussion of this and related results see Hardy and Wright (1960). They devote three chapters of their book to the value distribution properties of multiplicative functions. For further general results see Atkinson and Cherwell (1949).

In more recent times much effort has been expended on the asymptotic estimation of sums

$$\sum_{n \leqslant x} g(n)$$

Supported by N.S.F. Contract No. MCS 78–04374.

assuming that the behaviour of $g(p^k)$ on the prime-powers p^k is known. See, for example, Delange (1961), Wirsing (1961, 1967), Halász (1968), Levin and Fainleib (1967). Besides being of interest in their own right, such results have applications to probabilistic number theory (see, for example, Elliott (1980a)) and to the theory of Sieves (see, for example, Halberstam and Richert (1974)).

It is of equal interest to proceed in the opposite direction; assume familiarity with the average behaviour of $g(n)$ and then deduce the behaviour of g on the primes. Indeed, this is the route taken in the classical study of prime numbers.

In the present paper we concentrate our attention upon the problem of giving *necessary and sufficient* conditions in order that a finite mean-value

$$A = \lim_{x\to\infty} x^{-1} \sum_{n\leqslant x} g(n) \tag{1}$$

exists. This represents an interest to discover what is really needed in order to determine the average behaviour of a multiplicative function.

As an example we shall apply our results to the study of the function $\tau(n)$, defined by Ramanujan according to the equation

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x \prod_{j=1}^{\infty} (1-x^j)^{24}, \quad |x|<1.$$

It was conjectured by Ramanujan, and proved by Mordell (1917), that $\tau(n)$ is multiplicative. Already in 1918 Hardy proved that

$$c_1 x^{12} \leqslant \sum_{n\leqslant x} \tau(n)^2 \leqslant c_2 x^{12}$$

holds with certain positive constants c_1 and c_2, for all large enough values of x. This was refined by Rankin (1939), to an asymptotic estimate of the type

$$\sum_{n\leqslant x} \tau(n)^2 = Bx^{12} + O(x^{12-\delta})$$

for some $\delta > 0$. Hardy's earlier result may be found in Hardy (1927), and he gives a general account of these matters in Hardy (1940). We shall deduce from his upper bound that in a certain sense either $\tau(n)\, n^{-11/2} \to 0$ almost surely as $n \to \infty$, or $|\tau(p)|\, p^{-11/2} \to 1$ almost surely as $p(\text{prime}) \to \infty$.

In its full generality we cannot presently decide the question of when the mean-value (1) exists. Until recently progress was usually made by making some extra assumptions to the effect that $g(p^k)$ be 'not too large'. In the present paper we shall instead adopt the point of view of the author's paper (Elliott (1975)).

For each positive number α we say that $g(n)$ belongs to the class L^α if

$$\limsup_{x\to\infty} x^{-1} \sum_{n\leqslant x} |g(n)|^\alpha$$

is finite. This is an analogy with the Lebesgue measurable functions on the real line.

For functions $g(n)$ which assume only real non-negative values the existence of a mean-value (1) ensures that g belongs to the class L^1. For such functions only partial results are presently available (see, for example, Erdös and Rényi (1965)).

We shall show that if a multiplicative function $g(n)$ belongs to a class L^α with $\alpha > 1$, then one may completely decide, in terms of its behaviour on the prime-powers, when it has a mean-value (1).

For the duration of the paper $g(n)$ will denote a multiplicative function.

THEOREM 1. *Let $g(n)$ belong to the class L^α, $\alpha > 1$. In order that it should possess a non-zero mean-value*

$$\lim_{x\to\infty} x^{-1} \sum_{n\leq x} g(n)$$

it is necessary and sufficient that the series

$$(2) \qquad \sum \frac{g(p)-1}{p}, \quad \sum_{||g(p)|-1|\leq \frac{1}{2}} \frac{|g(p)-1|^2}{p}, \quad \sum_{||g(p)|-1|> \frac{1}{2}} \frac{|g(p)|^\alpha}{p}, \quad \sum_{p,m\geq 2}\sum \frac{|g(p^m)|^\alpha}{p^m}$$

converge, and that for each prime p

$$(3) \qquad \sum_{m=1}^{\infty} \frac{g(p^m)}{p^m} \neq -1.$$

REMARKS. The convergence of the fourth (and double) series at (2) ensures that the series

$$\sum_{m=1}^{\infty} \frac{g(p^m)}{p^m}$$

converges. If the series at (2) converge then, whether (3) holds or not, $g(n)$ will possess a mean-value, and

$$\lim_{x\to\infty} x^{-1} \sum_{n\leq x} |g(n)|^\alpha$$

exists. The mean-value (1) will then be non-zero if (3) holds.

The result of Theorem 1 generalizes the case $\alpha = 2$ which was established by the author (Elliott (1975)). A modified approach to the necessity of these conditions when $\alpha = 2$ was given by Daboussi and Delange (1976). An alternative proof of the sufficiency of the conditions when $\alpha = 2$ has been given by Schwarz (Frankfurt, unpublished). Apparently a result substantially equivalent to the present Theorem 1 has also been given by Daboussi and Delange (Orsay, unpublished), but with a different proof method.

THEOREM 2. *Let $g(n)$ belong to the class L^α, $\alpha > 1$. In order that $g(n)$ possess a zero mean-value*

$$\lim_{x\to\infty} x^{-1} \sum_{n\leq x} g(n) = 0$$

it is necessary and sufficient that one of the following four conditions be satisfied:

(i) *One of the series*

$$\sum_{||g(p)|-1|\leq\frac{1}{2}} p^{-1}|1-|g(p)||^2, \quad \sum_{||g(p)|-1|>\frac{1}{2}} p^{-1}|1-|g(p)||^\alpha \tag{4}$$

diverges.

(ii) *The condition* (i) *fails, but for each real value of t the series*

$$\sum_p p^{-1}(|g(p)| - \operatorname{Re} g(p)p^{-it}) \tag{5}$$

diverges.

(iii) *The conditions* (i) *and* (ii) *fail, but there is a real t so that the series* (5) *converges and*

$$\sum_{m=1}^{\infty} g(p^m)p^{-mit} = -1$$

for some prime p.

(iv) *The conditions* (i), (ii) *and* (iii) *fail, but*

$$\operatorname{Re}\left(\sum_{p\leq x} \frac{1-g(p)}{p}\right) \to \infty \tag{6}$$

as $x\to\infty$.

We round out these results by showing that in a certain sense we always have a near mean-value.

THEOREM 3. *Let $g(n)$ belong to the class L^α, $\alpha > 1$. Then there are constants A, t* (real), *and a slowly-oscillating function $S(x)$, so that as $x\to\infty$*

$$\sum_{n\leq x} g(n) = (A+o(1))x^{1+it}S(x).$$

REMARKS. This result may be compared with Theorem 2 of Halász (1968). Here for $S(x)$ to be slowly-oscillating is meant that $S(x)$ is non-zero for all sufficiently large values of x, and for each fixed positive value of u satisfies $S(ux)/S(x)\to 1$ as $x\to\infty$. We do not assert that $|S(x)| = 1$ holds. As we shall indicate, it is sometimes possible to give a reasonable description of $S(x)$.

We say that a set of integers E has a density δ if

$$\delta = \lim_{x\to\infty} x^{-1} \sum_{n\leq x,\, n\in E} 1$$

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

exists. We say that a set of primes P has a density λ if

$$\lambda = \lim_{x\to\infty} \frac{1}{\log x} \sum_{p\leq x,\, p\in P} \frac{\log p}{p}$$

exists.

Concerning Ramanujan's τ-function, we deduce from Hardy's upper bound and the fact that $\tau(n)$ is multiplicative the following.

THEOREM 4. *Either $\tau(n)n^{-11/2}\to 0$ as $n\to\infty$ in a sequence of integers of density* 1, *or $|\tau(p)|p^{-11/2}\to 1$ as $p\to\infty$ in a sequence of primes of density* 1.

We shall, in fact, prove more than this.

2

In this and Section 3 we investigate the consequences of the following hypothesis, which we shall call H:

There are constants α and β, $0<\beta<\alpha$, so that

$$\limsup_{x\to\infty} x^{-1} \sum_{n\leq x} |g(n)|^{\alpha} < \infty,$$

(7)

$$\limsup_{x\to\infty} x^{-1} \sum_{n\leq x} |g(n)|^{\beta} > 0.$$

The first part of this hypothesis asserts that $g(n)$ belongs to the class L^{α}. For the moment $\alpha>1$ is not required. Note that if g belongs to L^{α} and the hypothesis fails, then

$$x^{-1} \sum_{n\leq x} |g(n)|^{\delta} \to 0, \quad x\to\infty,$$

holds for each fixed δ, $0<\delta<\alpha$.

It is convenient to define the multiplicative function $h(n)=|g(n)|$. Note that $h(n)$ satisfies H whenever $g(n)$ does. The main conclusions in this and the following section will be stated in Lemma 1 and Lemma 4.

LEMMA 1. *If hypothesis* H *is satisfied then the series*

$$\sum_{|h(p)-1|>\frac{1}{2}} \frac{1}{p}, \quad \sum_{|h(p)-1|\leq\frac{1}{2}} \frac{(h(p)-1)^2}{p} \tag{8}$$

are convergent, and

$$\liminf_{x\to\infty} \left| \sum_{\substack{p\leq x\\ |h(p)-1|\leq\frac{1}{2}}} \frac{h(p)-1}{p} \right|$$

is finite.

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

This lemma will be established with the aid of a result (Lemma 2) from the probabilistic theory of numbers.

An arithmetic function $f(n)$ is said to be *additive* if it satisfies the relation $f(ab) = f(a)+f(b)$ for every pair of coprime positive integers a, b. In this paper all additive functions will be real-valued. $c_5, c_6, \ldots$ will denote positive constants.

LEMMA 2. *Let c_3, c_4 be positive real numbers. Assume that for each of an unbounded sequence of positive x-values, there is a sequence of positive integers*

$$a_1 < a_2 < \ldots < a_k \leqslant x,$$

with $k \geqslant c_3 x$, so that

$$|f(a_i)-f(a_j)| \leqslant c_4, \quad 1 \leqslant i \leqslant j \leqslant k. \tag{9}$$

Then there is a constant c so that for the function $l(p) = f(p) - c\log p$ the series

$$\sum_{|l(p)|>1} \frac{1}{p}, \quad \sum_{|l(p)|\leqslant 1} \frac{l(p)^2}{p} \tag{10}$$

are convergent.

PROOF. The condition (10) actually characterizes those additive functions which satisfy the hypotheses of the lemma. This was first proved by Erdös (1946). For an alternative proof see Ryavec (1970).

LEMMA 3. *In the notation of Lemma 2 replace the hypothesis* (9) *by $|f(a_i)| \leqslant (c_4)/2$, $1 \leqslant i \leqslant k$. Then the series*

$$\sum_{|f(p)|>1} \frac{1}{p}, \quad \sum_{|f(p)|\leqslant 1} \frac{f(p)^2}{p} \tag{11}$$

are convergent, and the partial sums

$$\sum_{\substack{p\leqslant x \\ |f(p)|\leqslant 1}} \frac{f(p)}{p} \tag{12}$$

are uniformly bounded for x belonging to the sequence of given values.

PROOF. Since the hypotheses of Lemma 2 are valid so is the conclusion (10). We prove that $c = 0$ and this will give the convergence of the two series (11).

Let ε be a positive real number. Choose p_1 so large that

$$\sum_{\substack{|l(p)|>1 \\ p>p_1}} \frac{1}{p} + \sum_{p>p_1} \frac{1}{p^2} < \varepsilon.$$

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

Then the number of integers up to x which are divisible by a prime p for which $|f(p)|>1$ or by the square of a prime $p>p_1$ is at most εx.

The number of integers up to x which are divisible by a prime-power p^m where $p\leqslant p_1$ and $m>t$, say, is at most

$$\sum_{p\leqslant p_1}\left[\frac{x}{p^t}\right]\leqslant x\sum_{p\leqslant p_1}p^{-t}<x\,2^{-t/2}\sum_p p^{-t/2}<\varepsilon x$$

if t is fixed at a sufficiently large value. On choosing ε so that $4\varepsilon<c_3$ we obtain a subsequence b_j of the integers a_i, containing at least $c_3x/2$ members, such that $f(b_j)=c\log b_j+O(\omega(b_j))$. Here $\omega(n)$ denotes the number of distinct prime divisors of the integer n. Hence

$$\begin{aligned}|c|\sum_{r\leqslant c_3x/2}\log r &\leqslant \sum_j|c|\log b_j\leqslant\sum_j\{|f(b_j)|+O(\omega(b_j))\}\\ &= O(\sum_{n\leqslant x}\omega(n))=O(x\log\log x),\end{aligned}$$

so that $c=O((\log\log x)/\log x)$. This shows that $c=0$ must hold.

We now apply the Turán–Kubilius inequality (Kubilius (1962))

$$\sum_{n\leqslant x}\left|w(n)-\sum_{p^m\leqslant x}p^{-m}w(p^m)\right|^2\leqslant c_5x\sum_{p^m\leqslant x}p^{-m}|w(p^m)|^2,$$

which is valid for every additive function $w(n)$ and real $x\geqslant 2$. With $w(p^m)=f(p^m)$ when $m=1$, $|f(p)|\leqslant 1$; or when $p\leqslant p_1$ and $m\leqslant t$, and $w(p^m)=0$ otherwise, we see that

$$\sum_{b_j\leqslant x}\left|f(b_j)-\sum_{\substack{p\leqslant x\\ |f(p)|\leqslant 1}}p^{-1}f(p)\right|^2\leqslant c_6x\{\sum_{\substack{p\leqslant x\\ |f(p)|\leqslant 1}}p^{-1}f(p)^2+1\}\leqslant c_7x$$

from the result of Lemma 2 (10) with $c=0$. Since the $f(b_j)$ are bounded independently of x (belonging to the special sequence) so are the partial sums at (12).

This completes the proof of Lemma 3.

Proof of Lemma 1. From (7), for all sufficiently large values of x belonging to a certain sequence

$$\sum_{n\leqslant x}h(n)^\beta>c_8x,$$

say. Hence, for any fixed $\varepsilon>0$,

$$\sum_{\substack{n\leqslant x\\ h(n)>\varepsilon}}h(n)^\beta>(c_8-\varepsilon^\beta)x.$$

Moreover, from our hypothesis H

$$\sum_{\substack{n\leqslant u\\ h(n)>1/\varepsilon}}h(n)^\beta\leqslant\varepsilon^{\alpha-\beta}\sum_{n\leqslant x}h(n)^\alpha<c_9\varepsilon^{\alpha-\beta}x,\quad x\geqslant 2,$$

for some constant c_9. We can thus find an unbounded sequence of x-values, and for each x a sequence of positive integers $a_1 < a_2 < \ldots < a_k \leqslant x$ on which $\varepsilon < h(a_i) \leqslant 1/\varepsilon$, where

$$k > x\varepsilon(c_8 - \varepsilon^{\beta} - c_9\,\varepsilon^{\alpha-\beta}) > c_{10}\,x$$

provided that we fix ε at a suitably small value.

Define an additive function $f(n)$ by

$$f(p^m) = \begin{cases} \log h(p^m) & \text{if } h(p^m) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then (typically)

$$|f(a_i)| \leqslant |-\log \varepsilon|$$

so that the hypotheses of Lemma 3 are satisfied. Hence the series

$$\sum_{|f(p)|>\delta} \frac{1}{p}, \quad \sum_{|f(p)|\leqslant\delta} \frac{f(p)^2}{p}$$

converge for each fixed $\delta > 0$, this being an equivalent form of (11).

If now $|h(p)-1| > \eta$ for some η, $0 < \eta < \frac{3}{4}$, then

$$|f(p)| \geqslant \min(\log(1+\eta), -\log(1-\eta), 1),$$

so that the series

$$\sum_{|h(p)-1|>\eta} \frac{1}{p}$$

converges.

Moreover, if $|h(p)-1| \leqslant \eta$ then

$$\log h(p) = \log(1 + h(p) - 1) = h(p) - 1 + O((h(p)-1)^2)$$

so that for η sufficiently small (but fixed)

$$|h(p)-1| \leqslant 2|\log h(p)| = 2|f(p)|$$

and

$$|f(p)| \leqslant 2|h(p)-1| \leqslant 2\eta.$$

The series

$$\sum_{|h(p)-1|\leqslant\eta} \frac{(h(p)-1)^2}{p}$$

is thus also convergent.

The proof of Lemma 1 is now readily completed.

3

We continue to investigate the consequences of hypothesis H.

LEMMA 4. *If the hypotheses* H *is satisfied with* $\alpha > 1$ *then the series*

$$\sum_{p,m\geq 2} p^{-m} h(p^m)^{\alpha}, \quad \sum_{p} p^{-1} |h(p)-1|^{\alpha}$$

converge.

The proof of this lemma also depends upon a result from the probabilistic theory of numbers, Lemma 5, due to Levin, Timofeev and Tuliaganov (1973).

To begin with we do not need the condition $\alpha > 1$.

LEMMA 5. *Let* $g(n)$ *be a real-valued multiplicative arithmetic function. In order that there exist functions* $\alpha(x)$ *and* $\beta(x) \neq 0$, *defined for all sufficiently large positive values of* x, *so that the frequencies*

$$F(x,z) = [x]^{-1} \sum_{n\leq x,\, g(n)-\alpha(x)\leq z\beta(x)} 1$$

possess a proper weak limiting distribution as $x \to \infty$, *it is both necessary and sufficient that* $g(n)$ *not be identically one, that the series*

$$\sum_{g(p)=0} \frac{1}{p} \tag{13}$$

converges, and that there is a constant c *so that the series*

$$\sum_{g(p)\neq 0} p^{-1} \| \log |g(p)| p^{-c} \|^2 \tag{14}$$

converges.

When these three conditions are satisfied one may take $\alpha(x) = 0$, *and*

$$\beta(x) = x^c \exp\Big(\sum_{\substack{p\leq x \\ g(p)\neq 0}} p^{-1} \| \log |g(p)| p^{-c} \| \Big). \tag{15}$$

REMARKS. A detailed proof of this result together with a discussion of background material and related topics may be found in Elliott (1980a). Here $\|y\|$ denotes y if $|y| \leq 1$ and 1 if $|y| > 1$. The lemma is valid for $g(n)$ real-valued, whether non-negative or not.

We shall apply Lemma 5 to the function $h(n)$ in order to establish the following result.

There is a function $w(x)$ *which satisfies*

$$\liminf_{x\to\infty} |w(x)| \leq c_{11} \tag{16}$$

and

$$\sup_{x^{\frac{1}{2}} \leqslant r \leqslant x} |w(x) - w(r)| \to 0, x \to \infty, \tag{17}$$

so that for each δ, $0 < \delta < \alpha$,

$$\sum_{n \leqslant x} h(n)^{\delta} = (1 + o(1))\, B(\delta)\, x \exp(\delta w(x)) \tag{18}$$

where the constant $B(\delta)$ *is positive.*

Consider the three conditions that need to be satisfied in order to ensure the weak convergence of the frequencies $F(x, z)$ of Lemma 5 (with $g(n)$ replaced by $h(n)$). We may assume that $h(n)$ is not identically one, otherwise the estimate (18) holds with $w(x) = 0$ and $B(\delta) = 1$.

Suppose now that the series (13) diverges. We maintain that the integers for which $h(n) = 0$ have density 1. Let us denote those integers for which $h(n)$ does not vanish by $n_1 < n_2 < \dots$. We shall apply the following

LEMMA 6. *The inequality*

$$\sum_{p \leqslant x} p \left| \sum_{m \leqslant x;\, p \| m} a_m - p^{-1} \sum_{m \leqslant x} a_m \right|^2 \leqslant 36x \sum_{m \leqslant x} |a_m|^2$$

holds for all complex numbers a_m, *and real* $x \geqslant 2$.

PROOF. This result is proved in Chapter 4 of Elliott (1980a). Here $p \| m$ means that p divides m but p^2 does not.

In our present circumstances we set $a_m = 1$ if $h(m) \neq 0$, and $a_m = 0$ otherwise. Let $N(x)$ denote the number of integers n_i not exceeding x. If $p \| m$ and $h(p) = 0$, then $h(m) = h(p)\, h(p^{-1} m) = 0$. For those primes p with $h(p) = 0$ we have

$$\left| \sum_{m \leqslant x;\, p \| m} a_m - p^{-1} \sum_{m \leqslant x} a_m \right|^2 = p^{-1} N(x)^2.$$

An application of lemma 6 now gives

$$N(x)^2 \sum_{p \leqslant x, h(p) = 0} \frac{1}{p} \leqslant 36x^2$$

and so $x^{-1} N(x) \to 0$ as $x \to \infty$. This justifies our earlier assertion.

Applying Hölder's inequality ($\mu = 1 - (\beta/\alpha) > 0$):

$$\sum_{n \leqslant x} h(n)^{\beta} = \sum_{n_j \leqslant x} h(n_j)^{\beta} \leqslant N(x)^{\mu} \sum_{n_j \leqslant x} h(n_j)^{\alpha} = o(x)$$

as $x \to \infty$. This contradicts the second part of our hypothesis H. The series at (13) converges.

Thus there is a distribution function $F(z)$ so that

$$F(x,z) \Rightarrow F(z), \quad x \to \infty,$$

where one may set $\alpha(x) = 0$ and

$$\beta(x) = \exp\Big(\sum_{\substack{p \leqslant x \\ h(p) \neq 0}} p^{-1} \| \log h(p) \| \Big).$$

Define $w(x)$ by $\exp(w(x)) = \beta(x)$.

Let t denote a positive continuity point of $F(z)$. Then from hypothesis H

$$\int_0^t z^\alpha \, dF(z) = \lim_{x\to\infty} x^{-1} \sum_{\substack{n \leqslant x \\ h(n) \leqslant t \exp(w(x))}} \{h(n)\exp(-w(x))\}^\alpha \leqslant c_{12},$$

where the constant c_{12} does not depend upon t. Hence

$$\lim_{t\to\infty} \int_0^t z^\alpha \, dF(z)$$

exists. This shows that $F(z)$ has moments of all orders up to and including the αth. If $0 \leqslant \delta < \alpha$ then a similar argument shows that

$$x^{-1} \sum_{n \leqslant x} \{h(n)\exp(-w(x))\}^\delta \to B(\delta), \quad x \to \infty,$$

where the constant $B(\delta)$ has the value

$$\int_0^\infty z^\delta dF(z).$$

Since $F(z)$ is proper, that is, does not consist of a single jump, it cannot be concentrated at the origin. Thus every $B(\delta)$ is positive.

We have established the asymptotic estimate (18). In order to obtain (17) note that

$$w(x) - w(r) = \sum_{\substack{r < p \leqslant x \\ h(p) \neq 0}} p^{-1} \| \log h(p) \|.$$

After an application of the Cauchy–Schwarz inequality,

$$|w(x) - w(r)|^2 \leqslant \sum_{\substack{r < p \leqslant x \\ h(p) \neq 0}} \frac{1}{p} \sum_{\substack{p \leqslant x \\ h(p) \neq 0}} p^{-1} \| \log h(p) \|^2 = o(1)$$

as $x \to \infty$, since the series

$$\sum_{h(p) \neq 0} \frac{1}{p}, \quad \sum_{h(p) \neq 0} \frac{1}{p} \| \log h(p) \|^2$$

have both been proven convergent.

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

Moreover, according to hypothesis H there is an unbounded sequence of x-values, $x_1 < x_2 < \dots$ say, so that

$$c_{13}x \leqslant \sum_{n \leqslant x} h(n) \leqslant c_{14}x$$

holds for certain positive constants c_{13}, c_{14}, and all $x = x_k$. This together with our estimate (18) gives

$$\tfrac{1}{2}c_{13} < B(\delta)\exp(\delta(w(x)) < 2c_{14}$$

for $x = x_k$ sufficiently large. This justifies (16) and so the validity of (16), (17) and (18).

We need a further result of the type given as Lemma 6. *We now assume that* $\alpha > 1$.

LEMMA 7. *For each* $\alpha > 1$ *the inequalities*

$$(19) \qquad \sum_{p^m \leqslant x} \sum_{p, m \geqslant 2} p^{m(\alpha-1)} \Big| \sum_{n \leqslant x;\, p^m \| n} a_n \Big|^\alpha \leqslant c_{15} x^{\alpha-1} \sum_{n \leqslant x} |a_n|^\alpha,$$

$$(20) \qquad \sum_{p \leqslant x,\, p \in P} p^{\alpha-1} \Big| \sum_{n \leqslant x;\, p \| n} a_n \Big|^\alpha \leqslant c_{16} x^{\alpha-1} \Big(1 + \sum_{p \leqslant x,\, p \in P} \frac{1}{p}\Big) \sum_{n \leqslant x} |a_n|^\alpha$$

hold uniformly for all real $x \geqslant 2$, *complex numbers* a_n, $1 \leqslant n \leqslant x$, *and sets of primes* P. *The constants* c_{15} *and* c_{16} *depend at most upon* α.

PROOF. See Elliott (1980b).

PROOF OF LEMMA 4. We apply the inequality (19) with $a_n = h(n)$. For a typical prime-power p^m which does not exceed $\sqrt{x}$,

$$(21) \qquad \sum_{n \leqslant x;\, p^m \| n} a_n = h(p^m) \sum_{\substack{s \leqslant p^{-m}x \\ (s,p)=1}} h(s) = h(p^m)\Big\{ \sum_{s \leqslant p^{-m}x} h(s) - \sum_{\substack{s \leqslant p^{-m}x \\ p|s}} h(s) \Big\}.$$

Our estimate (18) with $\delta = 1$, $B = B(1)$, making use of (17), gives

$$\sum_{s \leqslant p^{-m}x} h(s) = (1 + o(1))\, Bp^{-m} x \exp(w(x)).$$

Moreover, by Hölder's inequality

$$\sum_{\substack{s \leqslant p^{-m}x \\ p|s}} h(s) = O\Big(\Big(\sum_{\substack{s \leqslant p^{-m}x \\ p|s}} 1\Big)^{1-(1/\alpha)} \Big(\sum_{s \leqslant p^{-m}x} h(s)^\alpha\Big)^{1/\alpha}\Big)$$
$$= O(xp^{-m-1+(1/\alpha)}).$$

If we restrict x to a sequence $x_1 < x_2 < \dots$ for which $|w(x)| \leqslant a$, say (see (16)), and if p is sufficiently large, say $p > p_0$, then

$$\sum_{n \leqslant x;\, p^m \| n} a_n > \tfrac{1}{2} B \exp(-a)\, h(p^m)\, p^{-m} x.$$

It follows from (19) and the fact that g belongs to the class L^α that

$$\sum_{p^m \leqslant \sqrt{x};\, p, m \geqslant 2} p^{-m} h(p^m)^\alpha \leqslant c_{16} < \infty$$

uniformly for $x = (x_k)$ sufficiently large.

This establishes the first inequality of Lemma 4 insofar as it deals with primes which exceed p_0. To obtain the full inequality we prove that for each *fixed* prime $p \leqslant p_0$ the series

$$\sum_{m=2}^{\infty} p^{-m} h(p^m)^\alpha \tag{22}$$

converges.

Let p be such a prime. If m_0 is chosen suitably the density of those integers which are exactly divisible by p^m with $m > m_0$ is

$$\left(1 - \frac{1}{p}\right) \sum_{m > m_0} p^{-m} \leqslant p^{-m_0}.$$

If t_j denotes a typical such integer, then (once again by Hölder's inequality),

$$\sum_{t_j \leqslant x} h(t_j)^\beta \leqslant c_{18} (p^{-m_0} x)^\mu \Big(\sum_{t_j \leqslant x} h(t_j)^\alpha \Big)^{1/\alpha} < c_{19}\, p^{-m_0 \mu} x,$$

where $\mu = 1 - (\beta/\alpha) > 0$. If we fix m_0 at a sufficiently large value then the last coefficient of x will not exceed $c/2$, where

$$c = \limsup_{x \to \infty} x^{-1} \sum_{n \leqslant x} h(n)^\beta > 0.$$

Hence, if $'$ denotes that summation is confined to those integers n which are divisible only by prime-powers p^m with $m \leqslant m_0$, then

$$\limsup_{x \to \infty} x^{-1} \sum_{n \leqslant x}{}' h(n)^\beta \geqslant c/2 > 0. \tag{23}$$

Let $\sum^{(m)}$ denote summation over those integers n which are exactly divisible by p^m. It follows from (23) that

$$\sum_{m=0}^{m_0} \limsup_{x \to \infty} x^{-1} \sum_{n \leqslant x}{}^{(m)} h(n)^\beta > 0$$

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

and hence that for at least one value of m, say ω,

$$\limsup_{x\to\infty} x^{-1} \sum_{n\leqslant x}^{(\omega)} h(n)^{\beta} = c_{20} > 0. \tag{24}$$

Note that for each n which is counted in $\sum^{(\omega)}$, $h(n) = h(p^{\omega})h(p^{-\omega}n)$, so that $h(p^{\omega})$ cannot be zero.

Now let $''$ denote summation over those integers n which are prime to p. Then from (24)

$$\limsup_{x\to\infty} x^{-1} \sum_{n\leqslant x}'' h(n)^{\beta} \geqslant c_{20} p^{-\omega} h(p^{\omega})^{-1} > 0. \tag{25}$$

We define a multiplicative function $k(n)$ by $k(n) = h(n)$ $(= |g(n)|)$ if $(n,p) = 1$, and $k(n) = 0$ otherwise. From (25) and the fact that $k(n) \leqslant |g(n)|$ we see that $k(n)$ also satisfies the hypothesis H. We may therefore obtain analogues of the estimates (18) with (16) and (17) for it. In particular,

$$\sum_{n\leqslant x} k(n) = (1+o(1))\, Dx \exp(\eta(x)), \quad x\to\infty,$$

holds for some function $\eta(x)$ which satisfies the analogue of (17). Here D is non-zero. Expressed another way

$$\sum_{n\leqslant x,\,(n,p)=1} h(n) = (1+o(1))\, D \exp(\eta(x)), \quad x\to\infty.$$

Returning to (21), where p is now our particular fixed prime and $p^m \leqslant \sqrt{x}$, we obtain

$$\sum_{n\leqslant x;\; p^m \| n} a_n = h(p^m)(1+o(1))\, Dp^{-m} x \exp(\eta(x)) > c_{21} h(p^m) p^{-m} x,$$

provided that x belongs to a suitable (unbounded) sequence. Another application of the inequality (19) of Lemma 7 and we obtain the convergence of the series (22).

This completes the proof of the first part of Lemma 4.

To obtain the second part we apply inequality (20) of Lemma 7 with $a_n = h(n)$. Let P be the set of primes for which $|h(p)-1| > \frac{1}{2}$. According to (8) of Lemma 1, the series

$$\sum_{p\in P} \frac{1}{p} \tag{26}$$

converges. Thus we obtain

$$\sum_{p\leqslant x,\, p\in P} p^{\alpha-1} \left| \sum_{n\leqslant x;\; p\| n} h(n) \right|^{\alpha} \leqslant c_{22} x, \quad x \geqslant 2.$$

Here

$$\sum_{n\leqslant x;\, p\|n} h(n) = h(p) \sum_{\substack{s\leqslant p^{-1}x \\ (s,p)=1}} h(s) \geqslant h(p)\tfrac{1}{2}B\exp(-a)p^{-1}x,$$

arguing as in the earlier (uniform) treatment of prime-powers p^m with $p > p_0$. Then

$$\sum_{\substack{p_0<p\leqslant x \\ p\in P}} p^{-1}h(p)^\alpha \leqslant c_{23}$$

for an unbounded sequence of x-values. Since $|h(p)-1|^\alpha = O(h(p)^\alpha + 1)$ and the series (26) converges anyway, we obtain the convergence of

$$\sum_{p\in P} p^{-1}|h(p)-1|^\alpha.$$

The truth of Lemma 4 now follows from this result and that at (8) of Lemma 1.

4. Proof of Theorem 4

Define the multiplicative function $g(n) = |\tau(n)n^{-11/2}|$. An integration by parts allows us to deduce from Hardy's upper bound estimate that

$$\sum_{n\leqslant x} g(n)^2 = O(x), \quad x \geqslant 2,$$

so that g belongs to the class L^2. We shall not need Hardy's lower bound. We deduce from Lemma 4 that

either (H fails and)

$$x^{-1}\sum_{n\leqslant x} g(n)^\beta \to 0, \quad x\to\infty$$

for each fixed β, $0<\beta<2$

or (H holds and) *the series*

$$\sum_p p^{-1}(g(p)-1)^2$$

converges.

If the first possibility holds then (with $\beta = 1$) there is a non-increasing function $\mu(x)$ so that $\mu(x)\to 0$ as $x\to\infty$ and

$$\sum_{n\leqslant x} g(n) \leqslant x\mu(x)^2, \quad x \geqslant 2.$$

It follows easily that those integers for which $g(n) \geqslant \mu(n)$ holds have density zero.

Suppose now that the second possibility holds. Then from an application of the Cauchy–Schwarz inequality

$$\sum_{r<p\leqslant x} p^{-1}|g(p)-1|\log p \leqslant c_{24}\Big(\sum_{p>r} p^{-1}|g(p)-1|^2 \sum_{p\leqslant x} p^{-1}(\log p)^2\Big)^{\frac{1}{2}}$$

$$\leqslant c_{25}\,\varepsilon(r)\log x,$$

uniformly for $2\leqslant r\leqslant x$, where

$$\varepsilon(r)^2 = \sum_{p>r} p^{-1}|g(p)-1|^2 \to 0, \quad r\to\infty.$$

Moreover, for each fixed $r\geqslant 2$

$$\frac{1}{\log x}\sum_{p\leqslant r} p^{-1}|g(p)-1|\log p\to 0, \quad x\to\infty.$$

Hence

$$\limsup_{x\to\infty}\frac{1}{\log x}\sum_{p\leqslant x} p^{-1}|g(p)-1|\log p\leqslant c_{25}\,\varepsilon(r)$$

for every $r\geqslant 2$, so that the limit must be zero. There is then a decreasing function $\lambda(x)$, which approaches zero as x becomes unbounded, so that

$$\sum_{p\leqslant x} p^{-1}|g(p)-1|\log p\leqslant \lambda(x)\log x, \quad x\geqslant 2.$$

It is now clear that $|g(p)-1|\leqslant \lambda(p)^{\frac{1}{2}}\to 0$ as $p\to\infty$ save possibly on a set of primes of density zero.

This completes the proof of Theorem 4.

5. Proof of Theorem 1, necessity

Let $g(n)$, possibly complex-valued, have a non-zero mean-value A. Let $h(n)$ be the multiplicative function $|g(n)|$. This function also belongs to the class L^α. Moreover,

$$\sum_{n\leqslant x} h(n)\geqslant \Big|\sum_{n\leqslant x} g(n)\Big|\geqslant x|A|/2$$

for all x sufficiently large, so that $h(n)$ satisfies the hypothesis H (with $\beta=1$). For it the results of Lemma 1 and Lemma 4 are therefore valid. This gives at once the convergence of the third and fourth series at (2).

Consider the Dirichlet series

$$G(s)=\sum_{n=1}^{\infty} g(n)n^{-s}, \quad s=\sigma+it, \quad \sigma>1.$$

Since g is of class L^{α}, $\alpha > 1$, $G(s)$ is defined, and an analytic function of s in the half-plane $\sigma > 1$. It has in that half-plane an Euler product representation

$$\prod_p \Big(1+\sum_{m=1}^{\infty} p^{-ms} g(p^m)\Big) = \prod_p (1+\kappa(p)),$$

say. Since the series

$$\sum_{p,m\geqslant 2} p^{-m}|g(p^m)|, \quad \sum_{|h(p)-1|>\delta} p^{-1}|g(p)|$$

converge, we see that if $\delta \leqslant \frac{1}{2}$ and v is sufficiently large (in terms of δ) then $|\kappa(p)| \leqslant \frac{7}{8}$ for all $p > v$. Moreover,

$$\prod_{p>v}(1+|\kappa(p)|) \leqslant \exp\Big(\sum_{p>v}|\kappa(p)|\Big) \leqslant \exp\Big(\sum_{|h(p)-1|\leqslant\delta} h(p)p^{-\sigma} + c_{26}\Big)$$

$$\leqslant \exp\big((1+\delta)\sum p^{-\sigma} + c_{26}\big) \leqslant c_{27}\,\zeta(\sigma)^{1+\delta} \leqslant c_{28}(\sigma-1)^{-1-\delta}$$

uniformly for $1 < \sigma < 2$. Here, as usual, we denote by $\zeta(s)$ the Riemann zeta function

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.$$

Suppose now that one of the terms

$$1+\sum_{m=1}^{\infty} p^{-m} g(p^m)$$

has the value zero. Then $p \leqslant v$ must hold and $1+\kappa(p) = O((\sigma-1))$ as $s = \sigma \to 1+$. Here we have made use of the readily established fact that the function

$$1+\sum_{m=1}^{\infty} p^{-ms} g(p^m)$$

is analytic in the half-plane $\sigma > \frac{1}{\alpha}$ (apply Hölder's inequality again). Thus, as $\sigma \to 1+$

$$|G(\sigma)| \leqslant c_{29}(\sigma-1)^{-\delta}$$

where $0 < \delta \leqslant \frac{1}{2}$. However, since $g(n)$ has a non-zero mean-value

$$G(s) = s\int_1^{\infty} y^{-s-1} \sum_{n\leqslant y} g(n)\, dy \sim A(s-1)^{-1}$$

as $s \to 1$, $\sigma > 1$, and a contradiction is obtained.

This establishes the validity of the condition (3).

Moreover, as $s = \sigma \to 1+$,

$$\prod_{p \leqslant v} (1 + \kappa(p)) \to c_{30} \neq 0,$$

$$\prod_{\substack{p > v \\ |h(p)-1| > \frac{1}{2}}} (1 + \kappa(p)) \to c_{31} \neq 0,$$

and

$$G(\sigma) \sim c_{32} \exp\Big(\sum_{\substack{p > v \\ |h(p)-1| \leqslant \frac{1}{2}}} p^{-\sigma} g(p)\Big).$$

We have already shown that

$$G(\sigma) \sim A(\sigma - 1)^{-1}$$

in the same circumstances, and so may assert that

$$\sum_{|h(p)-1| \leqslant \frac{1}{2}} p^{-\sigma}\{g(p) - 1\} \to D, \quad \sigma \to 1+ \tag{27}$$

for some finite number D.

Let

$$\theta(x) = \sum_{\substack{p \leqslant \exp(x) \\ |h(p)-1| \leqslant \frac{1}{2}}} p^{-1}\{g(p) - 1\}.$$

Then we may write our condition (27) in the form

$$\int_1^\infty x^{-(\sigma-1)}\, d\theta(\log x) \to D, \quad \sigma \to 1+,$$

and, setting $x = e^y$, $u = \sigma - 1$,

$$\int_0^\infty e^{-yu}\, d\theta(y) \to D, \quad u \to 0+.$$

Note that if $x \leqslant y$, $x \to \infty$, $y/x \to 1$, then

$$|\theta(y) - \theta(x)| \leqslant \tfrac{5}{2} \sum_{\exp(x) < p \leqslant \exp(y)} p^{-1} = \tfrac{5}{2}\{\log(y/x) + O(x^{-1})\} = o(1).$$

By applying the Hardy–Littlewood Tauberian theorem (see Hardy (1949), or Elliott (1980a), Chapter 2) to the functions $\operatorname{Re}\theta(y)$ and $\operatorname{Im}\theta(y)$ in turn we deduce that

$$\lim_{y \to \infty} \theta(y) = D.$$

From this result we readily obtain the convergence of the first series at (2).

Finally,

$$|g(p) - 1|^2 = (h(p) - 1)^2 + 2(h(p) - 1) - 2(\operatorname{Re} g(p) - 1).$$

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

The series

$$\sum_{|h(p)-1|\leqslant\frac{1}{2}} p^{-1}(h(p)-1)^2, \quad \sum_{|h(p)-1|\leqslant\frac{1}{2}} p^{-1}\{\operatorname{Re} g(p)-1\}$$

are convergent (in particular, see Lemma 1), and the partial sums

$$\sum_{\substack{p\leqslant x\\ |h(p)-1|\leqslant\frac{1}{2}}} p^{-1}\{h(p)-1\}$$

are bounded uniformly for $x\geqslant 2$. (See Lemma 1 and Lemma 3 (12) upon which it rests.) Hence, for some constant c_{33} and all $x\geqslant 2$

$$\sum_{\substack{p\leqslant x\\ |h(p)-1|\leqslant\frac{1}{2}}} p^{-1}|g(p)-1|^2\leqslant c_{33}.$$

Letting $x\to\infty$ we obtain the convergence of the second (and so of all) the series at (2).

This completes the proof that the convergence of the series (2) and the condition (3) in the statement of Theorem 1 are necessary in order that g should belong to the class L^α and have a non-zero mean-value (1).

6. Proof of Theorem 1, sufficiency

We do not give a detailed proof here. One may readily modify the treatment given by the author (Elliott (1975)) for the case $\alpha=2$. As remarked earlier, the conditions at (2) and (3) guarantee both that g belongs to the class L^α and that it possesses a non-zero mean-value.

7. Proof of Theorem 2, necessity

Assume that g is of class L^α, $\alpha>1$ and has a zero mean-value. We shall further assume that conditions (i), (ii) and (iii) fail, and prove that (iv) must then hold. Accordingly we may assume the convergence of the series

$$\sum_{||g(p)|-1|\leqslant\frac{1}{2}} p^{-1}|1-|g(p)||^2, \quad \sum_{||g(p)|-1|>\frac{1}{2}} p^{-1}|1-|g(p)||^\alpha \tag{28}$$

and of

$$\sum_p p^{-1}(|g(p)|-\operatorname{Re} g(p)p^{-it})$$

for some real t. Moreover, for this value of t and each prime p

$$\sum_{m=1}^{\infty} g(p^m)p^{-mit}\neq -1. \tag{29}$$

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

LEMMA 8. *Let $g(n)$ be a multiplicative function for which the series*

$$(30)\qquad \sum_{|g(p)-1|\leq\frac{1}{2}} \frac{|g(p)-1|^2}{p}, \quad \sum_{|g(p)-1|>\frac{1}{2}} \frac{|g(p)-1|^\alpha}{p}, \quad \sum_{p,m\geq 2} \frac{|g(p^m)|}{p^m}$$

converge, $\alpha > 1$ fixed. Let the series

$$(31)\qquad \sum_p p^{-1}(|g(p)|-\operatorname{Re} g(p))$$

converge.

Then

$$(32)\qquad \{x\Lambda(\log x)\}^{-1} \sum_{n\leq x} g(n) \to J, \quad x\to\infty,$$

where

$$(33)\qquad \Lambda(u) = \exp\left(\sum_p p^{-1-1/u}(g(p)-1)\right)$$

is a slowly oscillating function of $\exp(u)$, *and the constant J is given by*

$$(34)\qquad J = \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\dots\right)\exp\left(\frac{1-g(p)}{p}\right).$$

PROOF. A proof of this result when $\alpha = 2$ is indicated in Elliott (1980), Chapter 10. Only slight changes are needed in order to obtain the present result.

In our present circumstances we apply Lemma 8 to the function $g(n)n^{-it}$. Let us temporarily denote this function by $r(n)$.

Since g is of class L^α, an application of Hölder's inequality together with the estimate

$$\sum_{p,m\geq 2;\, p^m\leq x} 1 = O(x^{\frac{1}{2}}/\log x)$$

gives the upper bound

$$\sum_{p^m\leq x} |r(p^m)| = O(x^{1-\nu}), \quad \nu = (\alpha-1)/(2\alpha) > 0.$$

An integration by parts allows one to deduce the convergence of the series

$$\sum_{p,m\geq 2} p^{-m}|r(p^m)|.$$

By means of the identity

$$|r(p)-1|^2 = (|g(p)|-1)^2 + 2(|g(p)|-\operatorname{Re} g(p)p^{-it})$$

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

we obtain the convergence of the series

$$\sum_{||r(p)|-1|\leqslant\frac{1}{2}} p^{-1}|r(p)-1|^2. \tag{35}$$

Note that $||r(p)|-1|\leqslant|r(p)-1|$ so that we have obtained the convergence of the first series at (30).

If $|r(p)-1|>\frac{1}{2}$ but $||r(p)|-1|\leqslant\frac{1}{2}$ then

$$|r(p)-1|^{\alpha}\leqslant 2^{2-\alpha}|r(p)-1|^2.$$

If $|r(p)-1|>\frac{1}{2}$ but $||r(p)|-1|>\frac{1}{2}$ then

$$|r(p)-1|^{\alpha}\leqslant(|r(p)|+1)^{\alpha}\leqslant(|r(p)|-1+2)^{\alpha}\leqslant 5^{\alpha}||r(p)|-1|^{\alpha}.$$

Hence we obtain from (35) and (28) the convergence of all the series at (30). We have at once the condition (31) and the lemma may be applied. We see that in our present circumstances

$$\sum_{n\leqslant x} g(n)n^{-it}=(1+o(1))Jx\Lambda(\log x), \quad x\to\infty,$$

since the condition (29) ensures that J is non-zero.

However, at the outset of this proof we assumed that $g(n)$ had a mean-value zero, and an integration by parts enables us to assert that

$$x^{-1}\sum_{n\leqslant x} g(n)n^{-it}\to 0, \quad x\to\infty,$$

must also hold. For example, for each fixed ε, $0<\varepsilon<1$,

$$\sum_{\varepsilon x<n\leqslant x} g(n)n^{-it}=\Big[y^{-it}\sum_{n\leqslant y} g(n)\Big]_{\varepsilon x}^{x}+it\int_{\varepsilon x}^{x} y^{-it-1}\sum_{n\leqslant y} g(n)\,dy$$

$$=o(x)+it\int_{\varepsilon x}^{x} y^{-it-1}o(y)\,dy=o(x)$$

as $x\to\infty$, whilst

$$\Big|\sum_{n\leqslant\varepsilon x} g(n)n^{-it}\Big|\leqslant\Big(\sum_{n\leqslant\varepsilon x}1\Big)^{1-(1/\alpha)}\Big(\sum_{n\leqslant x}|g(n)|^{\alpha}\Big)^{1/\alpha}=O(\varepsilon^{1-(1/\alpha)}x)$$

since g is of class L^{α}.

We deduce that

$$\Lambda(\log x)\to 0$$

as $x\to\infty$, or put another way

$$\operatorname{Re}\sum_{p} p^{-\sigma}(1-r(p))\to\infty \tag{36}$$

as $\sigma\to 1+$.

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

To set this in a more convenient form we apply the following

LEMMA 9. *Let α be fixed, $\alpha > 1$. Let $\psi(p)$ be complex numbers for which the series*

$$\sum_{|\psi(p)|\leqslant \frac{1}{2}} p^{-1}|\psi(p)|^2, \quad \sum_{|\psi(p)|>\frac{1}{2}} p^{-1}|\psi(p)|^\alpha$$

converge. Define the function

$$\alpha(u) = \sum_{p\leqslant \exp(u)} p^{-1}\psi(p).$$

Then there is a constant c so that

$$\alpha(\beta^{-1}) = \sum_p p^{-1-\beta}\psi(p) + R(\beta),$$

where

$$|R(\beta)| \leqslant \min\left(c, \int_0^\infty e^{-y}\left|\alpha\left(\frac{1}{\beta}\right)-\alpha\left(\frac{y}{\beta}\right)\right|dy\right)$$

for $0<\beta\leqslant \frac{1}{2}$.

REMARK. Compare with Deboussi and Delange (1976).

PROOF. After an integration by parts

$$\sum_p p^{-1-\beta}\psi(p) = \int_0^\infty e^{-\beta u}\,d\alpha(u) = \beta\int_0^\infty e^{-\beta u}\alpha(u)\,du.$$

With the change of variables $\beta u = y$ this last integral becomes

$$\int_0^\infty e^{-y}\alpha\left(\frac{y}{\beta}\right)dy.$$

Hence we obtain the representation

$$R(\beta) = \alpha\left(\frac{1}{\beta}\right) - \int_0^\infty e^{-y}\alpha\left(\frac{y}{\beta}\right)dy = \int_0^\infty e^{-y}\left\{\alpha\left(\frac{1}{\beta}\right)-\alpha\left(\frac{y}{\beta}\right)\right\}dy.$$

Consider first the range $y \geqslant \max(1, \beta\log 2)$. We write

$$\alpha\left(\frac{y}{\beta}\right)-\alpha\left(\frac{1}{\beta}\right) = \sum\nolimits_1 p^{-1}\psi(p) + \sum\nolimits_2 p^{-1}\psi(p),$$

where the first sum is over those primes p, $\exp(1/\beta) < p \leqslant \exp(y/\beta)$, say, for which $|\psi(p)| \leqslant \frac{1}{2}$, and the second sum is over the remaining primes in this same interval.

From the Cauchy–Schwarz inequality

$$\Sigma_1 p^{-1}|\psi(p)| \leqslant (\Sigma_1 p^{-1}|\psi(p)|^2 \Sigma_1 p^{-1})^{\frac{1}{2}} = O\left(\left(\left|\log\left(\frac{y/\beta}{1/\beta}\right)\right|+1\right)^{\frac{1}{2}}\right)$$

and by means of Hölder's inequality

$$\Sigma_2 p^{-1}|\psi(p)| \leqslant (\Sigma_2 p^{-1}|\psi(p)|^{\alpha})^{1/\alpha}(\Sigma_2 p^{-1})^{1-(1/\alpha)} = O((|\log y|+1)^{1-(1/\alpha)})$$

so that

$$\left|\alpha\left(\frac{y}{\beta}\right)-\alpha\left(\frac{1}{\beta}\right)\right| \leqslant c_{34}(\log y+1).$$

If $y<\max(1,\beta\log 2)$ we argue similarly to obtain the bound

$$\left|\alpha\left(\frac{y}{\beta}\right)-\alpha\left(\frac{1}{\beta}\right)\right| \leqslant c_{34}\left(\log\frac{1}{\beta}+1\right).$$

Notice that $\alpha(y/\beta)=0$ if $y<\beta\log 2$.

Hence

$$|R(\beta)| \leqslant c_{35}\int_0^{\beta\log 2}\left(\log\frac{1}{\beta}+1\right)dy + c_{35}\int_{\beta\log 2}^{\infty} e^{-y}(|\log y|+1)\,dy \leqslant c$$

and the lemma is proved.

In our present circumstances we have $\psi(p)=1-r(p)$, and

$$\sum_p p^{-1-\beta}(1-r(p)) = \sum_{p\leqslant\exp(1/\beta)} p^{-1}(1-r(p))+O(1)$$

uniformly for $0<\beta\leqslant\frac{1}{2}$. From (36) we obtain at once the validity of condition (iv) of Theorem 2.

This completes the considerations of this section.

8. Proof of Theorem 2, sufficiency

Let g belong to class L^{α}, $\alpha>1$, and let one of the conditions (i)–(iv) in the statement of Theorem 2 be satisfied. We wish to prove that g has a zero mean-value.

Suppose that it does not. Then

$$\limsup_{x\to\infty} x^{-1}\left|\sum_{n\leqslant x} g(n)\right|>0$$

and in particular both $g(n)$ and $|g(n)|$ satisfy hypothesis H with $\beta=1$. From an

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

application of Lemma 4 we obtain the convergence of

$$\sum_{p,m\geqslant 2} p^{-m}|g(p^m)|^{\alpha},$$

and from Lemma 1 and Lemma 4 together the convergence of both the series at (4). Hence condition (i) of Theorem 2 must fail.

With $h(n) = |g(n)|$ we have the estimate (18), where

$$w(x) = \sum_{\substack{p\leqslant x\\ h(p)\neq 0}} p^{-1}\|\log h(p)\|.$$

In particular (setting $\delta = 1$), since $h(n)$ belongs to L^{α}, $\alpha > 1$, there is a constant c_{36} so that

$$\sum_{\substack{p\leqslant x\\ h(p)\neq 0}} p^{-1}\|\log h(p)\| \leqslant c_{36}, \quad x\geqslant 2.$$

When $|h(p)-1|\leqslant \frac{1}{2}$ we apply the estimate

$$\log h(p) = h(p)-1+O((h(p)-1)^2)$$

(compare with the arguments in Section 2). In view of the convergence of the series at (4) we obtain

$$\sum_{p\leqslant x} p^{-1}(h(p)-1)\leqslant c_{37}, \quad x\geqslant 2.$$

To put this into a more useful form we integrate by parts, and obtain

$$\sum_{p} p^{-\sigma}(h(p)-1)\leqslant 2c_{37}, \quad 1<\sigma<2. \tag{37}$$

Alternatively, this may be deduced from an application of Lemma 9.

We now seek an analogue of Lemma 8, valid when the series

$$\sum_{p} p^{-1}(|g(p)| - \operatorname{Re} g(p)p^{-it})$$

diverges for each real t. We sketch the procedure.

Beginning with the representation

$$\sum_{n\leqslant x} g(n)\log n \log\frac{x}{n} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{G'}{G}(s)\frac{x^s}{s^2}\,ds, \tag{38}$$

where the integration is along the line $\operatorname{Re}(s) = \sigma > 1$ in the complex s-plane, where $s = \sigma + it$ we set $\sigma = 1+(1/\log x)$ and show that

$$\int_{|t|>M}\frac{G'}{G}(s)\frac{x^s}{s^2}\,ds = O(M^{-1}x\log x).$$

This requires straightforward modification of the argument given in Elliott (1980a) Chapter 10, the essential preliminary results being available in Halász (1968). Define

$$\Delta = \sup_{|t|\leqslant M} |F(\sigma+it)|\,\zeta(\sigma)^{-1}.$$

Then it is not difficult to show that the range $|t|\leqslant M$ of the integral in (38) contributes an amount which is

$$O(\Delta^{\rho}\, x\log x)$$

for a certain positive constant ρ.

Suppose for the moment that $\Delta\to 0$ as $x\to\infty$ (and so $\sigma\to 1+$). Then dividing both sides of the equation in (38) by $x\log x$, letting $x\to\infty$ and then $M\to\infty$ would lead to the estimate

$$\sum_{n\leqslant x} g(n)\log n\log\frac{x}{n} = o(x\log x), \quad x\to\infty. \tag{39}$$

Let us consider Δ more closely. Employing Euler products and taking advantage of the convergence of the series (4) we readily obtain an upper bound

$$|F(\sigma+it)|\,\zeta(\sigma)^{-1}\leqslant c_{38}\exp\Big(-\sum_p p^{-\sigma}(1-\operatorname{Re} g(p)p^{-it})\Big).$$

Moreover, the sum in the exponential has the alternative representation

$$-\sum_p p^{-\sigma}(|g(p)|-\operatorname{Re} g(p)p^{-it})+\sum_p p^{-\sigma}(|g(p)|-1).$$

According to our temporary hypothesis that (ii) is valid, the first of these series becomes negatively unbounded as $\sigma\to 1+$. However, from (37) the second sum does not exceed $2c_{37}$, $1<\sigma<2$. Hence for each real t

$$|F(\sigma+it)|\,\zeta(\sigma)^{-1}\to 0, \quad \sigma\to 1+$$

and $\Delta\to 0$ as $\sigma\to 1+$. It follows that (39) is indeed valid.

Let ε be temporarily fixed, $0<\varepsilon<1$. We apply the estimate (39) with x, and x replaced by $x(1+\varepsilon)$, and by subtraction obtain

$$\sum_{n\leqslant x} g(n)\log n\log(1+\varepsilon) = o(x)+O\Bigg(\sum_{x<n\leqslant x(1+\varepsilon)} |g(n)|\log n\log\frac{x(1+\varepsilon)}{n}\Bigg). \tag{40}$$

The sum which appears in the error term on the right-hand side of this equation may be estimated by means of Hölder's inequality to be

$$O\Big(\varepsilon\log x\sum_{x<n\leqslant x(1+\varepsilon)}|g(n)|\Big) = O\Big(\varepsilon\log x\,\varepsilon^{\mu}\sum_{n\leqslant x(1+\varepsilon)}|g(n)|^{\alpha}\Big) = O(\varepsilon^{1+\mu}\,x\log x),$$

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

where $\mu = 1-(1/\alpha) > 0$. Since $\log(1+\varepsilon) = \varepsilon + O(\varepsilon^2)$ we obtain the estimate (dividing both sides of (40) by ε)

$$\sum_{n\leqslant x} g(n)\log n = \{o(1)+O(\varepsilon^{\mu})\}\, x\log x, \quad x\to\infty.$$

However, ε may be chosen arbitrarily small, and therefore

$$\sum_{n\leqslant x} g(n)\log n = o(x\log x), \quad x\to\infty.$$

Integrating by parts we see that $g(n)$ has the mean value zero.

But this contradicts the assumption made at the beginning of this section. Hence condition (ii) fails and there is a real value of t for which the series (5) converges.

Consider next the possibility that (i) and (ii) fail, but (iii) does not. We may argue as in Section 7 and establish an estimate

$$\{x\Lambda(\log x)\}^{-1}\sum_{n\leqslant x} g(n)\, n^{-it} \to J, \quad x\to\infty,$$

where $g(n)$ is everywhere to be replaced by $g(n)\, n^{-it}$. Here J has a factor

$$1+\sum_{m=1}^{\infty} p^{-m} g(p^m)\, p^{-mit}$$

which presently has the value zero. Hence $J = 0$. Moreover, with $\sigma = 1+(1/\log x)$

$$(41) \qquad |\Lambda(\log x)| = \exp\Big(-\sum_p p^{-\sigma}(1-\operatorname{Re} g(p)p^{-it})\Big)$$
$$= \exp\Big(-\sum_p p^{-\sigma}(|g(p)|-\operatorname{Re} g(p)p^{-it})+\sum_p p^{-\sigma}(|g(p)|-1)\Big),$$

which is bounded above uniformly for $x > e$. Hence $g(n)\, n^{-it}$ and so $g(n)$ has the mean-value zero. We have reached a contradiction. Therefore condition (iii) must also fail.

We are now only left with condition (iv). Since conditions (i)–(iii) all fail there is a value of t so that

$$\{x\Lambda(\log x)\}^{-1}\sum_{n\leqslant x} g(n)\, n^{-it} \to J, \quad x\to\infty,$$

where J is non-zero. In this case (see (41))

$$|\Lambda(\log x)| \leqslant c_{39}\exp\Big(-\sum_p p^{-\sigma}(1-|g(p)|)\Big), \quad \sigma = 1+(1/\log x).$$

We apply Lemma 9 and deduce from condition (iv) that as σ approaches 1 from above the sum in the exponent in this last inequality becomes unbounded, and

$\Lambda(\log x)$ approaches zero. Hence $g(n)n^{-it}$ and so $g(n)$ has mean-value zero. This again contradicts our initial assumption.

Since in every case our initial assumption led to a contradiction it must have been false, that is to say, $g(n)$ has a mean-value zero.

This completes the proof of Theorem 2.

An interesting feature of the proof of this part of Theorem 2 is that we do not seem to be able to deduce directly from (i) that $g(n)$ has a mean-value zero.

9. Proof of Theorem 3

This is now easy. Either $g(n)$ has a mean-value zero or it satisfies hypothesis H. In the first case we set $A = 0$, $S(x) = 1$. In the second case either the series (5) diverges for each real t and g has again the mean-value zero, or there is a real value of t so that the function $g(n)n^{-it}$ meets the hypotheses of Lemma 8. Then (see Section 7)

$$\sum_{n\leqslant x} g(n)n^{-it} = (J+o(1))x\,\Lambda(\log x), \quad x\to\infty,$$

for some (possibly zero) constant J. Here we set $S(x) = \Lambda(\log x)$, and so define a slowly oscillating function of x. An integration by parts, treating ranges $0<n\leqslant \varepsilon x$, $\varepsilon x<n\leqslant x$ separately, leads to the estimate

$$\sum_{n\leqslant x} g(n) = (J+o(1))x^{1+it}(1+it)^{-1}S(x), \quad x\to\infty.$$

We define

$$A = J(1+it)^{-1}$$

and the proof of Theorem 3 is complete.

Concluding remarks

Theorems 1 and 2 are not quite the same in form. One can reformulate the result of Theorem 1 so as to give necessary and sufficient conditions in order that an arbitrary multiplicative function simultaneously belong to some class L^α with $\alpha>1$ and have a non-zero mean-value. In Theorem 2 one postulates at the outset that g belongs to some class L^α and has a zero mean-value and deduces the validity of one of the conditions (i)–(iv). In the other direction the results assert that *for functions of class* L^α any one of these conditions is also sufficient.

What are the necessary and sufficient conditions for a multiplicative function to belong to the class L^α, for $\alpha>1$; for $0<\alpha\leqslant 1$?

Note that the function $\tau(n)n^{-11/2}$ belongs to L^2 so that necessary and sufficient conditions in order for it to have a mean-value zero may be read off from Theorem 2. Since it is known to have a mean-value zero one of these conditions must be satisfied. Which?

The methods of the present paper enable the following result to be established: *Let $g(n)$ be a non-negative multiplicative arithmetic function and let $g^2(n)$ have a non-zero mean-value. Then in order for $g(n)$ not to have a zero mean-value it is both necessary and sufficient that the series*

$$\sum_p p^{-1}(g(p)-1)^2$$

converges.

Assuming that g does not have a mean-value zero the essential point is to deduce from the existence of the non-zero mean-value for g^2 that the series

$$\sum_p p^{-1}(g(p)^2-1)$$

converges. One then applies Theorem 2.

We proved earlier that *either*

$$x^{-1}\sum_{n\leqslant x}|\tau(n)|n^{-11/2}\to 0, \quad x\to\infty,$$

or the series

$$\sum_p \frac{1}{p}\left(\frac{|\tau(p)|}{p^{11/2}}-1\right)^2$$

converges. In view of Rankin's asymptotic estimate (See section 1) the above remark shows that both of these conditions cannot hold. Which does?

References

F. V. Atkinson and Lord Cherwell (1949), 'The mean-values of arithmetical functions', *Quart. J. Math. Oxford Ser.* **20**, 65–79.

H. Daboussi and H. Delange (1976), 'On a theorem of P. D. T. A. Elliott on multiplicative functions', *J. London Math. Soc.* **14**, 345–356.

H. Delange (1961), 'Un théorème sur les fonctions arithmétiques multiplicatives et ses applications', *Ann. Sci. Éc. Norm. Sup.*, *3e série*, **78**, 1–29.

P. G. L. Dirichlet (1849), 'Über die Bestimmung der mittleren Werthe in der Zahlentheorie', *Abh. Akad. Berlin*, 69–83.

P. D. T. A. Elliott (1975), 'A mean-value theorem for multiplicative functions', *Proc. Lond. Math. Soc.* (3) **31**, 418–438.

P. D. T. A. Elliott (1980a), *Probabilistic number theory* (Springer, Heidelberg, New York).

P. D. T. A. Elliott (1980b), 'High-power analogues of the Turán–Kubilius inequality, and an application to number theory', *Canad. J. Math.*, to appear.

https://doi.org/10.1017/S1446788700021182 Published online by Cambridge University Press

P. Erdös (1946), 'On the distribution function of additive functions', *Ann. of Math.* **47**, 1–20.
C. F. Gauss (1801), *Disquitiones arithmeticae*, §5, Lipsiae, apud Gerh. Fleischer Iun., 1801 = *Werke, Band 1*, Göttingen, 1870, 362–366.
G. Halász (1968), 'Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen', *Acta Math. Acad. Sci. Hungar.* **19**, 365–403.
G. H. Hardy (1927), 'Note on Ramanujan's arithmetic function $\tau(n)$', *Math. Proc. Cambridge Philos. Soc.* **23**, 675–680.
G. H. Hardy (1940), *Ramanujan. Twelve lectures on subjects suggested by his life and work* (Cambridge).
G. H. Hardy (1949), *Divergent series* (Oxford).
G. H. Hardy and E. M. Wright (1960), *An introduction to the theory of numbers*, 4th ed. (Oxford).
J. Kubilius (1962), *Probabilistic methods in the theory of numbers*, Amer. Math. Soc. Transl. of Math. Monographs no. 11 (Providence, 1964).
B. V. Levin and A. S. Fainleib (1967), 'Applications of some integral equations to problems in number theory', *Uspekhi Mat. Nauk* 22 (3) **135**, 119–198 = *Russian Math. Surveys* **22** (3), May–June 1967, 119–204.
B. V. Levin, N. M. Timofeev and S. T. Tuliagonov (1973), 'The distribution of the values of multiplicative functions', *Liet. Mat. Rinkinys = Lit. Mat. Sb.* **13** (1), 87–100.
L. J. Mordell (1917), 'On Mr. Ramanujan's empirical expansions of modular functions', *Math. Proc. Cambridge Philos. Soc.* **19**, 117–124.
R. A. Rankin (1934), 'Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions', *Math. Proc. Cambridge Philos. Soc.* **35**, 357–372.
C. Ryavec (1970), 'A characterization of finitely distributed additive functions', *J. Number Theory* **2**, 393–403.
E. Wirsing (1961), 'Das asymptotische Verhalten von Summen über multiplikative Funktionen', *Math. Ann.* **143**, 75–102.
E. Wirsing (1967), 'Das asymptotische Verhalten von Summen über multiplikative Funktionen II', *Acta Math. Acad. Sci. Hungar.* **18**, 411–467.

University of Colorado
Boulder, Colorado,
U.S.A.