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Cohomology of chain recurrent sets

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Abstract. Let φ be a flow on a compact metric space Λ and let $p \in \Lambda$ be chain recurrent. We show that $\check{H}^1(\Lambda; \mathbb{R}) \neq 0$ if $\dim_p \Lambda = 1$ or if p belongs to a section of φ. Applications to planar flows and to smooth flows are given.

Introduction

The classical Poincaré–Bendixson Theorem states that for a flow in the plane, any compact limit set Λ which contains no stationary point must be a periodic orbit. As a consequence, Λ separates the plane. This has been generalized to compact limit sets Λ of planar flows containing only finitely many stationary points; such a set is a connected graph (in the combinatorial sense) of which every edge belongs to a loop. See Hartman [**6**].

It is easy to construct a non-trivial compact limit set L of a planar flow φ which does not separate the plane. For example, L can be an interval of stationary points toward which the φ-trajectories spiral. But experimentation suggests that if a planar limit set Λ is not entirely stationary then it does separate. In fact this is true and the original motivation of our paper was to prove it.

It turns out that the only properties of Λ we need are

(1) Λ is 1-dimensional, φ-invariant, compact;

(2) Λ has a local section at any non-stationary point;

(3) points of Λ are chain recurrent.

These properties are valid for any compact non-stationary planar limit set and they are easily seen to imply

(4) Λ has a section E containing a chain recurrent point.

To say that E is a *section* of the flow φ on Λ means that $E \subset \Lambda$ is a compact set having a flowbox neighborhood $N \subset \Lambda$; i.e. the flow $\varphi : \mathbb{R} \times \Lambda \to \Lambda$ maps $[-\sigma, \sigma] \times E$ homeomorphically onto N and E lies in the *interior* of N in Λ. In the smooth case, a section is given by a closed, codimension one submanifold E to which the flow is transverse. The flow trajectories need not return to E.

Our main result, Theorem 1.1, says that

$$\text{condition (4) implies } \check{H}^1(\Lambda; \mathbb{R}) \neq 0$$

where $\check{H}$ denotes Alexander–Spanier cohomology.

By Alexander Duality, it follows that if such a Λ is contained in the plane then it separates. See theorems 1.5, 1.6. In this sense our result generalizes the Poincaré-Bendixson Theorem.

We also consider the local flow φ generated by a smooth vector field F on a manifold M. If Λ is a compact, chain recurrent, φ-invariant subset of M and if Λ meets some smooth, codimension one submanifold of M which is transverse to F then the natural homomorphism $\check{H}^1(M;\mathbb{R})\to\check{H}^1(\Lambda;\mathbb{R})$ is non-trivial. See theorem 1.4.

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1. *The main theorem and its applications*

Below, X will denote a metric space with metric d. A *local flow* φ on X is a continuous map $\varphi:\Omega\to X$, denoted $\varphi(t,x)=\varphi_t x$, such that

Ω is a neighborhood of $\{0\}\times X$ in $\mathbb{R}\times X$;

$\varphi_0(x)=x$ for all $x=X$;

$\varphi_{t+s}(x)=\varphi_t\circ\varphi_s(x)$ whenever the r.h.s. is defined.

If $\Omega=\mathbb{R}\times X$ then φ is a *flow*. It is convenient to use the notation

$$t\cdot x=\varphi_t(x)\quad\text{and}\quad A\cdot B=\varphi(A\times B)$$

where $A\times B\subset\mathbb{R}\times X$.

Let φ be a flow on X and let $T,\varepsilon>0$ be given. A (T,ε)*-pseudo orbit from* $p\in X$ *to* $q\in X$ is a finite, indexed set of partial trajectories of φ

$$t\cdot p_j,\qquad 0\le j\le n,\qquad 0\le t\le t_j,$$

such that $p=p_0$, $q=p_{n+1}$, and for $j=0,1,\dots,n$,

$$d(t_j\cdot p_j,p_{j+1})<\varepsilon\quad\text{and}\quad t_j>T.$$

If for every $T,\varepsilon>0$ there exists a (T,ε)-pseudo orbit from p to q then we write $p<q$. If $p<p$ then p is chain recurrent. If $p<q$ for all p,q in a compact invariant set Λ then Λ is a *chain recurrent set* of the flow.

The set of all chain recurrent points of a flow is closed, invariant, and includes all non-wandering points. For the theory of chain recurrence, see Bowen [**1**], Conley [**3**], Franke & Selgrade [**5**], or Hurley [**8**].

Let φ be a local flow on X. A closed subset $E\subset X$ is called a *local section at* $p\in E$ if there exists $\sigma>0$ such that φ maps $[-\sigma,\sigma]\times E$ homeomorphically onto a closed neighborhood B of p. In this case, $B=[-\sigma,\sigma]\cdot E$, and we call $B=B(\sigma,E)$ a *flowbox.*

It may happen that the flowbox $B=B(\sigma,E)$ is a compact neighborhood of *all* the points of E. We then call E a *section* of the flow and define a map $h:X\to\mathbb{R}/\mathbb{Z}$ as follows. Let $\pi:\mathbb{R}\to\mathbb{R}/\mathbb{Z}$ be the canonical projection onto $\mathbb{R}/\mathbb{Z}=\mathbb{T}$, the circle of unit length, and set

$$h(x)=\begin{cases}\pi(2s/\sigma) & \text{if } x\in s\cdot E \text{ and } 0\le s\le \frac{1}{2}\sigma\\ \pi(0) & \text{otherwise.}\end{cases}$$

The effect of h is to wrap the partial trajectory $t\cdot p$, $0\le t\le\frac{1}{2}\sigma$, $p\in E$, around the circle once, while crushing the rest of X to the point $\pi(0)$. Since B is a neighborhood of E, the boundary of $[0,\frac{1}{2}\sigma]\cdot E$ is $E\cup\frac{1}{2}\sigma\cdot E$, so h is continuous. We call h a *cosection map* associated to E. It is easy to see that any two cosection maps associated to E are homotopic.

Let $\check{H}^1(X;\mathbb{R})$ denote the first Alexander–Spanier cohomology group of the space X with real coefficients. (See § 2 or Spanier [**9**] for definitions.) If h is a cosection map associated to the section E then the induced map

$$h^*:\check{H}^1(\mathbb{T};\mathbb{R})\to\check{H}^1(X;\mathbb{R})$$

carries the fundamental class, Θ, of $\mathbb{T}$ to a 1-dimensional cohomology class which we denote by c_E. It is well defined since any two co-section maps of E are homotopic; we call c_E the *flow class* of E. The way that c_E depends on E is an interesting, open problem.

From now on, φ is a flow on the compact metric space X. The following is our main result.

THEOREM 1.1. *If the section E of φ contains a chain recurrent point then its flow class is a non-zero element of $\check{H}^1(X;\mathbb{R})$.*

Remark 1. A dual question would be this. Is there some kind of 1-dimensional homology class, u_E, in X (say an Alexander or Vietoris class [**10**]) such that the Kronecker index of c_E and u_E is non-zero when E contains chain recurrent points? We expect the answer is 'no'.

Remark 2. Is the converse of theorem 1.1 true or false?

THEOREM 1.2. *Suppose that $p\in X$ belongs to a local section and that X has dimension one at p. If p is chain recurrent then $\check{H}^1(X;\mathbb{R})\neq 0$.*

Proof. Let E_0 be a local section at p and $B_0=B(\sigma, E_0)$ be a flowbox. Then $\dim_p B_0=1$. Since B_0 is homeomorphic to $[-\sigma,\sigma]\times E_0$, it follows from Hurewicz and Wallman [**7**] that $\dim_p E_0=0$. Thus, there is a compact neighborhood E of p in E_0 having empty boundary in E_0. Consequently, E is a section at p and Theorem 1.1 applies. □

Since smooth flows have local sections at all non-stationary points, we obtain as a corollary:

THEOREM 1.3. *Let F be a smooth vector field on the manifold M and let X be a compact subset of M invariant under the local flow generated by F. If X contains a non-stationary, chain recurrent point at which X has dimension one then $\check{H}^1(X;\mathbb{R})\neq 0$.*

In the situation of theorem 1.3 we can interpret the flow class c_E as follows. Let D be a smooth $(m-1)$-dimensional disc transverse to F at the chain recurrent point p, m being the dimension of M. Then $D\cap X$ contains a section E of the F-flow φ

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on X. Let $h: X \to \mathbb{T}$ be a cosection map associated to E. Let $g: N_0 \to \mathbb{T}$ be a continuous extension of h to a neighborhood of X in M. Fix an orientation of a neighborhood of D so that the intersection number $([-\sigma, \sigma] \cdot x) \# D = +1$ for all $x \in D$ near X. (We assume the arc $[-\sigma, \sigma] \cdot x$ is oriented from $-\sigma \cdot x$ to $\sigma \cdot x$.) For any small neighborhood N of X in M, define a homomorphism

$$\mu_{E,N}: H_1(N; \mathbb{Z}) \to \mathbb{R}$$

by assigning to each singular cycle z in N, its intersection number $z \# D$ with D. Using the proof of theorem 1.1, it is not hard to show that $\mu_{E,N}$ corresponds to $(g|_N)^*\Theta$ under the natural isomorphisms

$$I_N: \check{H}^1(N; \mathbb{R}) \approx H^1(N; \mathbb{R}) \approx \operatorname{Hom}(H_1(N; \mathbb{Z}), \mathbb{R}).$$

Thus, c_E is the direct limit of the classes $I_N^{-1}(\mu_{E,N})$ under the natural isomorphism

$$\check{H}^1(X; \mathbb{R}) \approx \text{direct-limit}_{N \to X} \check{H}^1(N; \mathbb{R}).$$

The next result is like theorem 1.3 without X being 1-dimensional. We retain the other hypotheses on F, X, M.

THEOREM 1.4. *Suppose there is a smooth, codimension one submanifold $\Sigma \subset M$ such that Σ is closed as a subset of M, its normal bundle ν is trivial, and F is transverse to Σ at $X \cap \Sigma$, always with the same orientation respecting ν. Then the non-zero flow class c_E is in the image of $i_X^*: \check{H}^1(M; \mathbb{R}) \to \check{H}^1(X; \mathbb{R})$.*

Proof. Using the orientation of ν, we get a homomorphism $\mu: H_1(M; \mathbb{Z}) \to \mathbb{R}$ defined by $z \mapsto z \# \Sigma$. The previous discussion shows that

$$i_X^* \circ I_M^{-1}(\mu) = c_E. \qquad \square$$

Another application of theorem 1.1 is the following.

THEOREM 1.5. *If X has dimension one at the chain recurrent point p and S is any planar surface containing X then X disconnects S.*

Proof. We may assume S is an open subset of the sphere S^2. It need not be invariant under the flow. By Alexander Duality and theorem 1.3,

$$\begin{aligned} \text{the number of components of } S^2 \backslash X &= \operatorname{rank} H_0(S^2 \backslash X; \mathbb{R}) \\ &= 1 + \operatorname{rank} \check{H}^1(X; \mathbb{R}) \geq 2. \end{aligned}$$

Therefore $S \backslash X$ is also disconnected. $\square$

The following consequence of theorem 1.2 is a generalization of the Poincaré–Bendixson Theorem.

THEOREM 1.6. *Let Λ be a compact ω-limit set of a local flow defined in an open subset S of the plane. If some $p \in \Lambda$ is non-stationary then Λ separates S.*

Proof. H. Whitney [**11**] shows that there is a local section through any non-stationary point of a planar local flow. It is well known that every point of a compact ω-limit set is chain recurrent. Therefore, by theorem 1.2, it suffices to check that $\dim_p \Lambda = 1$. Since p is non-stationary, $\dim_p \Lambda \geq 1$. On the other hand, it is well known (and part of the classical proof of Poincaré–Bendixson) that p must be isolated in any local section at p. Thus, $\dim_p \Lambda = 1$. $\square$

2. *Alexander–Spanier cohomology*

We recall some concepts from Spanier [9]. A *cocycle* on a space X is a map (not necessarily continuous) $\alpha: U \to \mathbb{R}$ where U is a neighborhood of the diagonal in $X \times X$ such that

$$\alpha(x, y) + \alpha(y, z) + \alpha(z, x) = 0$$

whenever all three terms are defined. Thus,

$$\alpha(x, x) = 0 \quad \text{and} \quad \alpha(x, y) = -\alpha(y, x).$$

If α has the special form $f(y) - f(x)$ for some $f: X \to \mathbb{R}$ then it is the *coboundary* δf of f.

Two cocycles *have the same germ* if they agree on some neighborhood of the diagonal. An equivalence class under this equivalence relation is a *germ*. The set of germs of cocycles forms a group Z under the operation induced by addition of representative cocycles. The germs containing coboundaries form a subgroup B and the quotient group, Z/B, is by definition the Alexander–Spanier cohomology group in dimension one with real coefficients, $\check{H}^1(X; \mathbb{R})$. The coset of the germ of α is denoted by $[\alpha] \in \check{H}^1(X; \mathbb{R})$.

If $h: Y \to X$ is a continuous map then a homomorphism

$$h^*: \check{H}^1(X; \mathbb{R}) \to \check{H}^1(Y; \mathbb{R})$$

is induced by the operation of $h^\#$ on cocycles

$$h^\# \alpha(y, y') = \alpha(h(y), h(y'))$$

for $y, y' \in Y$. One then defines $h^*[\alpha] = [h^\# \alpha]$.

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle of unit length and let $\pi: \mathbb{R} \to \mathbb{T}$ be the projection. The *fundamental cocycle* $\theta: W \to \mathbb{R}$ on $\mathbb{T}$ is defined by

$$\theta(\pi x, \pi y) = y - x \qquad \text{if } |x - y| < \tfrac{1}{2}.$$

Thus,

$$W = \{(\pi x, \pi y) \in \mathbb{T} \times \mathbb{T} : |x - y| < \tfrac{1}{2}\}.$$

Let $V \subset X \times X$ be a neighborhood of the diagonal. A *V-chain* is a finite sequence of points in X, $x_0, x_1, \ldots, x_n$, such that $n \geq 1$ and each pair (x_{j-1}, x_j) lies in V, $1 \leq j \leq n$. If also $x_0 = x_n$ then it is a *V-cycle*. The natural way to evaluate a cocycle α on a V-cycle $z = \{x_0, \ldots, x_n\}$ is

$$\langle \alpha, z \rangle = \sum_j \alpha(x_{j-1}, x_j) \qquad 1 \leq j \leq n.$$

LEMMA 2.1. *The cohomology class of the cocycle $\alpha: U \to \mathbb{R}$ is non-zero provided that for every neighborhood $V \subset U$ of the diagonal there exists a V-cycle z such that $\langle \alpha, z \rangle \neq 0$.*

Proof. We prove the contrapositive. If the cohomology class of α is zero then it is a coboundary; i.e. there is some $f: X \to \mathbb{R}$ such that $\alpha(x, y) = f(y) - f(x)$ for all (x, y) in some neighborhood V of the diagonal. Then, for any V-cycle $z = \{x_0, \ldots, x_n\}$,

$$\langle \alpha, z \rangle = \sum_j \alpha(x_{j-1}, x_j) = \sum_j f(x_j) - f(x_{j-1}) = f(x_n) - f(x_0) = 0. \qquad \square$$

It is also convenient to evaluate a cocycle $\alpha: U \to \mathbb{R}$ on a singular 1-simplex (or path) $\gamma: [a, b] \to X$ as follows. Let $\varepsilon > 0$ be small enough that $(\gamma(s), \gamma(t)) \in U$

whenever $|s-t|<\varepsilon$. Let $a=t_0<t_1<\cdots<t_n=b$ be a partition of the interval $[a, b]$ with $t_j-t_{j-1}<\varepsilon$, $1\le j\le n$. It is easy to see that the number

$$\sum_j \alpha(\gamma(t_{j-1}), \gamma(t_j)) = \langle \alpha, \{\gamma(t_j)\}_j\rangle$$

is independent of the choice of the partition $\{t_j\}$, owing to the cocycle property of α and the fact that the partition has mesh $<\varepsilon$. We define this number to be $\langle\alpha, \gamma\rangle$.

LEMMA 2.2. *If $\gamma:[a, b]\to\mathbb{T}$ is a path and $\tilde{\gamma}:[a, b]\to\mathbb{R}$ lifts γ (i.e. $\pi\circ\tilde{\gamma}=\gamma$) then $\langle\theta, \gamma\rangle=\tilde{\gamma}(b)-\tilde{\gamma}(a)$, where θ is the fundamental cocycle on the circle $\mathbb{T}$.*

Proof. Left to the reader. □

One extends the idea of a V-chain by replacing the points x_j with paths γ_j, V being a neighborhood of the diagonal. An *extended V-chain* is a finite sequence $\Gamma=\{\gamma_j\}$ of paths $\gamma_j:[a_j, b_j]\to X$ such that $(\gamma_{j-1}(b_{j-1}), \gamma_j(a_j))\in V$, $1\le j\le n$. If also $\gamma_n(t)\equiv\gamma_0(a_0)$, $t\in[a_n, b_n]$, then we call Γ an *extended V-cycle.* The value of a cocycle $\alpha: V\to\mathbb{R}$ on Γ is

$$\langle\alpha, \Gamma\rangle=\sum_j \alpha(\gamma_{j-1}(b_{j-1}), \gamma_j(a_j))+\sum_i\langle\alpha, \gamma_i\rangle$$

where $1\le j\le n$ and $0\le i\le n$. Combining Lemmas 2.1 and 2.2, we get

LEMMA 2.3. *The cohomology class of a cocycle $\alpha: U\to\mathbb{R}$ is non-zero provided that for every neighborhood V of the diagonal there is an extended V-cycle Γ with $\langle\alpha, \Gamma\rangle\ne 0$.*

3. *Proof of Theorem* 1.1

Let $E\subset X$ be a section containing the chain recurrent point $p\in E$. Let $[-\sigma, \sigma]\cdot E=B$ be a flowbox neighborhood of E. To simplify notation, we assume $\sigma=2$. This is no loss of generality because we can replace φ by the re-parameterized flow $\psi(t, x)=\varphi(2t/\sigma, x)$, and all the hypotheses on φ hold also for ψ.

Let $h: X\to\mathbb{T}$ be the cosection map associated to $[-2, 2]\cdot E=B$. See § 2. Thus

$$h(t\cdot x)=\pi(t)\qquad \text{if } 0\le t\le 1 \text{ and } x\in E,$$

while h sends the rest of X to $\pi(0)$.

Recall that W is the domain of the fundamental cocycle θ on $\mathbb{T}$ and $\Theta=[\theta]$ is its cohomology class in $\check{H}^1(\mathbb{T};\mathbb{R})$. To prove theorem 1.1, we must show that the flow class $c_E=h^*(\Theta)\ne 0$. We do so via lemma 2.3. That is, for any neighborhood V of the diagonal in $X\times X$, we will find an extended V-cycle Γ such that $\langle c_E, \Gamma\rangle\ne 0$. Let such a V be given. There exists $\varepsilon>0$ so small that if $x, y\in X$ and $d(x, y)<\varepsilon$ then

(5) $(x, y)\in V$;

(6) $(hx, hy)\in W$;

(7) if $0\le r\le 1$ and $x\in r\cdot E$ then $y\in s\cdot E$ for some s with $|r-s|<1$.

It follows from (6) that $h^{\#}\theta(x, y)$ is defined if $d(x, y)<\varepsilon$.

Since p is chain recurrent, there exists a $(2, \varepsilon)$-pseudo-orbit of φ from p to itself, $\{\gamma_i\}$, $i=0, 1, \ldots, n-1$. Define $\gamma_n(t)\equiv p$ for $t\in[0, 3]$ and $\Gamma=\{\gamma_i: 0\le i\le n\}$. In view of lemma 2.3, it suffices to verify

(8) $\langle h^{\#}\theta, \Gamma\rangle>0$,

Call $\gamma_i(0)=p_i$ and $\gamma_i(t_i)=q_j$, $0\le i\le n$. Then

$$\langle h^{\#}\theta,\Gamma\rangle=\sum_j h^{\#}\theta(q_{j-1},p_j)+\sum_i\langle h^{\#}\theta,\gamma_i\rangle$$
$$=\sum_j \theta(h(q_{j-1}),h(p_j))+\sum_i\langle\theta,h\circ\gamma_i\rangle.$$

As before, $1\le j\le n$ and $0\le i\le n$. All terms in the last sum are ≥ 0. Indeed, for any forward-oriented partial trajectory $\gamma: t\mapsto t\cdot x$, $0\le t\le T$, the set

$$J(\gamma)=\{t\in[0,T]: t\cdot x\in[0,1]\cdot E\}$$

is a finite union of intervals, and from lemma 2.2 and the definition of h, their total length equals $\langle\theta,h\circ\gamma\rangle$. Since γ_0 starts at $p\in E$ and $t_0\ge 2$, we see that $[0,1]\subset J(\gamma_0)$, so

(9) $\langle\theta,h\circ\gamma_0\rangle\ge 1$.

We claim that for each i, $1\le i\le n-1$,

(10) $\theta(h(q_{i-1}),h(p_i))+\langle\theta,h\circ\gamma_i\rangle\ge 0$.

We need only prove (10) for those i with negative first term. Fix such an i. Since h sends the complement of $[0,1]\cdot E$ onto a single point, one or both of the points q_{i-1},p_i lie in $[0,1]\cdot E$. Since $d(q_{i-1},p_i)<\varepsilon$, it follows from (6), (7) that $q_{i-1}\in s\cdot E$ and $p_i\in r\cdot E$ for some r, s such that

$$0<s-r<1\quad\text{and}\quad\text{either } r \text{ or } s \text{ lies in } [0,1].$$

If $0\le r<s\le 1$ then

$$J(\gamma_i)\supset[r,1]\supset[r,s]\quad\text{and}\quad s-r=\theta(h(q_{i-1}),h(p_i)),$$

verifying (10). If $0\le r\le 1<s$ then

$$J(\gamma_i)\supset[r,1]\quad\text{and}\quad 1-r=\theta(h(q_{i-1}),h(p_i)),$$

verifying (10). If $r\le 0\le s<1$ then

$$J(\gamma_i)\supset[0,s]\quad\text{and}\quad s-1=\theta(h(q_{i-1}),h(p_i)),$$

verifying (10) in this case also.

From (9), (10), and the definition of h we see that

$$\langle h^{\#}\theta,\Gamma\rangle\ge\theta(h(q_{n-1}),h(p_n))+\langle\theta,h\circ\gamma_0\rangle \tag{11}$$
$$\ge\theta(h(q_{n-1}),\pi(0))+1.$$

Since $p\in E=0\cdot E$, and $d(q_{n-1},p_n)<\varepsilon$, it follows from (6), (7) that $q_{n-1}\in s\cdot E$ with

(12) $|s|<1$.

If $s<0$ then $\theta(h(q_{n-1}),\pi(0))=0$ and (11) implies (8). If $s>0$ then $\theta(h(q_{n-1}),\pi(0))=-s$ and (11), (12) imply (8). □

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