EXTENSION OPERATORS FOR SOBOLEV SPACES COMMUTING WITH A GIVEN TRANSFORM

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Abstract. We consider a real-valued function $r = M(t)$ on the real axis, such that $M(t) < 0$ for $t < 0$. Under appropriate assumptions on M, the pull-back operator M^* gives rise to a transform of Sobolev spaces $W^{s,p}(-\infty, 0)$ that restricts to a transform of $W^{s,p}(-\infty, \infty)$. We construct a bounded linear extension operator $W^{s,p}(-\infty, 0) \to W^{s,p}(-\infty, \infty)$, commuting with this transform.

1. Motivation. As described in Schulze [**5**], Sobolev embedding theorems may be treated in the framework of pseudodifferential operators with operator-valued symbols whose definition is based on the 'twisted'' homogeneity.

In particular, consider the strongly continuous group action $(\kappa_\lambda)_{\lambda \in (0,\infty)}$ on a space $L = H^s(R_-)$, $s \in R$, given by $\kappa_\lambda u(t) = \lambda^{1/2} u(\lambda t)$. Obviously, κ_λ acts continuously also on $V = H^s(R)$. It is easy to verify that

$$W^s(R^q, H^s(R_-)) = H^s(R_-{}^{q+1}),$$
$$W^s(R^q, H^s(R)) = H^s(R^{q+1}),$$

where $W^s(R^q, L)$ is defined to be the completion of $C^\infty_{comp}(R^q, L)$ with respect to the norm $\|u\| = (\int_{R^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} F_{y\mapsto\eta} u\|^2_L d\eta)^{\frac{1}{2}}$, F being the Fourier transform. Each continuous linear extension operator $T: H^s(R_-) \to H^s(R)$ commuting with κ_λ gives rise to a constant operator-valued symbol $a(y, n)$ in $\mathcal{S}^0_{cl}(T^*(R^q), \mathcal{L}(L \to V))$ simply by $a(y, \eta) = T$. The symbol space in question is defined on the base of the group action κ_λ, so that $a(y, \eta)$ satisfies

$$\|\kappa^{-1}_{\langle\eta\rangle} D^\alpha_y D^\beta_\eta a(y, \eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(L\to V)} \le c\langle\eta\rangle^{-|\beta|},$$

for all multi-indices α and β, uniformly in y on compact subsets of R^q and $\eta \in R^q$. Then, the corresponding pseudodifferential operator $\mathrm{op}(a) u = F^{-1}_{\eta\mapsto y} a(y, \eta) F_{y\mapsto\eta} u$ extends to a continuous mapping of $W^s(R^q, L) \to W^s(R^q, V)$. Moreover, it is an extension operator of $H^s(R_-{}^{q+1}) \to H^s(R^{q+1})$, for if $R: H^s(R) \to H^s(R_-)$ is the restriction mapping, then $\mathrm{op}(R)$ is the restriction operator of $H^s(R^{q+1}) \to H^s(R_-{}^{q+1})$ and

$$\begin{aligned}\mathrm{op}(R)\,\mathrm{op}(a) &= \mathrm{op}(RT)\\ &= 1\end{aligned}$$

on $H^s(R_-{}^{q+1})$. This operator-valued boundary symbol is of particular interest in Boutet de Monvel's algebra. (See [**5**, Subsection 4.2.2].)

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With this as our starting point, we are looking in this paper for a bounded extension operator of $H^s(R_-) \to H^s(R)$ commuting with a general transform of these spaces.

2. Statement of the main result. For $s \in Z_+$, $1 \leq p \leq \infty$ and $-\infty \leq a < b \leq \infty$, let $W^{s,p}(a, b)$ stand for the Sobolev space of all functions $f \in L^p(a, b)$ having weak derivatives $f^{(s)}$ of order s on (a, b), such that

$$||f||_{W^{s,p}(a,b)} = ||f||_{L^p(a,b)} + ||f^{(s)}||_{L^p(a,b)} < \infty.$$

It is well-known (see Nikol'skii [3], Babich [1]) that there exists a bounded linear extension operator

$$T: W^{s,p}(-\infty, 0) \to W^{s,p}(-\infty, \infty) \tag{2.1}$$

(i.e. $(Tf)(t) = f(t)$ if $t < 0$). It can be constructed in the following way: for $t > 0$,

$$(Tf)(t) = \sum_{j=1}^{s} \alpha_j f(-\beta_j t), \tag{2.2}$$

where β_j are arbitrary distinct positive numbers and α_j are defined by

$$\sum_{j=1}^{s} \alpha_j(-\beta_j)^i = 1 \qquad (i = 0, 1, \ldots, s-1).$$

(This construction was first used in Hestenes [2].)

Denote by κ a dilation transform of the type

$$(\kappa f)(t) = Af(\lambda t), \qquad t \in (-\infty, \infty),$$

where A and λ are positive numbers. Then the extension operator T defined by (2.2) commutes with κ:

$$T\kappa = \kappa T. \tag{2.3}$$

(Note that in the left side κ is considered as an operator acting from $W^{s,p}(-\infty, 0)$ to $W^{s,p}(-\infty, 0)$, while in the right side it is considered as an operator acting from $W^{s,p}(-\infty, \infty)$ to $W^{s,p}(-\infty, \infty)$.)

Below we consider a more general transform κ defined by

$$(\kappa f)(t) = Af(M(t)), \qquad x \in (-\infty, \infty), \tag{2.4}$$

where A is a positive number and M a function satisfying appropriate conditions. We construct a bounded linear extension operator commuting with this transform.

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THEOREM 2.1. *Suppose $s \in Z_{+}$, $1 \le p \le \infty$, and κ is a transform defined by (2.4), where $A > 0$ and M satisfies the following conditions:*

(1) *$M \in C^{s}_{loc}(-\infty, \infty)$ and all derivatives $M^{(i)}$, $i = 1, \ldots, s$, are bounded;*
(2) *M is odd;*
(3) *$M(t) > 0$, for all $t \in (0, \infty)$;*
(4) *there exists $c > 0$ such that $M'(t) > c$ for $t \in (-\infty, \infty)$; moreover $M'(0) \neq 1$;*
(5) *$M''(0) = \ldots = M^{(s-1)}(0) = 0$.*

Then, there exists a bounded linear extension operator (2.1) satisfying (2.3).

Proof. **1°**. For $f \in W^{s,p}(-\infty, 0)$, we set $f_{-}(t) = f(-t)$ and

$$(Tf)(t) = \sum_{j=1}^{s} \alpha_j (\kappa^j f_{-})(t) \qquad (t > 0),$$

where $\alpha_j, j = 1, \ldots, s$, are defined by

$$\sum_{j=1}^{s} \alpha_j A^j (M'(0))^{ij} = (-1)^i \qquad (i = 0, 1, \ldots, s-1). \tag{3.1}$$

We note that, since $M'(0) \neq 1$, the determinant of this system with respect to the variables $\alpha_j A^j$, being a Van-der-Mond determinant, is not equal to 0.

Put

$$M_j(t) = \underbrace{M(\cdots(M}_{j}(t))\cdots).$$

Then

$$(Tf)(t) = \sum_{j=1}^{s} \alpha_j A^j f(-M_j(t)) \quad (t > 0).$$

As, by condition (3), $M_j(t) > 0$ for $t > 0$, the value $(Tf)(t)$ is well-defined.

2.° Suppose $f \in W^{s,p}(-\infty, 0)$. In order to prove that $Tf \in W^{s,p}(-\infty, \infty)$ it is enough to prove that $Tf \in W^{s,p}(0, \infty)$ and

$$(Tf)^{(i)}(0+) = f^{(i)}(0-) \qquad (i = 0, 1, \ldots, l-1), \tag{3.2}$$

where $f^{(i)}(0-)$ and $(Tf)^{(i)}(0+)$ are boundary values of $f^{(i)}$ and $(Tf)^{(i)}$ respectively. (See for instance Nikol'skii [**4**], Triebel [**6**].)

3.° Since $f \in W^{s,p}(-\infty, 0)$, it is equivalent to a function F defined on $(-\infty, 0]$, such that the ordinary derivatives $F^{(i)}$, $i = 1, \ldots, s-1$, exist on $(-\infty, 0]$ and $F^{(s-1)}$ is absolutely continuous on $[a, 0]$ for each $a < 0$. Moreover, $f^{(i)}(0-) = F^{(i)}(0)$ for $i = 1, \ldots, s-1$. We note also that the ordinary derivative $F^{(s)}$ exists almost everywhere on $(-\infty, 0)$ and is equivalent to the weak derivative $f^{(s)}$. (See for example Nikol'skii [**4**].)

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It follows that Tf, defined on $(0, \infty)$, is equivalent to TF, defined on $[0, \infty)$, the ordinary derivatives $(TF)^{(i)}$, $i = 1, \ldots, s-1$, exist on $[0, \infty)$ and $(TF)^{(s-1)}$ is absolutely continuous on $[0, b]$ for each $b > 0$. The latter is due to the fact that the functions M_j are absolutely continuous and monotonic. Consequently, the ordinary derivative $(TF)^{(s)}$ exists almost everywhere on $(0, \infty)$, is equivalent to the weak derivative $(Tf)^{(s)}$ and

$$\|Tf\|_{W^{s,p}(0,\infty)} = \|TF\|_{L^p(0,\infty)} + \|(TF)^{(s)}\|_{L^p(0,\infty)}. \tag{3.3}$$

Moreover, condition (3.2) is equivalent to

$$(TF)^{(i)}(0) = F^{(i)}(0) \qquad (i = 0, 1, \ldots, l-1). \tag{3.4}$$

4.° Our next observation is that, for $i = 1, \ldots, s$ and $t > 0$, we have

$$\begin{aligned}(F(-M_j(t)))^{(i)} =& (-1)^i F^{(i)}(-M_j(t))\,(M'(M_{j-1}(t))M'(M_{j-2}(t))\cdots M'(t))^i \\ &+ \sum_{k=1}^{i-1} F^{(k)}(-M_j(t))A_{i,k}(t),\end{aligned}$$

where $A_{i,k}$ are linear combinations of products of some natural powers of derivatives $M^{(l)}(M_m(t))$, where $0 \le m \le j-1$ and $1 \le l \le i-k+1$. This equality is valid everywhere on $[0, \infty)$, if $i < s$, and almost everywhere, if $i = s$.

It is worth pointing out that every summand in $A_{i,k}$ contains as a factor at least one derivative of M of order greater than 1. Consequently, we can assert, by conditions (2) and (5), that

$$(F(-M_j(t)))^{(i)}|_{t=0} = (-1)^i (M'(0))^{ij} F^{(i)}(0),$$

for all $i = 0, 1, \ldots, s-1$. Hence it follows that

$$(Tf)^{(i)}(0) = (-1)^i \left(\sum_{j=1}^{s} \alpha_j A^j (M'(0))^{ij} \right) F^i(0), \tag{3.5}$$

for $i = 0, 1, \ldots, s-1$.

Moreover, since the derivatives $M^{(1)}, \ldots, M^{(s)}$ are bounded, there exists a constant $c_1 > 0$ such that

$$|(F(-M_j(t)))^{(i)}| \le c_1 \sum_{k=1}^{i} |F^{(k)}(-M_j(t))|, \qquad t \ge 0,$$

for $i = 1, \ldots, s$. Thus,

$$|(TF)(t)| \le c_2 \sum_{j=1}^{s} |F(-M_j(t))|, \qquad t \ge 0,$$

and

$$|(TF)^{(i)}(t)| \le c_2 \sum_{j=1}^{s} \sum_{k=1}^{i} |F^{(i)}(-M_j(t))|, \qquad t \ge 0,$$

for $i = 1, \ldots, s$, the constant c_2 being independent of F.

5.° By condition (4), there is a constant $c_3 > 0$ with the property that

$$M_j'(t) \geq c_3, \qquad t \in (-\infty, \infty),$$

for $j = 1, \ldots, s$. Consequently,

$$\begin{aligned} \|TF\|_{L^p(0,\infty)} &\leq c_2 \sum_{j=1}^{s} \|F(-M_j(t))\|_{L^p(0,\infty)} \\ &= c_2 \sum_{j=1}^{2} \left(\int_{-\infty}^{0} |F(r)|^p \frac{dr}{M_j'(M_j^{-1}(r))} \right)^{\frac{1}{p}} \\ &\leq c_2 c_3^{-\frac{1}{p}} \sum_{j=1}^{s} \|F\|_{L^p(-\infty,0)} \\ &= c_4 \|F\|_{L^p(-\infty,0)}, \end{aligned} \tag{3.6}$$

where $c_4 = c_2 c_3^{-\frac{1}{p}} s$. Similarly,

$$\|(TF)^{(s)}\|_{L^p(0,\infty)} \leq c_5 \sum_{k=1}^{s} \|F^{(k)}\|_{L^p(-\infty,0)} \tag{3.7}$$

with c_5 a constant independent of F.

Now we invoke a well-known result that

$$\|F^{(k)}\|_{L^p(-\infty,0)} \leq c_6 \big(\|F\|_{L^p(-\infty,0)} + \|F^{(s)}\|_{L^p(-\infty,0)} \big) \tag{3.8}$$

for all $k \leq s$, where the constant c_6 depends only on s. (See Nikol'skii [**4**].) Thus, combining (3.3), (3.6), (3.7) and (3.8) we obtain

$$\|Tf\|_{W^{s,p}(0,\infty)} \leq c_7 \|f\|_{W^{s,p}(-\infty,0)}, \tag{3.9}$$

where c_7 is independent of f.

6.° According to (3.5) condition (3.4) and, hence, (3.2) is equivalent to (3.1). Thus, from what has been said in item **2°** it follows that $Tf \in W^{s,p}(-\infty, \infty)$. The estimate (3.9) now shows that the operator T is bounded.

7.° Finally, equality (2.3) is equivalent to

$$\sum_{j=1}^{s} \alpha_j \kappa^j (\kappa f)_- = \sum_{j=1}^{s} \alpha_j \kappa^{j+1} f_-$$

on $(0, \infty)$. The latter equality is valid for, by condition (2),

$$\begin{aligned} (\kappa f)_-(t) &= (Af(M(t)))_- \\ &= Af(M(-t)) \\ &= Af(-M(t)) \\ &= (\kappa f_-)(t), \end{aligned}$$

which completes the proof.

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