A TWO-POINT BOUNDARY PROBLEM FOR ORDINARY SELF-ADJOINT DIFFERENTIAL EQUATIONS OF FOURTH ORDER

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1. Introduction. The purpose of this note is to establish Theorem A below for the two-point homogeneous vector boundary problem

$$[P_0(x)u'']'' - [P_1(x)u']' + P_2(x)u = 0, \tag{1.1}$$

$$u(x_1) = u'(x_1) = 0 = u'(x_2) = u(x_2), \tag{1.2}$$

where the $P_i(x)$ are given real $m \times m$ symmetric matrix functions of x with $P_0(x)$ positive definite and $P_i(x)$ of class C^{2-i} on an infinite interval $[a, \infty)$, and where by a solution of (1.1) – (1.2) for $a \leqslant x_1 < x_2 < \infty$ we understand a real m-dimensional column vector $u = u(x)$ of class C^2 on $[a, \infty)$ which is such that $P_i(x)u^{(2-i)}$ is of class C^{2-i} on $[a, \infty)$ and which satisfies (1.1) – (1.2) with the former a vector identity on $[a, \infty)$.

THEOREM A. *If for some real number $k_0 > 0$ each of the matrices $P_0(x) - k_0I$, $P_1(x)$ and $P_2(x)$ is negative semi-definite on $[a, \infty)$ and if there exists an $a_0 \geqslant a$ such that for arbitrary x_1, x_2 satisfying $a_0 \leqslant x_1 < x_2 < \infty$ the only solution $u = u(x)$ of (1.1) – (1.2) is the trivial solution $u(x) \equiv 0$ on $[a, \infty)$ then the improper matrix integrals*

$$\int_a^\infty P_1(x)\,dx, \qquad \int_a^\infty x^2 P_2(x)\,dx, \tag{1.3}$$

exist and for each m-dimensional constant vector π satisfying $\pi^*\pi = 1$ we have

$$\limsup_{x\to\infty} |\, x\pi^* \int_x^\infty P_1(t)\,dt\pi \,| \leqslant 2k_0, \quad \limsup_{x\to\infty} |\, x\pi^* \int_x^\infty t^2P_2(t)\,dt\pi \,| \leqslant 8k_0. \tag{1.4}$$

We note in passing that Theorem A can be considered to be an analogue for the present problem of non-oscillation theorems of Hille **(1)** and Wintner **(4)** for an ordinary linear differential equation of the second order; see in particular **(1**, §1**)**. This analogy follows at once from the process employed in the proof below in which (1.1) is transformed into a vector differential system of the second order and twice the dimensionality for which the identical vanishing of the only solution of (1.1) – (1.2) corresponds to non-oscillation of the transformed system in the sense of Sternberg **(3**, §2**)**. Theorem A is also a partial converse of a theorem essentially given by Kaufman and Sternberg

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(2, §§1 and 3**)** for a higher order homogeneous two-point boundary problem of which (1.1) — (1.2) is a special case.

2. Proof. We may assume without loss of generality that $a_0 > 0$. It has been noted in effect in **(2**, §§2 and 3**)** that a solution $u = u(x)$ of (1.1) defines by the relations $y_1 = u$, $y_2 = u'$, $\mu = P_1(x)u' - [P_0(x)u'']'$ a solution $y = y(x) \equiv (y_j(x))$, $(j = 1, 2)$, $\mu = \mu(x)$ of the system of $2m + m$ ordinary linear differential equations

$$(2.1)\qquad \begin{aligned} L[y, \mu] \equiv \left[\begin{pmatrix} 0 & 0 \\ 0 & P_0(x)\end{pmatrix}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}' + \begin{pmatrix} I \\ 0\end{pmatrix}\mu\right]' \\ - \left[\begin{pmatrix} P_2(x) & 0 \\ 0 & P_1(x)\end{pmatrix}\begin{pmatrix} y_1 \\ y_2\end{pmatrix} + \begin{pmatrix} 0 \\ -I\end{pmatrix}\mu\right] = 0, \\ \Phi[y] \equiv (I\;0)\begin{pmatrix} y_1 \\ y_2\end{pmatrix}' + (0\; -I)\begin{pmatrix} y_1 \\ y_2\end{pmatrix} = 0, \end{aligned}$$

in the sense of **(3**, §2**)** and that this system is a special case of the system (2.1) of **(3**, §2**)**, the elements $y_1 = y_1(x)$, $y_2 = y_2(x)$ and $\mu = \mu(x)$ being m-dimensional column vectors. We now observe conversely that a solution $y = y(x)$, $\mu = \mu(x)$ of (2.1) in the sense of **(3**, §2**)** defines by the relation $u = y_1$ a solution $u = u(x)$ of (1.1) in the sense stated earlier; moreover, from the form of (1.2) and the equation $\Phi[y] = 0$ in (2.1) it follows that if for $a_0 \leqslant x_1 < x_2 < \infty$ the only solution $u = u(x)$ of (1.1) — (1.2) is the trivial solution $u(x) \equiv 0$ then the system (2.1) is non-oscillatory on $[a_0, \infty)$ in the sense of **(3**, §2**)**. Hence, under the hypotheses of Theorem A we have by Lemma 3.2 of **(3**, §3**)** that

$$(2.2)\qquad J[\eta; a_0, b_0] \equiv \int_{a_0}^{b_0} [\eta^{*\prime} G(x)\eta' - \eta^* F(x)\eta]\, dx > 0$$

for all $b_0 > a_0$ and arbitrary admissible variations $\eta = \eta(x) \equiv (\eta_j(x)) \not\equiv 0$, $(j = 1, 2)$ on $[a_0, b_0]$ satisfying $\eta(a_0) = 0 = \eta(b_0)$ and $\Psi[\eta] \equiv \psi(x)\eta' = 0$ where similarly as in **(2**, §2**)**

$$(2.3)\qquad G(x) = \begin{pmatrix} 0 & 0 \\ 0 & P_0(x)\end{pmatrix}, \quad F(x) = \begin{pmatrix} -P_2(x), & -xP_2(x) \\ -xP_2(x), & -P_1(x) - x^2P_2(x)\end{pmatrix}, \quad \psi(x) = (I\; xI),$$

and by definition in **(3**, §3**)** $\eta_1(x)$ and $\eta_2(x)$ are m-dimensional column vectors of class D', the equation $\Psi[\eta] = 0$ being required to hold merely in a piecewise manner on $[a_0, b_0]$.

Next, employing the symmetry and negative semi-definiteness of the matrices $P_1(x)$ and $P_2(x)$ on $[a, \infty)$ to establish the existence of real $m \times m$ matrices $Q_i(x)$ such that $P_i(x) = -Q_i^*(x)\, Q_i(x)$ on $[a, \infty)$ and, consequently, also such that

$$(2.4)\qquad F(x) = K^*(x)K(x), \quad K(x) = \begin{pmatrix} Q_2(x) & xQ_2(x) \\ 0 & Q_1(x)\end{pmatrix} \text{ on } [a, \infty),$$

we note that the matrix $F(x)$ is positive semi-definite on $[a, \infty)$.

Now consider for $a_0 < x_1 < x_2 < x_3 < x_4 < b_0$ the admissible variation $\eta = \eta(x) \equiv (\eta_j(x))$, $(j = 1, 2)$ defined on $[a_0, b_0]$ for each m-dimensional constant vector π satisfying $\pi^*\pi = 1$ as

$$\tag{2.5} \eta_1(x) = \begin{cases} -\frac{1}{2}\left(\dfrac{x^2 - a_0^2}{x_1 - a_0}\right)\pi, & \text{on } [a_0, x_1], \\ -\frac{1}{2}(x_1 + a_0)\,\pi, & \text{on } [x_1, x_2], \\ -\frac{1}{2}(x_1 + a_0)\,\pi + \left(\dfrac{x^2 - x_2^2}{x_3 - x_2}\right)\pi, & \text{on } [x_2, x_3], \\ -\frac{1}{2}(x_1 + a_0)\,\pi + (x_3 + x_2)\,\pi, & \text{on } [x_3, x_4], \\ -\frac{1}{2}(x_1 + a_0)\,\pi + (x_3 + x_2)\,\pi - \frac{1}{2}\left(\dfrac{x^2 - x_4^2}{b_0 - x_4}\right)\pi, & \text{on } [x_4, b_0], \end{cases}$$

$$\eta_2(x) = \begin{cases} \left(\dfrac{x - a_0}{x_1 - a_0}\right)\pi, & \text{on } [a_0, x_1], \\ \pi, & \text{on } [x_1, x_2], \\ \pi - 2\left(\dfrac{x - x_2}{x_3 - x_2}\right)\pi, & \text{on } [x_2, x_3], \\ -\pi, & \text{on } [x_3, x_4], \\ \left(\dfrac{x - b_0}{b_0 - x_4}\right)\pi, & \text{on } [x_4, b_0], \end{cases}$$

with $x_4 = x_3 + (x_2 - x_1)$ and $b_0 = x_4 + (x_1 - a_0)$ and where x_1 and x_2 are not to be confused with the x_1 and x_2 in the statement of the theorem. One readily verifies that for each π as described the admissible variation $\eta = \eta(x)$ given by (2.5) satisfies the conditions set forth below (2.2). Hence, employing the positive semi-definiteness of the matrix $F(x)$ on $[a, \infty)$ to obtain from (2.2) the relation

$$\tag{2.6} 0 \leqslant \int_{x_1}^{x_2} \eta^* F(x)\,\eta\,dx \leqslant \int_{a_0}^{b_0} \eta^* F(x)\,\eta\,dx < \int_{a_0}^{b_0} \eta^{*\prime} G(x)\,\eta'\,dx$$

we substitute (2.3) and (2.5) in (2.6) and use the negative semi-definiteness of the matrix $P_0(x) - k_0 I$ on $[a, \infty)$ to establish the chain of relations

$$\tag{2.7} \begin{aligned} 0 &\leqslant -\int_{x_1}^{x_2} \pi^* P_1(x)\,\pi\,dx - \tfrac{1}{4}\int_{x_1}^{x_2} (2x - x_1 - a_0)^2 \pi^* P_2(x)\,\pi\,dx \\ &< \int_{a_0}^{x_1} \frac{\pi^* P_0(x)\,\pi}{(x_1 - a_0)^2}\,dx + 4\int_{x_2}^{x_3} \frac{\pi^* P_0(x)\,\pi}{(x_3 - x_2)^2}\,dx + \int_{x_4}^{b_0} \frac{\pi^* P_0(x)\,\pi}{(b_0 - x_4)^2}\,dx \\ &\leqslant \frac{k_0}{x_1 - a_0} + \frac{4k_0}{x_3 - x_2} + \frac{k_0}{b_0 - x_4} = \frac{2k_0}{x_1 - a_0} + \frac{4k_0}{x_3 - x_2} \end{aligned}$$

for each m-dimensional constant vector π satisfying $\pi^*\pi = 1$. Since x_1, x_2 and x_3 are arbitrary numbers satisfying $a_0 < x_1 < x_2 < x_3$ it is clear that we may make x_2, x_3 and $x_3 - x_2$ as large as we please. Hence, employing the symmetry

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and negative semi-definiteness of the matrices $P_1(x)$ and $P_2(x)$ on $[a, \infty)$ once more it follows from (2.7) that each of the improper matrix integrals

$$\int_{x_1}^{\infty} P_1(x)\,dx \quad \text{and} \quad \int_{x_1}^{\infty} (2x - x_1 - a_0)^2 P_2(x)\,dx \tag{2.8}$$

exists and that

$$\begin{aligned} 0 &\leqslant -\,(x_1 - a_0)\ \pi^* \int_{x_1}^{\infty} P_1(x)\,dx\ \pi \leqslant 2k_0, \\ 0 &\leqslant -\,(x_1 - a_0)\ \pi^* \int_{x_1}^{\infty} (2x - x_1 - a_0)^2 P_2(x)\,dx\ \pi \leqslant 8k_0, \end{aligned} \tag{2.9}$$

for each π satisfying $\pi^*\pi = 1$ as before. The conclusions of Theorem A now follow readily.

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