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A NOTE ON GROUPS OF REE TYPE

BY
PETER LORIMER

The nonsolvable R-groups as defined by Walter [3] are groups of orders $(q^3+1)q^3(q-1)$, $q=3^{2n+1}$, $n\geq 0$. These are the groups of Ree type discussed by Ward [4] together with the Ree group $R(3)$ of order 28.27.2. The R-group with parameter q has a doubly transitive representation of degree q^3+1 but in this note we will prove that it cannot contain a sharply doubly transitive subset. This result is of interest in the theory of projective planes for if such a subset existed in the R-group of order $(q^3+1)q^3(q-1)$ there would be a projective plane with q^3+2 points on a line, see for example [1, p. 140].

If G is a group of permutations on a finite set Σ and U is a subset of G then U is said to be sharply doubly transitive on Σ if $1 \in U$ and if whenever $a, b, c, d \in \Sigma$, $a\neq b$, $c\neq d$ there is a unique permutation $u \in U$ with $u(a)=c$, $u(b)=d$. If G has degree n such a subset has $n(n-1)$ members.

THEOREM. *Let G be an R-group of order $(q^3+1)q^3(q-1)$, $q=3^{2n+1}$, $n\geq 0$, represented as a doubly transitive group on a set Σ containing q^3+1 members. Then G does not have a subset which is sharply doubly transitive on Σ.*

Proof. Let G be an R-group as mentioned in the theorem. We will make use of the following properties of G, see [4, pp. 62–3] and [3, pp. 332–5].

(1) G has one class of involutions.

(2) A Sylow 2-subgroup of G is elementary abelian of order 8.

(3) The stabilizer of any two points of Σ in G contains a unique involution.

(4) Each involution of G fixes $q+1$ points and apart from the identity the involutions are the only members of G fixing more than 2 symbols.

(5) If t is an involution of G, $C(t)=\{1, t\}\times K$ where K is isomorphic to PSL$(2, q)$ and acts on the $q+1$ points fixed by t as PSL$(2, q)$ in its usual representation.

Suppose now that U is a sharply doubly transitive subset of G.

Let α, β be any two points of Σ, H the stabilizer of α and β in G and t the (unique) involution of H. t fixes $q+1$ points of Σ, say those in the subset Ω of Σ.

Let a, b be any two members of Ω. We show now that

$$C(t) = \{g\in G \mid g(a)\in\Omega,\ g(b)\in\Omega\}.$$

If g is any member of G, gtg^{-1} is an involution of the stabilizer of $g(a)$ and $g(b)$ in G. If $g(a), g(b) \in \Omega$, t is also an involution of this subgroup and because there is

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only one such involution we get $gtg^{-1}=t$ or $g \in C(t)$. The converse result is straightforward.

Now consider the representation of $C(t)$ as a permutation group on Ω. It is clear from the preceding paragraph that $C(t)$ is doubly transitive on Ω and that the set $U \cap C(t)$ is sharply doubly transitive on Ω. From the properties of R-groups we have $C(t)=\{1, t\} \times K$ where K is isomorphic to the group PSL$(2, q)$. Consideration of this group shows that the representation of K on Ω is the usual representation of PSL$(2, q)$. Because of this no member of K except 1 fixes more than two symbols of Ω. Hence the kernel of the representation we are considering is $\{1, t\}$ and we may take the image of the representation as PSL$(2, q)$ in its usual representation. The image of $U \cap C(t)$ is then a sharply doubly transitive subset of PSL$(2, q)$. This contradicts the result of [2] except when $q=3$.

If $q=3$, G has order 28.27.2 and every involution of G fixes four symbols. U has 28.27 members and as U is sharply doubly transitive, $r^{-1}s$ cannot be an involution for $r, s \in U$. But this is impossible as the Sylow 2-subgroups of G are elementary abelian of order 8.

This proves the theorem.

References

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University of Auckland,
Auckland, New Zealand