ON INTEGRAL FUNCTIONS HAVING PRESCRIBED ASYMPTOTIC GROWTH

J. CLUNIE

1. In this paper I shall prove the following theorem:

THEOREM. *Let $\phi(r)$ be increasing and convex in* $\log r$ *with*

$$\phi(r) \neq O(\log r) \; (r \to \infty).$$

(This condition is imposed to exclude certain trivial cases.) *Then there is an integral function $f(z)$ such that*

(i) $\log M(r,f) \sim \phi(r) \qquad (r \to \infty)$,

(ii) $T(r,f) \sim \phi(r) \qquad (r \to \infty)$.

This paper is intended to be read as a sequel to the previous one by Edrei and Fuchs and so I shall not enter into any discussion of the theorem.

I should like to express my indebtedness to Professor Edrei for stimulating my interest in the subject of this paper.

2. We assume that

$$\phi(r) = \int_1^r \frac{\psi(t)}{t}\, dt,$$

where $\psi(t)$ is continuous, strictly increasing, and unbounded with $\psi(1) = 0$. This involves no loss of generality since to any function which is increasing and convex in $\log r$ and not $O(\log r)$ $(r \to \infty)$ there corresponds a $\phi(r)$ of the above kind, to which it is asymptotic as $r \to \infty$.

First I shall construct a function for which (i) is true and later one for which both (i) and (ii) are true. Though they are similar, the first of these constructions is much simpler than the second.

Let $r_1 < r_2 < \ldots$ be the unbounded sequence defined by $\psi(r_n) = n$. We define

$$F(z) = \sum_1^\infty a_n z^n,$$

where

$$a_n = \frac{1}{r_1 r_2 \ldots r_n} \qquad (n \geqslant 1).$$

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Then $F(z)$ is an integral function, and for $r_n \leqslant r \leqslant r_{n+1}$ it is clear that

$$\mu(r, F) = a_n r^n,$$

where

$$\mu(r, F) = \max_{k \geqslant 1} a_k r^k$$

is the maximum term of $F(z)$ for $|z| = r$. Since

$$\phi(r) \neq O(\log r) \qquad (r \to \infty),$$

it is not difficult to show that

(1) $$\log \mu(r, F) \sim \phi(r) \qquad (r \to \infty).$$

Now we define a sequence of integers $\lambda_1 < \lambda_2 < \ldots$ as follows: Take $\lambda_1 = 1$ and assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ have been specified. If

$$a_{\lambda_{n+1}} r_{\lambda_{n+1}}^{\lambda_{n+1}} > 2a_{\lambda_n} r_{\lambda_n}^{\lambda_n},$$

take $\lambda_{n+1} = \lambda_n + 1$. Otherwise take λ_{n+1} to be the largest integer m for which

$$a_m r_m^m \leqslant 2a_{\lambda_n} r_{\lambda_n}^{\lambda_n}.$$

Since the sequence $\{a_n r_n^n\}$, $n = 1, \ldots, \infty$, increases strictly to ∞, the sequence $\{\lambda_n\}$, $n = 1, \ldots, \infty$, is well defined.

Put

$$f(z) = \sum_1^\infty \frac{a_{\lambda_n} z^{\lambda_n}}{n^2}.$$

It will be proved that $f(z)$ satisfies Condition (i) of the theorem.

LEMMA 1.

$$\log n = o(\log a_{\lambda_n} r_{\lambda_n}^{\lambda_n}) \qquad (n \to \infty).$$

Proof. From the construction of the λ_n it follows that

$$a_{\lambda_{n+2}} r_{\lambda_{n+2}}^{\lambda_{n+2}} > 2a_{\lambda_n} r_{\lambda_n}^{\lambda_n} \qquad (n \geqslant 1).$$

Using these inequalities, it is easy to prove that for any $\delta > 0$, the series

$$\sum_1^\infty (a_{\lambda_n} r_{\lambda_n}^{\lambda_n})^{-\delta}$$

converges. As the terms of this series decrease monotonically, we find, using a theorem of Abel, that

$$n(a_{\lambda_n} r_{\lambda_n}^{\lambda_n})^{-\delta} \to 0 \qquad (n \to \infty).$$

Since $\delta > 0$ is arbitrary, this is equivalent to the lemma.

From (1) and the next lemma the result follows.

LEMMA 2.

$$\log M(r, f) \sim \log \mu(r, F) \qquad (r \to \infty).$$

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Proof. Consider the interval $r_\nu \leqslant r \leqslant r_{\nu+1}$. If for some n we have $\nu = \lambda_n$, then

$$a_{\lambda_n} r^{\lambda_n} = \mu(r, F) \qquad (r_\nu \leqslant r \leqslant r_{\nu+1}).$$

Now, clearly,

$$a_{\lambda_m} r^{\lambda_m} \leqslant \mu(r, F) \qquad (m \geqslant 1; r \geqslant 0),$$

and so we obtain

$$\frac{\mu(r, F)}{n^2} \leqslant M(r, f) \leqslant \mu(r, F) \sum_1^\infty \frac{1}{n^2}. \tag{2}$$

From Lemma 1 we see that

$$\log n = o\{\log \mu(r, F)\} \qquad (r_\nu \leqslant r \leqslant r_{\nu+1}, r \to \infty). \tag{3}$$

Together, (2) and (3) give Lemma 2 as $r \to \infty$ through values under consideration.

Suppose now that there is no k such that $\lambda_k = \nu$, and let λ_n be the largest λ_k satisfying $\lambda_k < \nu$. Then, by construction,

$$a_{\nu+1} r_{\nu+1}^{\nu+1} \leqslant 2a_{\lambda_n} r_{\lambda_n}^{\lambda_n}.$$

Hence it follows that

$$a_{\lambda_n} r_{\lambda_n}^{\lambda_n} \leqslant \mu(r, F) \leqslant 2a_{\lambda_n} r_{\lambda_n}^{\lambda_n} \qquad (r_\nu \leqslant r \leqslant r_{\nu+1}), \tag{4}$$

and so

$$\frac{\mu(r, F)}{2n^2} \leqslant M(r, f) \leqslant \mu(r, F) \sum_1^\infty \frac{1}{n^2} \qquad (r_\nu \leqslant r \leqslant r_{\nu+1}). \tag{5}$$

From Lemma 1 and from (4) we again obtain (3). Consequently Lemma 2 follows from (5) as $r \to \infty$ through values under consideration. This completes the proof of Lemma 2.

3. Now I shall give the more complicated construction that leads to a function which satisfies both (i) and (ii).

First we define a sequence of integers $\nu_1 < \nu_2 < \ldots$ in the following manner: Take $\nu_1 = 1$ and assume that $\nu_1, \nu_2, \ldots, \nu_n$ have been specified. If

$$\log a_{\nu_n+1} r_{\nu_n+1}^{\nu_n+1} > e^{3/n} \log a_{\nu_n} r_{\nu_n}^{\nu_n},$$

take $\nu_{n+1} = \nu_n + 1$. Otherwise take ν_{n+1} to be the largest integer k such that

$$\log a_k r_k^k \leqslant e^{3/n} \log a_{\nu_n} r_{\nu_n}^{\nu_n}.$$

Since the sequence $\{a_k r_k^k\}$, $k = 1, \ldots, \infty$, increases strictly to ∞, the sequence $\{\nu_n\}$, $n = 1, \ldots, \infty$, is well defined.

LEMMA 3.

$$n = o(\log a_{\nu_n} r_{\nu_n}^{\nu_n}) \qquad (n \to \infty).$$

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Proof. By construction,

$$\log a_{\nu_{n+2}} r_{\nu_{n+2}}^{\nu_{n+2}} > e^{3/n} \log a_{\nu_n} r_{\nu_n}^{\nu_n} \qquad (n \geqslant 1).$$

From these inequalities it is not difficult to show that

$$\sum_{2}^{\infty} (\log a_{\nu_n} r_{\nu_n}^{\nu_n})^{-1}$$

converges. As the terms of this series decrease monotonically, we find, applying again the theorem of Abel previously used, that

$$n(\log a_{\nu_n} r_{\nu_n}^{\nu_n})^{-1} \to 0 \qquad (n \to \infty).$$

Hence the lemma is proved.

We now construct a sub-sequence $\{k_n\}, n = 1, \ldots, \infty$, of $\{\nu_n\}, n = 1, \ldots, \infty$, as follows. Let $\{\kappa_n\}$, $n = 1, \ldots, \infty$, be an auxiliary increasing, unbounded sequence with $\kappa_1 > 1$ and $\kappa_{n+1} \sim \kappa_n$ $(n \to \infty)$, such that

$$n \log \kappa_n = o(\log a_{\nu_n} r_{\nu_n}^{\nu_n}) \qquad (n \to \infty).$$

It is not difficult to see that there is such a sequence $\{\kappa_n\}$, $n = 1, \ldots, \infty$. Take $k_1 = \nu_1$ and assume that $k_1, k_2, \ldots, k_n$ have been defined. Let ν_m be the smallest ν_s satisfying $\nu_s > k_n$, if

$$\sum_{l=1}^{n} a_{k_l} r_{\nu_m}^{k_l} \leqslant \kappa_{n+2}^{-n-2} a_{\nu_m} r_{\nu_m}^{\nu_m}, \tag{6}$$

take $k_{n+1} = \nu_m$. Otherwise take k_{n+1} to be the largest ν_s such that

$$\sum_{l=1}^{n} a_{k_l} r_{\nu_p}^{k_l} > \kappa_{n+2}^{-n-2} a_{\nu_p} r_{\nu_p}^{\nu_p} \qquad (k_n \leqslant \nu_p \leqslant \nu_s). \tag{7}$$

One can see that the sequence $\{k_n\}$, $n = 1, \ldots, \infty$, is well defined by noting that the maximum term of an integral function grows more quickly than any power of r.

I shall now prove that

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{k_n} z^{k_n}}{\kappa_n^n}$$

satisfies (i) and (ii) of the theorem.

LEMMA 4.

$$\sum_{l=1}^{n} a_{k_l} r_{k_{n+2}}^{k_l} \leqslant \kappa_{n+2}^{-n-2} a_{k_{n+2}} r_{k_{n+2}}^{k_{n+2}} \qquad (n \geqslant 1).$$

Proof. Suppose at first that k_{n+1} is defined by (6). The left-hand side of (6) is of degree k_n and the right-hand side is of degree $k_{n+1} > k_n$ and so, since $r_{k_{n+2}} > r_{k_{n+1}}$,

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$$\sum_{l=1}^{n} a_{k_l} r_{k_n+2}^{k_l} \leqslant \kappa_{n+2}^{-n-2} a_{k_n+1} r_{k_n+2}^{k_n+1}. \tag{8}$$

By definition,

$$a_m r_{k_n+2}^{m} \leqslant a_{k_n+2} r_{k_n+2}^{k_n+2} \qquad (m \geqslant 1). \tag{9}$$

From (8) and (9) the lemma follows in this case.

Suppose now that k_{n+1} is defined by (7), and let $k_{n+1} = \nu_m$. Then

$$\sum_{l=1}^{n} a_{k_l} r_{\nu_m+1}^{k_l} \leqslant \kappa_{n+2}^{-n-2} a_{\nu_m+1} r_{\nu_m+1}^{\nu_m+1}.$$

The degree of the left-hand side of this inequality is less than that of the right. As $k_{n+2} \geqslant \nu_{m+1}$, it follows that $r_{k_n+2} \geqslant r_{\nu_m+1}$ and so

$$\sum_{l=1}^{n} a_{k_l} r_{k_n+2}^{k_l} \leqslant \kappa_{n+2}^{-n-2} a_{\nu_m+1} r_{k_n+2}^{\nu_m+1}.$$

Hence, using (9), the lemma follows in this case also.

LEMMA 5. *For* $r_{k_\nu} \leqslant r \leqslant r_{k_\nu+1}$ $(\nu \geqslant 2)$,

$$f(z) = \frac{a_{k_{\nu-1}} z^{k_{\nu-1}}}{\kappa_{\nu-1}^{\nu-1}} + \frac{a_{k_\nu} z^{k_\nu}}{\kappa_\nu^{\nu}} + \frac{a_{k_\nu+1} z^{k_\nu+1}}{\kappa_{\nu+1}^{\nu+1}} + o\{\mu(r,f)\} \qquad (r \to \infty).$$

Proof. Since

$$a_{k_\nu+1} r_{k_\nu+1}^{k_\nu+1} \geqslant a_{k_n} r_{k_\nu+1}^{k_n} \qquad (n \geqslant \nu + 1),$$

we find that for $r \leqslant r_{k_\nu+1}$,

$$\frac{\displaystyle\sum_{n=\nu+2}^{\infty} \frac{a_{k_n} r^{k_n}}{\kappa_n^{n}}}{\dfrac{a_{k_\nu+1} r^{k_\nu+1}}{\kappa_{\nu+1}^{\nu+1}}} \leqslant \frac{\displaystyle\sum_{n=\nu+2}^{\infty} \frac{a_{k_n} r_{k_\nu+1}^{k_n}}{\kappa_n^{n}}}{\dfrac{a_{k_\nu+1} r_{k_\nu+1}^{k_\nu+1}}{\kappa_{\nu+1}^{\nu+1}}} \tag{10}$$

$$\leqslant \kappa_{\nu+1}^{\nu+1} \sum_{n=\nu+2}^{\infty} \kappa_n^{-n}$$

$$\leqslant \kappa_{\nu+1}^{\nu+1} \kappa_{\nu+2}^{-\nu-2} (1 - \kappa_{\nu+2}^{-1})^{-1}$$

$$\leqslant (\kappa_{\nu+2} - 1)^{-1} = o(1) \qquad (\nu \to \infty).$$

From Lemma 4, we have

$$\sum_{l=1}^{\nu-2} a_{k_l} r_{k_\nu}^{k_l} \leqslant \kappa_\nu^{-\nu} a_{k_\nu} r_{k_\nu}^{k_\nu}.$$

Given $\epsilon > 0$, we choose $m = m(\epsilon)$ such that $\kappa_l^{-l} < \epsilon$ $(l > m)$. Next we choose $\nu_0 = \nu_0(\epsilon)$ such that when $\nu > \nu_0$,

$$\sum_{l=1}^{m} a_{k_l} r_{k_\nu}^{k_l} < \epsilon\kappa_\nu^{-\nu} a_{k_\nu} r_{k_\nu}^{k_\nu}.$$

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Hence, when $\nu > \nu_0$, we obtain

$$\sum_{l=1}^{\nu-2} \frac{a_{k_l} r_{k_\nu}^{k_l}}{\kappa_l^l} \leqslant \sum_{l=1}^{m} a_{k_l} r_{k_\nu}^{k_l} + \epsilon \sum_{m+1}^{\nu-2} a_{k_l} r_{k_\nu}^{k_l}$$

$$< 2\epsilon\kappa_\nu^{-\nu} a_{k_\nu} r_{k_\nu}^{k_\nu}.$$

Therefore, for $r \geqslant r_{k_\nu}$,

(11) $$\sum_{l=1}^{\nu-2} \frac{a_{k_\nu} r^{k_l}}{\kappa_l^l} = o(1)\kappa_\nu^{-\nu} a_{k_\nu} r^{k_\nu} \qquad (\nu \to \infty),$$

since the degree of the sum on the left is less than k_ν and its terms are positive.
From (10) and (11) the lemma follows.

LEMMA 6. *If*

$$g(z) = \sum_1^\infty a_{\nu_n} z^{\nu_n},$$

then

$$\log \mu(r, g) \sim \log \mu(r, F) \qquad (r \to \infty).$$

Proof. Consider $r_n \leqslant r \leqslant r_{n+1}$. If for some m we have $\nu_m = n$, then

$$\mu(r, g) = \mu(r, F) \qquad (r_n \leqslant r \leqslant r_{n+1}),$$

and so the lemma is true as $r \to \infty$ through such values.

Suppose that there is no s such that $\nu_s = n$, and let ν_m be the largest ν_s satisfying $\nu_s < n$. By construction,

(12) $$\log a_{n+1} r_{n+1}^{n+1} \leqslant e^{3/m} \log a_{\nu_m} r_{\nu_m}^{\nu_m}.$$

We also have

(13) $$\log a_{\nu_m} r_{\nu_m}^{\nu_m} \leqslant \log \mu(r, g) \qquad (r_n \leqslant r \leqslant r_{n+1}).$$

From (12) and (13) it follows that

$$e^{-3/m} \log \mu(r, F) \leqslant \log \mu(r, g) \qquad (r_n \leqslant r \leqslant r_{n+1}),$$

and since, clearly,

$$\log \mu(r, g) \leqslant \log \mu(r, F) \qquad (r \geqslant 0),$$

the lemma follows as $r \to \infty$ through values under consideration.

LEMMA 7.

$$\log \mu(r, f) \sim \log \mu(r, F) \qquad (r \to \infty).$$

Proof. Consider $r_{k_s} \leqslant r \leqslant r_{k_s+1}$. Suppose that $k_s = \nu_m$ and $k_{s+1} = \nu_n$. If $n = m + 1$, then

$$\log \mu(r, f) \geqslant \log \mu(r, g) - (s + 1) \log \kappa_{s+1} \qquad (r_{k_s} \leqslant r \leqslant r_{k_s+1}),$$

since in the range of r either

$$\mu(r, g) = a_{k_s} r^{k_s} \quad \text{or} \quad \mu(r, g) = a_{k_s+1} r^{k_s+1}.$$

Also for $r_{k_s} \leqslant r \leqslant r_{k_s+1}$,

$$\frac{(s+1)\log \kappa_{s+1}}{\log \mu(r, g)} \leqslant \frac{(s+1)\log \kappa_{s+1}}{\log a_{k_s} r_{k_s}^{k_s}} = o(1) \qquad (s \to \infty),$$

by Lemma 3 and the choice of the κ_n, In any case,

$$\log \mu(r, f) \leqslant \log \mu(r, g), \tag{14}$$

so that we obtain Lemma 7 as $r \to \infty$ through values under consideration.

Suppose now that $n > m + 1$. Then, by construction,

$$\sum_{l=1}^{s} a_{kl}\, r_{\nu_p}^{kl} > \kappa_{s+2}^{-s-2}\, a_{\nu_p}\, r_{\nu_p}^{\nu_p} \qquad (p = m, m+1, \ldots, n),$$

and so, as a little consideration shows,

$$\sum_{l=1}^{s} a_{kl}\, r^{kl} > \frac{\mu(r, g)}{\kappa_{s+2}^{s+2}} \qquad (r_{k_s} \leqslant r \leqslant r_{k_s+1}).$$

Hence

$$\sum_{l=1}^{s} \frac{a_{kl}\, r^{kl}}{\kappa_l^{l}} > \frac{\mu(r, g)}{\kappa_{s+2}^{s+2}\, \kappa_s^{s}} \qquad (r_{k_s} \leqslant r \leqslant r_{k_s+1}),$$

which gives, using Lemma 5,

$$(2 + o(1))\mu(r, f) > \frac{\mu(r, g)}{\kappa_{s+2}^{s+2}\, \kappa_s^{s}} \qquad (r_{k_s} \leqslant r \leqslant r_{k_s+1},\ s \to \infty).$$

Making use of Lemma 3 and the conditions on the κ_n, we obtain

$$\log \mu(r, f) > (1 + o(1)) \log \mu(r, g) \qquad (r_{k_s} \leqslant r \leqslant r_{k_s+1},\ s \to \infty).$$

Together with (14), the lemma follows as $r \to \infty$ through values under consideration.

This completes the proof of Lemma 7.

From (1) and Lemmas 5, 6, and 7, it follows that $f(z)$ satisfies Condition (i) of the theorem.

LEMMA 8. *If $h(z) = \sum b_n z^n$ is an integral function such that*

$$h(z) = b_{n_1} z^{n_1} + b_{n_2} z^{n_2} + b_{n_3} z^{n_3} + o\{\mu(r, h)\} \qquad (r \to \infty),$$

where $n_1 < n_2 < n_3$ depend on r, then

$$T(r, h) \sim \log \mu(r, h) \qquad (r \to \infty).$$

Proof. By Cauchy's inequalities for the terms of a Taylor series, it follows that for all large r each term of $h(z)$ concealed in $o\{\mu(r, h)\}$ is less than $\mu(r, h)$

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in modulus. In what follows we assume that we are dealing with r for which this is true, so that the central index of $h(z)$ is n_1, n_2, or n_3.

Let $\mu(r, h) = |b_{n_k}| r^{n_k}$, where $k = 1, 2,$ or 3. If

$$|b_{n_j}| r^{n_j} \leqslant \tfrac{1}{4}\mu(r, h) \qquad (j \neq k; j = 1, 2, 3),$$

then on $|z| = r$, when r is sufficiently large,

$$2\mu(r, h) \geqslant |h(z)| \geqslant \tfrac{1}{2}\mu(r, h).$$

Hence the result follows in this case.

Suppose now that for $j \neq k$ we have

$$|b_{n_j}| r^{n_j} > \tfrac{1}{4}\mu(r, h).$$

Then one of j and k is 1 or 3. Assume, in fact, that one of them is 1. The other case is similar. If

$$\phi(z) = b_{n_1} z^{n_1} + b_{n_2} z^{n_2} + b_{n_3} z^{n_3},$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\phi(re^{i\theta})|d\theta > \log\frac{\mu(r, h)}{4} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|1 + \frac{b_{n_2}}{b_{n_1}}(re^{i\theta})^{n_2 - n_1} + \frac{b_{n_3}}{b_{n_1}}(re^{i\theta})^{n_3 - n_1}\right| d\theta$$

$$\geqslant \log\frac{\mu(r, h)}{4}, \tag{15}$$

since the integral on the right is non-negative as can be seen by applying Jensen's theorem to

$$1 + \frac{b_{n_2}}{b_{n_1}} z^{n_2 - n_1} + \frac{b_{n_3}}{b_{n_1}} z^{n_3 - n_1}$$

with $|z| = r$. Let η $(0 < \eta < 1)$ be given, and let the set

$$E = \{\theta : |\phi(re^{i\theta})| \leqslant \eta\mu(r, h)\}$$

be of measure $2\pi\delta$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\phi(re^{i\theta})|d\theta \leqslant (1 - \delta)\log 3\mu(r, h) + \delta\log\eta\mu(r, h).$$

Together with (15) this gives

$$\log\mu(r, h) - \log 4 \leqslant (1 - \delta)\{\log\mu(r, h) + \log 3\} + \delta\{\log\mu(r, h) - \log(1/\eta)\},$$

and so

$$\delta \leqslant \frac{\log 12}{\log(1/\eta)}.$$

Put $h(z) = \phi(z) + R(z)$ and let

$$\eta = \max\left[2 \max_{|z|=r} \frac{|R(z)|}{\mu(r, h)}, \exp[-\{\log \mu(r, h)\}^{\frac{1}{2}}]\right].$$

Outside the set E we have

$$|h(z)| \geqslant |\phi(z)| - |R(z)| \geqslant \tfrac{1}{2}|\phi(z)|,$$

and so, if F is the complement of E in $(0, 2\pi)$, then

$$\begin{aligned}(16)\qquad T(r, h) &\geqslant \frac{1}{2\pi}\int_F \log |h(re^{i\theta})|d\theta \\ &\geqslant \frac{1}{2\pi}\int_F \log |\phi(re^{i\theta})|d\theta - \log 2 \\ &\geqslant (1 - \delta) \log\{\eta\mu(r, h)\} - \log 2 \\ &= (1 + o(1)) \log \mu(r, h) \qquad (r \to \infty).\end{aligned}$$

On the other hand, $|h(z)| \leqslant (3 + o(1))\mu(r, h)$ $(r \to \infty)$, so that

$$(17)\qquad T(r, h) \leqslant (1 + o(1)) \log \mu(r, h) \qquad (r \to \infty).$$

From (16) and (17), the lemma follows.

From Lemmas 5, 6, 7, and 8 it follows that $f(z)$ satisfies Condition (ii) of the theorem.

Imperial College, London