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Analysis of semilocal convergence for ameliorated super-Halley methods with less computation for inversion

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Abstract

In this paper, the semilocal convergence for ameliorated super-Halley methods in Banach spaces is considered. Different from the results in [J. M. Gutiérrez and M. A. Hernández, *Comput. Math. Appl.* 36 (1998) 1–8], these ameliorated methods do not need to compute a second derivative, the computation for inversion is reduced and the R-order is also heightened. Under a weaker condition, an existence–uniqueness theorem for the solution is proved.

1. *Introduction*

Finding the solution of nonlinear equations in Banach spaces is important in the areas of scientific and engineering computing. Such equations can be written as $F(x) = 0$, where $F : \Omega \subseteq X \to Y$ is a nonlinear operator in a non-empty open convex subset Ω, and where X and Y are Banach spaces.

The second-order Newton's method [10] is widely applied for solving this equation. Recently, third-order Chebyshev–Halley methods and some of their variants have been developed [1–9]. In reference [8], Gutiérrez and Hernández studied the convergence of super-Halley method given by

$$x_{n+1} = x_n - [I + \tfrac{1}{2}L_F(x_n)[I - L_F(x_n)]^{-1}]F'(x_n)^{-1}F(x_n), \tag{1.1}$$

where $L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x)$. By assuming that:
(A1) $\|\Gamma_0\| \leqslant \beta$;
(A2) $\|\Gamma_0 F(x_0)\| \leqslant \eta$;
(A3) $\|F''(x)\| \leqslant M$, $x \in \Omega_0$; and
(A4) $\|F''(x) - F''(y)\| \leqslant L_1\|x - y\|$, $x, y \in \Omega_0$,
where $\Omega_0 \subseteq \Omega$ is a non-empty open convex subset, $\Gamma_0 = F'(x_0)^{-1}$ exists at some $x_0 \in \Omega_0$. Gutiérrez and Hernández proved that the super-Halley method converges with R-order at least three.

In reference [6], Ezquerro and Hernández studied convergence of the Halley method given by

$$x_{n+1} = x_n - [I + \tfrac{1}{2}L_F(x_n)[I - \tfrac{1}{2}L_F(x_n)]^{-1}]F'(x_n)^{-1}F(x_n), \quad n \geqslant 0. \tag{1.2}$$

They used the assumptions that:
(B3) $\|F''(x)\| \leqslant N$, $x \in \Omega$;
(B4) $\|F''(x) - F''(y)\| \leqslant \omega(\|x - y\|)$, $x, y \in \Omega$,
where $\omega(0) \geqslant 0$, for $z > 0$, and $\omega(z)$ is a non-decreasing continuous real function; and
(B5) there exists a positive real function $\nu \in C[0, 1]$, such that $\nu(t) \leqslant 1$, $\omega(tz) \leqslant \nu(t)\omega(z)$, for $t \in [0, 1]$ and $z \in (0, +\infty)$.
Under assumptions (A1)–(A2) and (B3)–(B5), Ezquerro and Hernández proved that the Halley method is of R-order at least two. When $\omega(z) = \sum_{i=1}^{m}(L_i z^{q_i})$, they proved that

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the Halley sequence converges with R-order at least $2+q$, where $q=\min\{q_1,q_2,\dots,q_m\}$ and $q_i\in[0,1], i=1,2,\dots,m$.

Notice that the super-Halley method and Halley method need the second Fréchet derivative of an operator to be computed, but when the computational cost of F'' is large or it is hard to compute F'', the super-Halley method and Halley method are less useful. Hernández [9] studied a second-derivative-free variant for the Chebyshev method given by

$$\begin{cases} y_n = x_n - \Gamma_n F(x_n),\\ z_n = x_n + (1/2)(y_n - x_n),\\ x_{n+1} = y_n - \Gamma_n[F'(z_n) - F'(x_n)](y_n - x_n), \quad n \geqslant 0, \end{cases} \tag{1.3}$$

where $\Gamma_n = F'(x_n)^{-1}$.

Under conditions (A1)–(A4), Hernández proved that the method (1.3) converges R-cubically.

Moreover, the assumptions (A3) and (A4) are replaced by

(C3) $\|F'(x) - F'(y)\| \leqslant L_2\|x-y\|$, for all $x, y \in \Omega_0$.

Under assumptions (A1)–(A2) and (C3), Hernández studied the convergence of method (1.3).

Applying a technique similar to the one in reference [9], let $u_n = x_n - \frac{3}{4}\Gamma_n F(x_n)$, where $\Gamma_n = F'(x_n)^{-1}$. Then

$$F'(u_n) \approx F'(x_n) + F''(x_n)(u_n - x_n) = F'(x_n) - \tfrac{3}{4}F''(x_n)\Gamma_n F(x_n).$$

It shows that $L_F(x_n) \approx -\frac{4}{3}\Gamma_n[F'(u_n) - F'(x_n)]$. Define $K(x_n) = \Gamma_n[F'(u_n) - F'(x_n)]$. Replacing $L_F(x_n)$ by $-\frac{4}{3}K(x_n)$ in the super-Halley method gives

$$x_{n+1} = x_n - [I - \tfrac{2}{3}K(x_n)[I + \tfrac{4}{3}K(x_n)]^{-1}]F'(x_n)^{-1}F(x_n). \tag{1.4}$$

Notice that the method in (1.4) needs the computation of the inversion for operator $I + \frac{4}{3}K(x_n)$. Generally, the computational cost of inversion for this operator is large. Apply $I - \frac{4}{3}K(x_n)$ to approximate $[I + \frac{4}{3}K(x_n)]^{-1}$ in the method given by (1.4). Then

$$x_{n+1} = x_n - [I - \tfrac{2}{3}K(x_n) + \tfrac{8}{9}K(x_n)^2]F'(x_n)^{-1}F(x_n). \tag{1.5}$$

To improve R-order, and also to reduce the computation for the inversion and the second derivative, we consider the semilocal convergence for ameliorated super-Halley methods in Banach spaces

$$\begin{cases} z_n = x_n - [I - \frac{2}{3}K(x_n) + K(x_n)^2\Phi(K(x_n))]\Gamma_n F(x_n),\\ x_{n+1} = z_n - [I - \frac{4}{3}K(x_n) + K(x_n)^\delta]\Gamma_n F(z_n), \end{cases} \tag{1.6}$$

where $n \geqslant 0$, $K(x_n) = \Gamma_n[F'(u_n) - F'(x_n)]$, $\Gamma_n = F'(x_n)^{-1}$, $u_n = x_n - \frac{3}{4}\Gamma_n F(x_n)$ and $\delta \geqslant 2$. In the methods in (1.6), Φ is an operator which does not need to compute other inversions except $F'(x_n)^{-1}$. Moreover, there exists a real non-negative and non-decreasing continuous function $\chi(t)$ such that $\|\Phi(K(x_n))\| \leqslant \chi(\|K(x_n)\|)$ and $\chi(t)$ is bounded for $t \in (0, s^*)$, where s^* will be defined in § 2. Obviously, the methods (1.6) do not need to compute the second derivative. Under the conditions (A1)–(A4), the R-order for the methods in (1.6) can reach to five, which is higher than for the super-Halley method, the Halley method and the method given in (1.3).

To relax the assumptions (A3) and (C3), consider

(D3) $\|F'(x) - F'(y)\| \leqslant L\|x-y\|^q$, $0 < q \leqslant 1$, $x, y \in \Omega_0$, $L > 0$.

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Obviously, condition (D3) is weaker than assumption (A3) and (C3). Under conditions (A1)–(A2) and (D3), we analyze the semilocal convergence of the methods in (1.6). Moreover, we prove a convergence theorem to show the existence and uniqueness of the solution.

Apply a condition similar to the one in reference [6], and consider the condition
(E4) $\|F''(x) - F''(y)\| \leqslant \omega(\|x - y\|)$, x, $y \in \Omega_0$,
where, for $s > 0$, $\omega(s)$ is a non-decreasing continuous real function that satisfies $\omega(0) \geqslant 0$, $\omega(ts) \leqslant t^q \omega(s)$ for $t \in [0, 1], s \in (0, +\infty)$ and $q \in (0, 1]$.

Obviously, the condition (E4) generalizes (A4) by choosing $\omega(s) = L_1 s$. Under the conditions (A1)–(A3) and (E4), the R-order for methods (1.6) is proved to be at least $3 + 2q$.

The semilocal convergence analysis here is different from the local convergence studied in references [2, 3]. The local convergence requires the assumptions around a solution, whereas the semilocal convergence needs the conditions around an initial point. In references [2, 3], to establish the local convergence for Chebyshev–Halley-type methods, two of the required assumptions are as listed below.
($\mathscr{B}1$) There exists $x^* \in \Omega$ such that $F(x^*) = 0$, $F'(x^*)^{-1}$ exists.
($\mathscr{B}2$) $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leqslant L_3\|x - y\|$, for all $x, y \in \Omega$, where $L_3 > 0$.
On the one hand, for the equations for which solutions are hard to compute, it is difficult to test the assumptions ($\mathscr{B}1$) and ($\mathscr{B}2$). On the other hand, for the equations for which solutions are easy to find, there are some equations that cannot satisfy the assumption ($\mathscr{B}2$), whereas (D3) can be satisfied, for example $f(x) = x^{3/2} - 1.03 = 0$, $x \in [0, 2]$. A solution of this equation is $x^* = 1.03^{2/3}$. Notice that $f'(x) = \frac{3}{2}x^{1/2}$. Then

$$|f'(x^*)^{-1}(f'(x) - f'(y))| = \frac{1}{\sqrt[3]{1.03}}|x^{1/2} - y^{1/2}| = \frac{1}{2\sqrt[3]{1.03}}\frac{1}{\sqrt{z}}|x - y|,$$

where $z \in (y, x)$ for $y < x$; $z \in (x, y)$ for $y > x$. If $x \to 0$ and $y \to 0$, then $(1/\sqrt{z}) \to +\infty$. Therefore the assumption ($\mathscr{B}2$) cannot be satisfied. Choosing $x_0 = 1$, it follows that if $|f(x_0)| = 0.03$, $|f'(x_0)^{-1}| = \frac{2}{3} \equiv \beta$, $|f'(x_0)^{-1}f(x_0)| \leqslant 0.02 \equiv \eta$ and $|f'(x) - f'(y)| \leqslant \frac{3}{2}|x - y|^{1/2}$, then the conditions (A1)–(A2) and (D3) are satisfied, where $L = \frac{3}{2}$ and $q = \frac{1}{2}$. Moreover, choosing $\Phi = 0, \delta = 9, \Omega_0 = (0, 2)$, it can be tested that all the conditions of Theorem 1 can be satisfied.

2. *Preliminary results*

Define X and Y as Banach spaces, $B(x, r) = \{y \in X : \|y - x\| < r\}$ and $\overline{B(x, r)} = \{y \in X : \|y - x\| \leqslant r\}$. Let the nonlinear operator $F : \Omega \subseteq X \to Y$ be Fréchet differentiable in a non-empty open and convex subset $\Omega_0 \subseteq \Omega$. Choose $x_0 \in \Omega_0$ and, moreover, suppose that the conditions (A1)–(A2) and (D3) hold.

Define the functions

$$h(t) = g(t) + [1 + (3/4)^{q-1}t + ((3/4)^q t)^\delta]\varphi_1(t), \tag{2.1}$$

$$p(t) = \left[\frac{1}{1 - h(t)^q t}\right]^{1/q}, \tag{2.2}$$

$$\begin{aligned}\varphi_2(t) = t[(3/4)^{q-1} + (3/4)^{\delta q}t^{\delta-1}]\varphi_1(t) + g(t)^q t[1 + (3/4)^{q-1}t + ((3/4)^q t)^\delta]\varphi_1(t) \\ + \frac{t}{q+1}[1 + (3/4)^{q-1}t + ((3/4)^q t)^\delta]^{1+q}\varphi_1(t)^{1+q},\end{aligned} \tag{2.3}$$

where

$$g(t) = 1 + (2/3)(3/4)^q t + (3/4)^{2q}t^2\chi((3/4)^q t),$$
$$\varphi_1(t) = (2/3)(3/4)^q t + (3/4)^{2q}t^2\chi((3/4)^q t) + \frac{t}{q+1}g(t)^{1+q}.$$

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Let $\xi(t) = h(t)^q t - 1$. Since $\xi(0) = -1 < 0$ and $\xi(1) > 0$, it follows that $\xi(t) = 0$ has at least a root in $(0, 1)$. Let s^* be the smallest positive root of $h(t)^q t - 1 = 0$. Then $s^* < 1$.

LEMMA 1. *Let the functions h, p and φ_2 be defined as in (2.1)–(2.3). Then:*
(a) *$h(t)$ and $p(t)$ are increasing, $h(t) > 1$, $p(t) > 1$ for $t \in (0, s^*)$;*
(b) *for $t \in (0, s^*)$, $\varphi_2(t)$ is increasing; and*
(c) *$h(\theta^q t) < h(t)$, $p(\theta^q t) < p(t)$, $\varphi_2(\theta^q t) < \theta^{2q}\varphi_2(t)$ for $t \in (0, s^*)$ and $0 < \theta < 1$.*

Define the sequences

$$\eta_{n+1} = d_n \eta_n, \tag{2.4}$$

$$\beta_{n+1} = p(a_n)^q \beta_n, \tag{2.5}$$

$$a_{n+1} = L\beta_{n+1}\eta_{n+1}^q, \tag{2.6}$$

$$d_{n+1} = p(a_{n+1})^q \varphi_2(a_{n+1}), \tag{2.7}$$

where $n \geqslant 0$. Choose $\eta_0 = \eta$, $\beta_0 = \beta$, $a_0 = L\beta\eta^q$ and $d_0 = p(a_0)^q\varphi_2(a_0)$. Then, from the definition of a_{n+1} and (2.4)–(2.5), it follows that

$$a_{n+1} = [p(a_n)d_n]^q a_n. \tag{2.8}$$

LEMMA 2. *If*

$$a_0 < s^* \quad \text{and} \quad p(a_0)d_0 < 1, \tag{2.9}$$

then, for $n \geqslant 0$:
(a) *$p(a_n) > 1$, $d_n < 1$;*
(b) *the sequences $\{\eta_n\}, \{a_n\}, \{d_n\}$ are decreasing; and*
(c) *$h(a_n)^q a_n < 1$ and $p(a_n)d_n < 1$.*

3. *Analysis for semilocal convergence*

Since Γ_0 exists, from the definition of u_0, it follows that u_0 exists and

$$\|u_0 - x_0\| \leqslant \tfrac{3}{4}\eta_0. \tag{3.1}$$

Then $u_0 \in B(x_0, R\eta)$, where $R = h(a_0)/(1 - d_0)$.
Moreover,

$$\|K(x_0)\| \leqslant L\beta_0\|u_0 - x_0\|^q \leqslant (3/4)^q L\beta_0\eta_0^q = (3/4)^q a_0, \tag{3.2}$$

$$\|z_0 - x_0\| \leqslant g(a_0)\|\Gamma_0 F(x_0)\| \tag{3.3}$$

and

$$\|x_1 - z_0\| \leqslant [1 + (3/4)^{q-1}a_0 + ((3/4)^q a_0)^\delta]\beta_0\|F(z_0)\|. \tag{3.4}$$

Since

$$\begin{aligned} F(z_n) = &\frac{2}{3}[F'(u_n) - F'(x_n)]\Gamma_n F(x_n) - [F'(u_n) - F'(x_n)]K(x_n)\Phi(K(x_n))\Gamma_n F(x_n) \\ &+ \int_0^1 [F'(x_n + t(z_n - x_n)) - F'(x_n)](z_n - x_n)\,dt, \end{aligned} \tag{3.5}$$

then

$$\|F(z_0)\| \leqslant \left[\frac{2}{3}\left(\frac{3}{4}\right)^q + \left(\frac{3}{4}\right)^{2q} a_0\chi((3/4)^q a_0) + \frac{1}{q+1}g(a_0)^{1+q}\right] L\eta_0^q\|\Gamma_0 F(x_0)\| \tag{3.6}$$

and

$$\beta_0\|F(z_0)\| \leqslant \varphi_1(a_0)\|\Gamma_0 F(x_0)\|. \tag{3.7}$$

Furthermore,

$$\|x_1 - x_0\| \leqslant \|x_1 - z_0\| + \|z_0 - x_0\| \leqslant h(a_0)\|\Gamma_0 F(x_0)\| \leqslant h(a_0)\eta_0. \tag{3.8}$$

Since $d_0 > 0$ and if we assume that $d_0 < 1/p(a_0) < 1$, then $x_1 \in B(x_0, R\eta)$.

Notice that as $a_0 < s^*$ and $h(a_0)^q < h(s^*)^q$,

$$\|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leqslant L\beta_0\|x_1 - x_0\|^q \leqslant h(a_0)^q a_0 < 1.$$

By the Banach lemma, it follows that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\begin{aligned}\|\Gamma_1\| &\leqslant \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\|\|F'(x_0) - F'(x_1)\|} \\ &\leqslant \frac{\|\Gamma_0\|}{1 - h(a_0)^q a_0} = p(a_0)^q\|\Gamma_0\| \\ &\leqslant p(a_0)^q\beta_0 = \beta_1.\end{aligned} \tag{3.9}$$

Then u_1 is well defined.

$$\begin{aligned}F(x_{n+1}) = {}& \frac{4}{3}[F'(u_n) - F'(x_n)]\Gamma_n F(z_n) \\ &+ [F'(z_n) - F'(x_n)](x_{n+1} - z_n) - [F'(u_n) - F'(x_n)]K(x_n)^{\delta-1}\Gamma_n F(z_n) \\ &+ \int_0^1 [F'(z_n + t(x_{n+1} - z_n)) - F'(z_n)](x_{n+1} - z_n)\,dt.\end{aligned} \tag{3.10}$$

Then

$$\begin{aligned}\|F(x_1)\| \leqslant {}& [4/3 + ((3/4)^q a_0)^{\delta-1}]L\|u_0 - x_0\|^q\beta_0\|F(z_0)\| \\ &+ L\|z_0 - x_0\|^q\|x_1 - z_0\| + \frac{1}{q+1}L\|x_1 - z_0\|^{1+q}.\end{aligned} \tag{3.11}$$

Moreover,

$$\beta_0\|F(x_1)\| \leqslant \varphi_2(a_0)\|\Gamma_0 F(x_0)\|. \tag{3.12}$$

From (3.9) and (3.12), it follows that

$$\begin{aligned}\|u_1 - x_1\| &= \|-\tfrac{3}{4}\Gamma_1 F(x_1)\| < \|\Gamma_1\|\|F(x_1)\| \\ &\leqslant p(a_0)^q\varphi_2(a_0)\|\Gamma_0 F(x_0)\| \leqslant p(a_0)^q\varphi_2(a_0)\eta_0 \\ &= d_0\eta_0 = \eta_1.\end{aligned} \tag{3.13}$$

Since $h(a_0) > 1$, then

$$\begin{aligned}\|u_1 - x_0\| &\leqslant \|u_1 - x_1\| + \|x_1 - x_0\| \\ &< (h(a_0) + d_0)\eta_0 < h(a_0)(1 + d_0)\eta_0 < R\eta,\end{aligned} \tag{3.14}$$

which shows that $u_1 \in B(x_0, R\eta)$.

In addition,

$$L\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^q \leqslant [p(a_0)d_0]^q a_0 = a_1. \tag{3.15}$$

Applying induction, it can be proved that $\Gamma_{n+1} = [F'(x_{n+1})]^{-1}$ exists and that:

(I) $\|\Gamma_{n+1}\| \leqslant p(a_n)^q\|\Gamma_n\| \leqslant \beta_{n+1}$;

(II) $\|\Gamma_{n+1}F(x_{n+1})\| \leqslant p(a_n)^q\varphi_2(a_n)\|\Gamma_n F(x_n)\| \leqslant \eta_{n+1}$;

(III) $L\|\Gamma_n\|\|\Gamma_n F(x_n)\|^q \leqslant a_n$;

(IV) $\|u_n - x_n\| = \|-\frac{3}{4}\Gamma_n F(x_n)\| < \|\Gamma_n F(x_n)\|$;
(V) $\|z_n - x_n\| \leqslant g(a_n)\|\Gamma_n F(x_n)\|$;
(VI) $\|x_{n+1} - x_n\| \leqslant h(a_n)\|\Gamma_n F(x_n)\| \leqslant h(a_n)\eta_n$,
where $n \geqslant 0$.

LEMMA 3. *Let the assumptions of Lemma 2 and conditions* (A1)–(A2), (D3) *hold. Then, for* $n \geqslant 0$, u_n, z_n *and* x_{n+1} *belong to* $B(x_0, R\eta)$, *where* $R = h(a_0)/(1 - d_0)$.

To prove Lemma 3, we need to apply the following lemma.

LEMMA 4. *Under the assumptions of Lemma 2, let* $\gamma = p(a_0)d_0$ *and* $\lambda = 1/p(a_0)$. *Then*

$$\prod_{i=0}^{n} d_i \leqslant \lambda^{n+1}\gamma^{((1+2q)^{n+1}-1)/2q}, \tag{3.16}$$

$$\sum_{i=n}^{n+m} \eta_i \leqslant \eta\lambda^n\gamma^{((1+2q)^n-1)/2q}\frac{1 - \lambda^{m+1}\gamma^{(1+2q)^n((1+2q)^m+2q-1)/2q}}{1 - \lambda\gamma^{(1+2q)^n}}, \quad n \geqslant 0, m \geqslant 1. \tag{3.17}$$

Proof. Since $a_1 = \gamma^q a_0$, from Lemma 1,

$$d_1 < p(a_0)^q\varphi_2(\gamma^q a_0) < \gamma^{2q}d_0 = \gamma^{(1+2q)^1-1}d_0 = \lambda\gamma^{(1+2q)^1}.$$

Suppose that $d_k \leqslant \lambda\gamma^{(1+2q)^k}$, $k \geqslant 1$. By Lemma 2, it follows that $a_{k+1} < a_k$ and $p(a_k)d_k < 1$. Then

$$d_{k+1} < p(a_k)^q\varphi_2((p(a_k)d_k)^q a_k) < p(a_0)^{2q}d_k^{(1+2q)} \leqslant \lambda\gamma^{(1+2q)^{k+1}}.$$

Therefore $d_n \leqslant \lambda\gamma^{(1+2q)^n}$, where $n \geqslant 0$. Furthermore, (3.16) holds. From (2.4) and (3.16), it follows that

$$\eta_n = \eta\left(\prod_{j=0}^{n-1} d_j\right) \leqslant \eta\lambda^n\gamma^{((1+2q)^n-1)/2q}, \quad n \geqslant 1.$$

Since $\eta_0 = \eta$, then, for $n \geqslant 0$, $\eta_n \leqslant \eta\lambda^n\gamma^{((1+2q)^n-1)/2q}$. Moreover, (3.17) can be obtained. □

Next we prove Lemma 3.

Proof. For $n = 0$, $\|u_0 - x_0\| = \|-\frac{3}{4}\Gamma_0 F(x_0)\| < R\eta$. When $n \geqslant 1$, from (IV) and (3.17), it follows that

$$\begin{aligned}\|u_n - x_0\| &\leqslant \|u_n - x_n\| + \sum_{i=0}^{n-1}\|x_{i+1} - x_i\| \\ &< \eta_n + \sum_{i=0}^{n-1} h(a_i)\eta_i \leqslant h(a_0)\sum_{i=0}^{n}\eta\lambda^i\gamma^{((1+2q)^i-1)/2q} \\ &\leqslant h(a_0)\eta\frac{1 - \lambda^{n+1}\gamma^{((1+2q)^n+2q-1)/2q}}{1 - d_0} < R\eta.\end{aligned}$$

Then, for $n \geqslant 0$, u_n belong to $B(x_0, R\eta)$. Similarly, z_n and x_{n+1} all belong to $B(x_0, R\eta)$. □

THEOREM 1. *Let the nonlinear operator* $F : \Omega \subseteq X \to Y$ *be Fréchet differentiable in a non-empty open and convex subset* $\Omega_0 \subseteq \Omega$, *where* X *and* Y *are Banach spaces. Assume that* $x_0 \in \Omega_0$ *and all conditions* (A1)–(A2) *and* (D3) *hold. Let* $a_0 = L\beta\eta^q$ *and* $d_0 = p(a_0)^q\varphi_2(a_0)$ *satisfy* $a_0 < s^*$ *and* $p(a_0)d_0 < 1$. *Then, starting at* x_0, *the sequence* $\{x_n\}$ *generated from*

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methods (1.6) *converges to a solution* x^* *for* $F(x) = 0$, *where* x_n, x^* *belong to* $\overline{B(x_0, R\eta)}$, x^* *is the unique solution in* $B(x_0, r^*) \cap \Omega_0$ *and* r^* *is the biggest positive root of the following equation with variable* z

$$L\beta \int_{R\eta}^{z} y^q \, dy = z - R\eta.$$

Furthermore, an error estimate is given by

$$\|x_n - x^*\| \leqslant h(a_0)\eta\lambda^n \gamma^{((1+2q)^n - 1)/2q} \frac{1}{1 - \lambda\gamma^{(1+2q)^n}}, \tag{3.18}$$

where $\gamma = p(a_0)d_0$ *and* $\lambda = 1/p(a_0)$.

Proof. From Lemma 3, it follows that the sequence $\{x_n\}$ is well defined in $\overline{B(x_0, R\eta)}$. For $n \geqslant 0, m \geqslant 1$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leqslant \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \leqslant h(a_0) \sum_{i=n}^{n+m-1} \eta_i \\ &\leqslant h(a_0)\eta\lambda^n\gamma^{((1+2q)^n-1)/2q} \frac{1 - \lambda^m \gamma^{((1+2q)^n((1+2q)^{m-1}+2q-1))/2q}}{1 - \lambda\gamma^{(1+2q)^n}}. \end{aligned} \tag{3.19}$$

Then there exists a x^* such that $\lim_{n\to\infty} x_n = x^*$.

Let $n = 0, m \to +\infty$ in (3.19). It follows that

$$\|x^* - x_0\| \leqslant R\eta. \tag{3.20}$$

Then $x^* \in \overline{B(x_0, R\eta)}$.

From (3.10),

$$\begin{aligned} \|F(x_{n+1})\| \leqslant\ & [(3/4)^{q-1} + (3/4)^{\delta q} a_0^{\delta-1}]\varphi_1(a_0)L\eta_n^{1+q} \\ &+ g(a_0)^q[1 + (3/4)^{q-1}a_0 + ((3/4)^q a_0)^\delta]\varphi_1(a_0)L\eta_n^{1+q} \\ &+ \frac{\varphi_1(a_0)^{1+q}}{q+1}[1 + (3/4)^{q-1}a_0 + ((3/4)^q a_0)^\delta]^{1+q} L\eta_n^{1+q}. \end{aligned} \tag{3.21}$$

Let $n \to +\infty$ in (3.21). Then $\|F(x_{n+1})\| \to 0$ since $\eta_n \to 0$. By the continuity for $F(x)$ in Ω_0, one knows that $F(x^*) = 0$. □

Next we prove the uniqueness of x^* in $B(x_0, r^*) \cap \Omega_0$.

Let $x^{**} \in B(x_0, r^*) \cap \Omega_0$ and $F(x^{**}) = 0$. Then

$$\int_0^1 F'((1-t)x^* + tx^{**})\, dt(x^{**} - x^*) = F(x^{**}) - F(x^*) = 0. \tag{3.22}$$

Since

$$\begin{aligned} \|\Gamma_0\| \left\| \int_0^1 [F'((1-t)x^* + tx^{**}) - F'(x_0)]\, dt \right\| &\leqslant L\beta \int_0^1 \|(1-t)(x^* - x_0) + t(x^{**} - x_0)\|^q \, dt \\ &< L\beta \int_0^1 \|(1-t)R\eta + tr^*\|^q \, dt \\ &= L\beta \int_0^1 \|R\eta + t(r^* - R\eta)\|^q \, dt = 1, \end{aligned} \tag{3.23}$$

by the Banach lemma, it follows that $\int_0^1 F'((1-t)x^* + tx^{**})\, dt$ is invertible. So $x^{**} = x^*$.

Let $m \to +\infty$ in (3.19). Then (3.18) can be obtained.

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4. *R-order of convergence for methods* (1.6)

Suppose that the nonlinear operator $F:\Omega\subseteq X\to Y$ is twice Fréchet differentiable in a non-empty open and convex subset $\Omega_0\subseteq\Omega$, where X and Y are Banach spaces. Let all of the conditions (A1)–(A3) and (E4) hold.

Define the functions as

$$\widetilde{h}(u)=\widetilde{g}(u)+[1+u+(3u/4)^{\delta}]\widetilde{\varphi}_1(u), \tag{4.1}$$

$$\widetilde{p}(u)=\frac{1}{1-\widetilde{h}(u)u}, \tag{4.2}$$

$$\begin{aligned}\psi_2(u,v)=&\left[\frac{3^q}{(q+1)4^q}v+(3u/4)^{\delta}(1+u)+u^2\right]\psi_1(u,v)\\&+\frac{v}{q+1}[1+u+(3u/4)^{\delta}]\psi_1(u,v)+\frac{u^2}{2}\left[1+\frac{9u}{8}\chi(3u/4)\right][1+u+(3u/4)^{\delta}]\psi_1(u,v)\\&+\frac{u}{2}[1+u+(3u/4)^{\delta}]^2\psi_1(u,v)^2,\end{aligned} \tag{4.3}$$

where

$$\widetilde{g}(u)=1+\frac{u}{2}+\frac{9u^2}{16}\chi(3u/4),$$

$$\widetilde{\varphi}_1(u)=\frac{u}{2}+\frac{9u^2}{16}\chi(3u/4)+\frac{u}{2}\widetilde{g}(u)^2,$$

$$\begin{aligned}\psi_1(u,v)=&\frac{v}{(q+1)(q+2)}+\frac{3^q}{2(q+1)4^q}v+\frac{9u^2}{16}\chi(3u/4)\\&+\frac{u^2}{2}\left[1+\frac{9u}{8}\chi(3u/4)\right]+\frac{u^3}{8}\left[1+\frac{9u}{8}\chi(3u/4)\right]^2.\end{aligned} \tag{4.4}$$

Let $\theta\in(0,1)$. Then $\psi_2(\theta u,\theta^{1+q}v)<\theta^{2+2q}\psi_2(u,v)$ for $u\in(0,\widetilde{s}),v>0$, where $\widetilde{s}$ is the smallest positive root of $\widetilde{h}(t)t-1=0$.

Define $\widetilde{\eta}_0=\eta$, $\widetilde{\beta}_0=\beta$, $b_0=M\beta\eta$, $c_0=\beta\eta\omega(\eta)$ and $\widetilde{d}_0=\widetilde{p}(b_0)\psi_2(b_0,c_0)$. Moreover, let

$$\widetilde{\eta}_{n+1}=\widetilde{d}_n\widetilde{\eta}_n,\quad \widetilde{\beta}_{n+1}=\widetilde{p}(b_n)\widetilde{\beta}_n, \tag{4.5}$$

$$b_{n+1}=M\widetilde{\beta}_{n+1}\widetilde{\eta}_{n+1},\quad c_{n+1}=\widetilde{\beta}_{n+1}\widetilde{\eta}_{n+1}\omega(\widetilde{\eta}_{n+1}), \tag{4.6}$$

$$\widetilde{d}_{n+1}=\widetilde{p}(b_{n+1})\psi_2(b_{n+1},c_{n+1}), \tag{4.7}$$

where $n\geqslant 0$. From the definitions of b_{n+1},c_{n+1} and equations (4.5), it follows that

$$b_{n+1}=\widetilde{p}(b_n)\widetilde{d}_nb_n,\quad c_{n+1}\leqslant\widetilde{p}(b_n)\widetilde{d}_n^{1+q}c_n. \tag{4.8}$$

Similarly to the derivation in § 3, under the conditions (A1)–(A3) and (E4), the semilocal convergence for methods (1.6) can be analyzed. Furthermore, an *a priori* error estimate can be given by

$$\|x_n-x^*\|\leqslant\frac{\widetilde{h}(b_0)\eta}{\widetilde{\gamma}^{1/(2+2q)}(1-\widetilde{d}_0)}(\widetilde{\gamma}^{1/(2+2q)})^{(3+2q)^n}, \tag{4.9}$$

where $\widetilde{\gamma}=\widetilde{p}(b_0)\widetilde{d}_0$, $\widetilde{\lambda}=1/\widetilde{p}(b_0)$. From (4.9), one knows that under the conditions (A1)–(A3) and (E4) the methods (1.6) have, at least, R-order $3+2q$. When $q=1$, the R-order becomes five.

5. *Numerical results*

EXAMPLE 1. Consider a nonlinear integral equation given by

$$x(s) = 1 + 1.6 \int_0^1 G(s,t)x(t)^{3/2}\, dt, \quad s \in [0,1], \tag{5.1}$$

where

$$G(s,t) = \begin{cases} (1-s)t & t \leqslant s, \\ s(1-t) & s \leqslant t, \end{cases}$$

$x \in C[0,1]$, $t \in [0,1]$. Finding the solution of equation (5.1) is equivalent to solving $F(x) = 0$, where $F : \Omega \subseteq C[0,1] \to C[0,1]$ and

$$[F(x)](s) = x(s) - 1 - 1.6 \int_0^1 G(s,t)x(t)^{3/2}\, dt, \quad s \in [0,1]. \tag{5.2}$$

Choose $\Omega_0 = \{x \in B(0,2); x \geqslant 0\}$, $\Phi = 0$ and $\delta = 4$. The Fréchet derivatives for F are given by

$$[F'(x)y](s) = y(s) - 2.4 \int_0^1 G(s,t)x(t)^{1/2}y(t)\, dt, \quad y \in \Omega_0,$$
$$[F''(x)yz](s) = -1.2 \int_0^1 G(s,t)x(t)^{-1/2}y(t)z(t)\, dt, \quad y,z \in \Omega_0.$$

Note that F'' can not satisfy assumptions (A3) and (C3), whereas condition (D3) can be satisfied. Because

$$\|F'(x) - F'(v)\| \leqslant \tfrac{3}{10}\|x - v\|^{1/2}, \quad x, v \in \Omega_0,$$

$L = \frac{3}{10}$, $q = \frac{1}{2}$. Choosing the function $x_0(t) = 1$ as the initial approximate solution, it follows that

$$\|F(x_0)\| = 0.2, \quad \|\Gamma_0\| \leqslant \tfrac{10}{7} \equiv \beta, \quad \|\Gamma_0 F(x_0)\| \leqslant \tfrac{2}{7} \equiv \eta.$$

Here, the max norm is applied. Moreover, $a_0 = 0.229\ldots$, since $h(a_0)^q a_0 < 1$, and then $a_0 < s^*$. Notice that $p(a_0)d_0 = 0.599\ldots < 1$ and $R\eta < 1$, and then $\overline{B(x_0, R\eta)} \subset \Omega_0$. As a result, the conditions of Theorem 1 are satisfied.

EXAMPLE 2. Consider the minimizer of the chained Rosenbrock function [**11**]

$$C(\mathbf{x}) = \sum_{i=1}^{m} [4(x_i - x_{i+1}^2)^2 + (1 - x_{i+1})^2], \quad \mathbf{x} \in \mathbb{R}^m.$$

To achieve the minimum of C, one needs to solve the nonlinear system $F(\mathbf{x}) = 0$, where $F(\mathbf{x}) = \nabla C(\mathbf{x})$. Here, we apply the methods of (1.6) with $\Phi(K(x_n)) = 2.5$ and $\delta = 5$ (PM). Moreover, PM is compared with Halley method (HM), the super-Halley method (SHM) and the method (1.3) (VCM). Choose $m = 10$ and $\mathbf{x}_0 = (1.2, 1.2, \ldots, 1.2)^T$ as the initial value for all methods tested. In Table 1, the iteration errors $\|\mathbf{x}_n - \mathbf{x}^*\|_2$ of the compared methods are listed, where $\mathbf{x}^* = (1, 1, \ldots, 1)^T$ is the exact solution.

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TABLE 1. *The iteration errors for different methods.*

Iteration	HM	SHM	VCM	PM
0	1.4142e+00	1.4142e+00	1.4142e+00	1.4142e+00
1	2.2666e−01	6.0105e−02	3.3600e−01	5.2918e−02
2	4.5064e−03	1.9130e−05	1.7245e−02	2.0789e−06
3	2.0487e−08	1.4649e−14	4.6991e−06	0

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