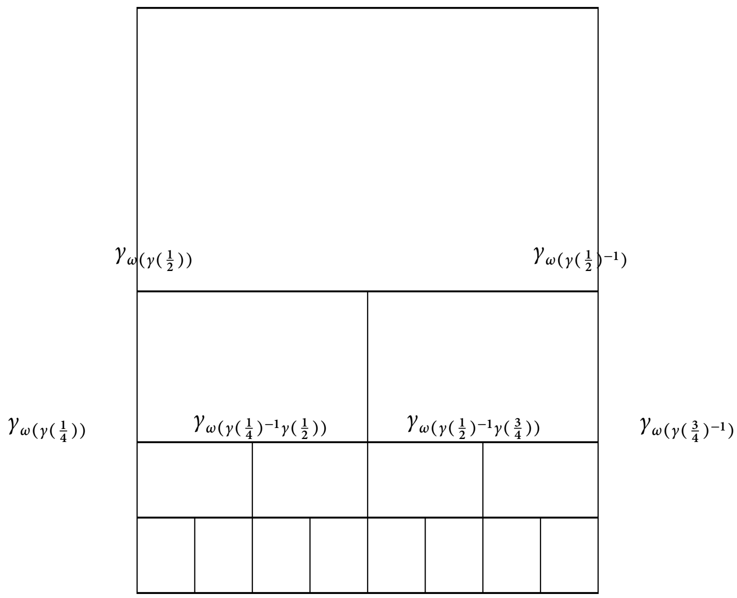

Figure 2: This figure indicates how to fill $\gamma = \gamma_w$ for an arbitrary relation $w \in \mathcal{S}^*$, given that one knows how to fill ω-triangles. The top edge and all the vertical edges are taken to be constant, and each horizontal edge is understood to be an appropriate translate of its label, so that each rectangle represents an ω-triangle, except the bottom row. The bottom edge is taken to be γ_w, and there is a row of squares along the bottom that can be filled by Proposition 2.6

4.1 Standard Solvable Groups

Observe that A acts on the abelianization $U/[U,U]$. Fix a norm on the vector space $U/[U,U]$. If there is no vector X such that $\lim_{n\to\infty} \frac{1}{n} \log \|a^n \cdot X\| \to 0$, for all $a \in A$, we say that G is standard solvable.

In this section we will be interested in the structure and geometry of standard solvable groups. Section 4.2 will describe the so-called standard tame subgroups of a standard solvable group. Lemma 4.9 will show how to find Lipschitz fillings for words that already represent the identity in the free product of the standard tame subgroups. Theorem 4.12, quoted from [4], will give conditions under which G can be presented as the free product of its standard tame subgroups modulo certain easily understood amalgamation relations.

4.2 Weights

Let $G = U \rtimes A$ be standard solvable, and let $\mathfrak{u}$ be the Lie algebra of U, identified as usual with the tangent space of U at 1_U, and fix any norm $\|\cdot\|$ on $\mathfrak{u}$. For $a \in A$, we denote the conjugation action of a on $\mathfrak{u}$ by $\mathrm{Ad}(a)$, so that $\mathrm{Ad}(a)X = \frac{d}{dt}|_{t=0} a^{-1}\exp(tX)a$. Observe that $\mathrm{Hom}(A, \mathbb{R})$ is a vector space.

Definition 4.1 ([4, §4.B]) For a homomorphism $\alpha: A \to \mathbb{R}$, define the α-weight space $\mathfrak{u}_\alpha \subset \mathfrak{u}$ to consist of 0 together with all $X \in \mathfrak{u}$ such that for all $a \in A$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(a)^n X\| = \alpha(a).$$

Define the set of weights $\mathcal{W}$ to consist of all $\alpha \in \mathrm{Hom}(A, \mathbb{R})$ for which $\dim \mathfrak{u}_\alpha > 0$. By a conic subset, we mean the intersection of $\mathcal{W}$ with an open, convex cone in $\mathrm{Hom}(A, \mathbb{R})$ that does not contain 0. Denote the set of all conic subsets by $\mathcal{C}$. For $C \in \mathcal{C}$, let U_C denote the closed connected subgroup of U whose Lie algebra is $\bigoplus_{\alpha\in C} \mathfrak{u}_\alpha$, and let $G_C = U_C \rtimes A$. These groups G_C are referred to as standard tame subgroups of G (see remarks in §4.4).

As an exercise, the reader may wish to compute the weights and weight spaces for a group of SOL type. We have that $\mathcal{C}$ is finite, that $\mathfrak{u} = \bigoplus_{\alpha\in\mathcal{W}} \mathfrak{u}_\alpha$, and that $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\alpha+\beta}$ for $\alpha, \beta \in \mathcal{W}$ [4, §4.B]. We now define $H_2(\mathfrak{u})$ and recall the definition of $\mathrm{Kill}(\mathfrak{u})$ so that we will be able to state Theorem 5.1, our main theorem.

Definition 4.2 Let $d_3: \bigwedge^3 \mathfrak{u} \to \bigwedge^2 \mathfrak{u}$ and $d_2: \bigwedge^2 \mathfrak{u} \to \mathfrak{u}$ be the maps of A-modules induced by taking

$$d_3(x \wedge y \wedge z) = [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y \quad d_2(x \wedge y) = -[x, y].$$

Define $H_2(\mathfrak{u}) = \ker(d_2)/\mathrm{image}(d_3)$. Define $\mathrm{Kill}(\mathfrak{u})$ to be the quotient of the symmetric square $\mathfrak{u}\odot\mathfrak{u}$ by the subspace spanned by elements of the form $[x, y]\odot z - y\odot[x, z]$.

4.3 $H_2(\mathfrak{u})$ and $\mathrm{Kill}(\mathfrak{u})$ Are A-representations

Observe that the natural A-action on $\bigwedge^3 \mathfrak{u}$ descends to an A-action on $H_2(\mathfrak{u})$ because, by the Jacobi-like identity

$$\mathrm{Ad}(a)[X, Y] = [\mathrm{Ad}(a)X, Y] + [X, \mathrm{Ad}(a)Y],$$

the subspaces $\mathrm{image}(d_3)$ and $\ker(d_2)$ are preserved by the action of A. Similarly, $\mathrm{Kill}(\mathfrak{u})$ is also an A-representation. Recall that for an A-representation V we define V_0 to consist of 0 together with vectors X such that $\lim_{n\to\infty} \frac{1}{n} \log \|a^n \cdot X\| = 0$ for all $a \in A$. We thus define the subspaces $H_2(\mathfrak{u})_0$ and $\mathrm{Kill}(\mathfrak{u})_0$.

4.4 Tame Subgroups

Definition 4.3 Given $a \in A$, a vacuum subset for a is a compact $\Omega \subset U$ such that for every compact $K \subset U$, there is some $n > 0$ with $\mathrm{Ad}(a)^n K \subset \Omega$. We say that G is tame if there exists $a \in A$ with a vacuum subset.

We say that G is tame if and only if there is some $a \in A$ with $\alpha(a) < 0$ for all $\alpha \in \mathcal{W}$, so G_C is tame for $C \in \mathcal{C}$ [4, Proposition 4.B.5]. We now wish to show that if G is tame, then it is Lipschitz 1-connected. Our starting point is the following.

Proposition 4.4 *Suppose G is tame. Then there is some $a \in A$ and a compact generating set $\mathcal{S} \subset U$ for U such that* $\mathrm{Ad}(a)\mathcal{S}^2 \subset \mathcal{S}$.

Proof By hypothesis, there is some $b \in A$ with a vacuum subset Ω. Let $\mathcal{S}_0$ be a compact generating set for U. As in the proof of Proposition 3.4, we see that some power of $\mathcal{S}_0$ contains an open ball around the identity. Hence, for some $M > 0$, the set $\mathcal{S}_0^M$ contains Ω. As Ω is a vacuum set for b, there exists L such that $\mathrm{Ad}(b)^L \mathcal{S}_0^{2M} \subset \Omega$. Taking $a = b^L$ and $\mathcal{S} = \mathcal{S}_0^M$, we have that $\mathcal{S}$ is a generating set because it contains $\mathcal{S}_0$ (because $1 \in \mathcal{S}_0$ by our standing assumption that generating sets contain the identity), and $\mathrm{Ad}(a)\mathcal{S}^2 = \mathrm{Ad}(b)^L \mathcal{S}_0^{2M} \subset \Omega \subset \mathcal{S}$ as desired. ■

Proposition 4.5 *If $G = U \rtimes A$ is tame, then G is Lipschitz 1-connected.*

Remark A tame group G is probably CAT(0) for some choice of metric, but we do not know how to prove this, so we give a combinatorial proof using Lemma 3.8.

Proof As in Proposition 4.4, fix $a \in A$ and a compact generating set $\mathcal{S}_U \subset U$ such that $\mathrm{Ad}(a)\mathcal{S}_U^2 \subset \mathcal{S}_U$. Let $\mathcal{S}_A$ be a generating set for A with $a \in \mathcal{S}_A$, and let $\mathcal{S} = \mathcal{S}_U \cup \mathcal{S}_A$, so that $\mathcal{S}$ is a generating set for G. Note that $\mathrm{Ad}(a)(s) = s$ for $s \in \mathcal{S}_A$, and observe that if $\ell > \lceil \log_2 j \rceil > 0$, then

$$\mathrm{Ad}(a)^{\ell}(\mathcal{S}_U^j) \subset \mathrm{Ad}(a)^{\ell-1}(\mathcal{S}_U^{\lceil \frac{j}{2} \rceil}) \subset \mathrm{Ad}(a)^{\ell-2}(\mathcal{S}_U^{\lceil \frac{1}{2}\lceil \frac{j}{2} \rceil\rceil}) \subset \cdots$$
$$\cdots \subset \mathrm{Ad}(a)^{\ell - \lceil \log_2 j \rceil}(\mathcal{S}_U) \subset \mathcal{S}_U,$$

because the function $f\colon j \mapsto \lceil \frac{j}{2} \rceil$ satisfies $f^{\lceil \log_2 j \rceil} j = 1$ for natural numbers j. In other words, given $u \in U$, there exist $k \le \lceil \log_2 |u|_{\mathcal{S}_U} \rceil$ and $s \in \mathcal{S}_U$ such that $u =_G a^k s a^{-k}$. It follows, letting $\phi_A\colon G \to A$ denote projection, that we may define a normal form $\omega\colon G \to \mathcal{S}^*$ such that

- for any $g \in G$, the word $\omega(g)$ is given by $\omega(\phi_A(g))\omega(\phi_A(g)^{-1}g)$,
- for $g \in A$, the word $\omega(g) \in \mathcal{S}_A^*$ is a minimal length word representing g,
- and for $g \in U \smallsetminus \{1\}$, $\omega(g)$ is of the form $a^k s a^{-k}$, where $s \in \mathcal{S}_U$ and $0 \le k = O(\log |g|_{\mathcal{S}_U})$.

To check that this is a normal form, *i.e.*, that $\ell(\omega(g)) = O(|g|_{\mathcal{S}})$, note that $|\phi_A(g)|_{\mathcal{S}} \le |g|_{\mathcal{S}}$ and $|\phi_A(g)^{-1}g|_{\mathcal{S}_U} = O(\exp |g|_{\mathcal{S}})$, so that

$$|\phi_A(g)^{-1}(g)|_{\mathcal{S}} = O(\log(O(\exp |g|_{\mathcal{S}}))) = O(|g|_{\mathcal{S}}),$$

because U is at most exponentially distorted in G [4, Proposition 6.B.2].

It will suffice to show that $\Delta \rightsquigarrow \varepsilon$ for ω-triangles $\Delta \in \mathcal{S}^*$. An ω-triangle Δ has the form $a_1\omega(u_1)a_2\omega(u_2)a_3\omega(u_3)$, where for $i = 1, 2, 3$, we have $u_i \in U$ and $a_i \in \mathcal{S}_A^*$ with the a_i and u_i satisfying $(\mathrm{Ad}(a_3a_2)u_1)(\mathrm{Ad}(a_3)u_2)u_3 = 1$. To show that $\Delta \rightsquigarrow \varepsilon$, it thus suffices to establish the following two facts:

- $\omega(u)b \rightsquigarrow b\omega(b^{-1}ub)$ for $u \in U$ and $b \in \mathcal{S}_A^*$,
- $\omega(u_1)\omega(u_2)\omega(u_3) \rightsquigarrow \varepsilon$ for $u_1, u_2, u_3 \in U$ such that $u_1u_2u_3 = 1$.

Lemma 3.6 will be crucial for showing the first fact, in particular, we use the lemma to provide homotopies between words that stay inside a bounded neighborhood of the identity.

4.5 Conjugation

To show that $\omega(u)b \rightsquigarrow b\omega(b^{-1}ub)$ for $u \in U$ and $b \in \mathcal{S}_A^*$, note that $\omega(u)$ may be written as $a^k s a^{-k}$ for some $s \in \mathcal{S}_U$ and $k \geq 0$. Because $\mathrm{Ad}(a)\mathcal{S}^2 \subset \mathcal{S}$, there is some $C \geq 1$ such that, for all $a' \in \mathcal{S}_A$, we have $\mathrm{Ad}(a)^C \,\mathrm{Ad}(a')\mathcal{S} \subset \mathcal{S}$. Let $K = C\ell(b) + k$, so that $\mathrm{Ad}(a)^{K-k}\,\mathrm{Ad}(b')\mathcal{S} \subset \mathcal{S}$ for any word $b' \in \mathcal{S}_A^*$ with $\ell(b') \leq \ell(b)$, and let $s', s'' \in \mathcal{S}$ be given by $s' = \mathrm{Ad}(a)^{K-k}s$ and $s'' = b^{-1}s'b$. We homotope as follows, liberally using Lemma 3.6.

$$\begin{aligned}\omega(u)b = a^k s a^{-k} b &\rightsquigarrow a^K a^{k-K} s a^{K-k} a^{-K} b \\ &\rightsquigarrow a^K s' b a^{-K} \rightsquigarrow a^K b b^{-1} s' b a^{-K} \rightsquigarrow b a^K s'' a^{-K} \rightsquigarrow b\omega(b^{-1}ub).\end{aligned}$$

4.6 Filling ω-triangles in U

To show that $\omega(u_1)\omega(u_2)\omega(u_3) \rightsquigarrow \varepsilon$ for $u_1, u_2, u_3 \in U$ such that $u_1u_2u_3 = 1$, write $\omega(u_i)$ as $a^{k_i} s_i a^{-k_i}$ for $i = 1, 2, 3$, with $k_i \geq 0$ and $s_i \in \mathcal{S}_U$. Let $K = k_1 + k_2 + k_3$ and let $s_i' = \mathrm{Ad}(a^{k_i - K})s_i \in \mathcal{S}_U$. We homotope as follows.

$$\begin{aligned}\omega(u_1)\omega(u_2)\omega(u_3) &= (a^{k_1}s_1a^{-k_1})(a^{k_2}s_2a^{-k_2})(a^{k_3}s_3a^{-k_3}) \\ &\rightsquigarrow (a^K a^{k_1-K} s_1 a^{K-k_1} a^{-K})(a^K a^{k_2-K} s_2 a^{K-k_2} a^{-K})(a^K a^{k_3-K} s_3 a^{K-k_3} a^{-K}) \\ &\rightsquigarrow (a^K s_1' a^{-K})(a^K s_2' a^{-K})(a^K s_3' a^{-K}) \rightsquigarrow a^K s_1's_2's_3' a^{-K} \rightsquigarrow a^K a^{-K} \rightsquigarrow \varepsilon. \quad \blacksquare\end{aligned}$$

4.7 Filling Freely Trivial Words

We now return to the case where the standard solvable group $G = U \rtimes A$ is not necessarily tame. Recall that the collection of conic subsets $\mathcal{C}$ is finite. For $C \in \mathcal{C}$, let G_C be the tame group $U_C \rtimes A$, and let $\mathcal{S}_{G_C}$ be a compact generating set for this group. Let $H = *_{C\in\mathcal{C}} G_C$, and let $\mathcal{S}_H \subset H$ be the union of the $\mathcal{S}_{G_C}$. There is a natural map from H to G. Lemma 4.9 will show that if $w \in \mathcal{S}_H^*$ represents the identity in H, then its image in G admits an $O(\ell(w))$-Lipschitz filling.

We will need the following auxiliary results first. (The reader should probably skip directly to the proof of Lemma 4.9 to understand the point of these propositions). Proposition 4.6 shows that a word $w \in \mathcal{S}_H^*$ representing the identity in H may be reduced to the identity by repeated deletion of subwords $r_j \in \mathcal{S}_{G_{C_j}}^*$ such that r_j represents the identity in G_{C_j}. Proposition 4.8 describes an appropriately Lipschitz rectangular homotopy between words obtained by these deletions. Proposition 4.7 describes a part of the homotopy given in Proposition 4.8.

Given $v, w \in \mathcal{S}_H^*$, we will write $w =_{\mathcal{S}_H^*} v$ if v and w are the same word and write $w =_H v$ if v and w represent the same element of H. Similar notation will be used for equality in other groups when there is any ambiguity.

Proposition 4.6 *Given a word $w \in \mathcal{S}_H^*$ such that $w =_H 1$, there exists a natural number $n \le \ell(w)$ and, for $j = 0, \dots, n$, words $a_j, r_j, b_j \in \mathcal{S}_H^*$ with the following properties.*

(i) $w =_{\mathcal{S}_H^*} a_0 r_0 b_0$.

(ii) $a_n, r_n, b_n = \varepsilon$.

(iii) *For all $j = 0, \dots, n$, there is some $C_j \in \mathcal{C}$ such that $r_j \in \mathcal{S}_{G_{C_j}}^*$ and $r_j =_{G_{C_j}} 1$.*

(iv) $a_j b_j =_{\mathcal{S}_H^*} a_{j+1} r_{j+1} b_{j+1}$ *for $j = 0, \dots, n-1$.*

Proof Note that each element of $\mathcal{S}_H$ lies in some G_C. If $w \neq \varepsilon$ represents the identity, then by the theory of free products, w has a (nonempty) subword that is comprised of elements of some $\mathcal{S}_{G_C}$ and represents the identity in G_C. Thus, we may write $w =_{\mathcal{S}_H^*} a_0 r_0 b_0$, where $r_0 \in \mathcal{S}_{G_{C_0}}^*$ as desired. Applying this argument recursively, we obtain a_j, r_j, b_j as desired. ∎

Proposition 4.7 *Given a natural number k and $w \in \mathcal{S}_H^*$, there exists an $O(k + \ell(w))$ Lipschitz map $f\colon [0,1] \times \left[0, \frac{k}{k+\ell(w)}\right] \to G$ with the following properties.*

(i) *Along the bottom, f is given by $1^k w$, meaning $f(t,0) = \gamma_{1^k w}(t)$, for $0 \le t \le 1$.*

(ii) *Along the top, f is given by $w1^k$, meaning $f\left(t, \frac{k}{k+\ell(w)}\right) = \gamma_{w1^k}(t)$, for $0 \le t \le 1$.*

(iii) *f is constant on the sides, so $f(0,s)$ and $f(1,s)$ do not depend on s.*

Proof Let $\gamma\colon \mathbb{R} \to G$ be given as follows: $\gamma(t) = 1_G$, for $t \le 0$, $\gamma(t) = \gamma_{1^k w}(t)$,for $0 < t \le 1$, and $\gamma(t) = \gamma_{1^k w}(1)$, for $t > 1$. Observe that $\mathrm{Lip}(\gamma) = O(k + \ell(w))$ Then we may take $f(t,s) = \gamma(t+s)$ as our desired filling. ∎

Proposition 4.8 *Given $a, b \in \mathcal{S}_H^*$, $r \in \mathcal{S}_{G_C}^*$ a relation in G_C for some $C \in \mathcal{C}$, and any natural number k, let $\ell = \ell(arb1^k)$ and $h = \frac{\ell(r)}{\ell}$. There exists an $O(\ell)$-Lipschitz map $f\colon [0,1] \times [0,h] \to G$ with the following properties.*

(i) *Along the bottom, f is given by $arb1^k$, meaning $f(t,0) = \gamma_{arb1^k}(t)$, for $t \in [0,1]$.*

(ii) *Along the top, f is given by $ab1^{k+\ell(r)}$, meaning*

$$f(t,h) = \gamma_{ab1^{k+\ell(r)}}(t), \quad \text{for } t \in [0,1].$$

(iii) *f is constant on the sides, so $f(0,s)$ and $f(1,s)$ do not depend on s.*

Proof (See the right-hand side of Figure 3.) Subdivide the rectangle into $[0,1] \times [0,h/2]$ and $[0,1] \times [h/2,1]$. Define f to be $a1^{\ell(r)}b1^k$ on $[0,1] \times \{h/2\}$, *i.e.*, $f(t,h/2) = \gamma_{a1^{\ell(r)}b1^k}(t)$.

First, we extend f over the top rectangle $[0,1] \times [h/2,1]$. For

$$(t,s) \in \left[0, \frac{\ell(a)}{\ell}\right] \times [h/2,1] \cup \left[1 - \frac{k}{\ell}, 1\right] \times [h/2,1],$$

we have that $\gamma_{a1^r b1^k}(t) = \gamma_{ab1^{\ell(r)+k}}(t)$, so we can just set f to be constant vertically, *i.e.*, we define $f(t,s) = \gamma_{ab1^{\ell(r)+k}}(t)$ for these (t,s). To extend f to

$$(t,s) \in \left[\frac{\ell(a)}{\ell}, 1 - \frac{k}{\ell}\right] \times [h/2,1],$$

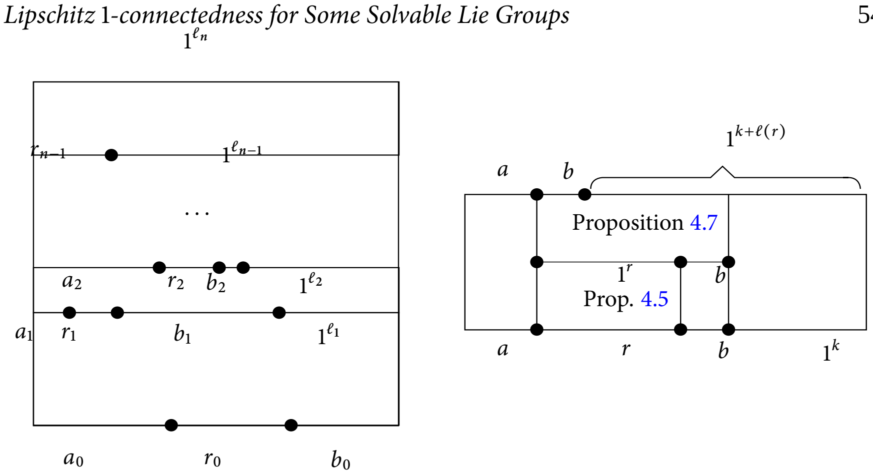

Figure 3: The figure on the left depicts our strategy for filling of the freely trivial word $w = a_0 r_0 b_0$, where the a_j, r_j, b_j are as in Proposition 4.6. The figure on the right depicts the proof of Proposition 4.8 that allows us to fill in each rectangle in the left-hand figure.

we simply apply Proposition 4.7. Thus, we have given an $O(\ell)$-Lipschitz extension of f over the top rectangle.

Now we extend over the bottom rectangle $[0,1] \times [0, h/2]$. For

$$(t,s) \in \Big[0, \frac{\ell(a)}{\ell}\Big] \times [0, h/2] \cup \Big[\frac{\ell(ar)}{\ell}, 1\Big] \times [0, h/2],$$

define $f(t,s) = \gamma_{arb1^k}(t)$. Finally, we apply Proposition 4.5 to extend f over

$$\Big[\frac{\ell(a)}{\ell}, \frac{\ell(ar)}{\ell}\Big] \times [0, h/2],$$

since this is equivalent to filling r. ■

Lemma 4.9 *Recalling the notation introduced at the start of Section 4.7, we have $w \rightsquigarrow \varepsilon$ (in G) for all $w \in \mathcal{S}_H^*$ such that $w =_H 1$.*

Proof (See Figure 3.) Let $w \in \mathcal{S}_H^*$ be a relation in H and take a sequence of words a_j, r_j, b_j, $j = 0, \dots, n$ as in Proposition 4.6. We will define an $O(\ell(w))$-Lipschitz filling $f\colon [0,1] \times [0,1] \to G$ of γ_w as follows. Let $\ell_k = \sum_{j<k} \ell(r_j)$, so that $\ell_0 = 0$ and $\ell_n = \ell(w)$, and subdivide $[0,1] \times [0,1]$ into rectangles $[0,1] \times [\ell_j, \ell_{j+1}]$ for $j = 0, \dots, n-1$. Set $f(t, \ell_j) = \gamma_{a_j r_j b_j 1^{\ell_j}}(t)$, noting that $\ell(a_j r_j b_j 1^{\ell_j}) = \ell(w)$.

Proposition 4.8 now shows that f may be extended over each rectangle $[0,1] \times [\ell_j, \ell_{j+1}]$ with Lipschitz constant $O(\ell(w))$. ■

4.8 Generalizing Lemma 4.9.

A careful examination of the proof of Lemma 4.9 shows that we have not used most of our hypotheses. In particular, the same proof shows that if $G_1, \dots, G_n$ are compactly presented groups whose presentation complexes have Lipschitz 1-connected universal covers, then the presentation complex of the free product $G_1 * \cdots * G_n$ has Lipschitz 1-connected universal cover as well.

4.9 Distortion

We now see that if G is standard solvable, then elements of U may be expressed much more efficiently in the generators of G than in the generators of U.

Proposition 4.10 *Suppose $G = U \rtimes A$ is standard solvable, $\mathcal{S}$ is a compact generating set for G, and $\mathcal{S}_U$ is a compact generating set for U. There exists $C > 1$ such that if $u \in U \smallsetminus \{1_U\}$, then $\frac{1}{C}\log(1+|u|_{\mathcal{S}_U}) \leq |u|_{\mathcal{S}} \leq C\log(1+|u|_{\mathcal{S}_U})$.*

Proof This follows, with some effort, from [4, Proposition 6.B.2]. ∎

4.10 The Multiamalgam

In this subsection, we will define the multiamalgam $\widehat{G}$ of a standard solvable group $G = U \rtimes A$ (first introduced by Abels [1]), and quote a key theorem of Cornulier and Tessera that states certain conditions under which $G \cong \widehat{G}$. This means that G is put together from its standard tame subgroups in a nice way, which will eventually let us build fillings in G from fillings in standard tame subgroups. In order to state this theorem in the proper generality, we must briefly discuss the theory of unipotent groups.

4.11 Unipotent Groups

For a commutative $\mathbb{R}$-algebra $\mathcal{P}$, and a real unipotent group U, *i.e.*, a closed group of upper triangular real matrices with diagonal entries equal to 1, the theory of algebraic groups allows us to define a group $U(\mathcal{P})$ [2, §1.4]. In particular, if $U \subset \mathrm{GL}(n;\mathbb{R})$ consists of all upper triangular matrices with diagonal entries equal to 1, then $U(\mathcal{P})$ consists of all upper triangular $n \times n$ matrices over $\mathcal{P}$ with diagonal entries equal to 1; such matrices are certainly invertible, having determinant equal to 1. Suppose $\mathcal{P} = \mathbb{R}^Y$, so that $\mathcal{P}$ consists of all functions $f\colon Y \to \mathbb{R}$. Then there is an obvious bijection $U(\mathcal{P}) \leftrightarrow U^Y$, and for $y \in Y$ and $\widetilde{u} \in U(\mathcal{P})$ we may speak of $\widetilde{u}(y) \in U$.

Definition 4.11 ([4, §10.B]) Let $G = U \rtimes A$ be real standard solvable. The multiamalgam $\widehat{G}$ of the standard tame subgroups G_C is defined by $\widehat{G} = \ast_{C\in\mathcal{C}} G_C/\langle\langle R_G\rangle\rangle$, where $R_G = \{i_C(u)^{-1} i_{C'}(u) : u \in G_C \cap G_{C'}\}$ and i_C denotes the inclusion of G_C in the direct product.

Similarly, the multiamalgam $\widehat{U}$ is defined by $\widehat{U} = \ast_{C\in\mathcal{C}} U_C/\langle\langle R_U\rangle\rangle$, where $R_U = \{i_C(u)^{-1} i_{C'}(u) : u \in U_C \cap U_{C'}\}$ and i_C denotes the inclusion of U_C in the direct product.

For any commutative $\mathbb{R}$-algebra $\mathcal{P}$, we define $\widehat{U}(\mathcal{P})$ and $\widehat{G}(\mathcal{P})$ similarly, where $G_C(\mathcal{P})$ is understood to be $U_C(\mathcal{P}) \rtimes A$.

Of course, $\widehat{U} \rtimes A \cong \widehat{G}$. Recall that G admits the SOL obstruction if it surjects onto a group of SOL type. Cornulier and Tessera give conditions under which $\widehat{U}$ is isomorphic to U.

Theorem 4.12 *Let $G = U \rtimes A$ be a standard solvable real Lie group. If $H_2(\mathfrak{u})_0 = 0$, $\mathrm{Kill}_2(\mathfrak{u})_0 = 0$, and G does not admit the SOL obstruction, then $\widehat{U}(\mathcal{P}) \cong U(\mathcal{P})$ for all commutative $\mathbb{R}$-algebras $\mathcal{P}$.*

Proof This follows from [4, Corollary 9.D.4]. The 2-tameness hypotheses of the corollary is satisfied from [4, Proposition 4.C.3]. ∎

5 Proof of the Main Theorem

The rest of this paper is devoted to the proof of the following theorem.

Theorem 5.1 *Let $G = U \rtimes A$, where U and A are contractible real Lie groups, A is abelian, and U is a real unipotent group,* i.e., *a closed group of strictly upper triangular real matrices. If G is standard solvable and does not surject onto a group of SOL type, and $H_2(\mathfrak{u})_0$ and $\mathrm{Kill}(\mathfrak{u})_0$ are trivial, then G is Lipschitz 1-connected.*

Proof Lemma 5.2 will show that there exists a generating set $\mathcal{S}$ for G and normal form $\omega\colon G \to \mathcal{S}^*$ with certain properties. Lemma 5.6 will show that if ω has these properties, then $\Delta \rightsquigarrow \varepsilon$ for ω-triangles Δ. By Lemma 3.8, this will suffice to prove the theorem. ∎

5.1 Defining ω

Assumption *Throughout the rest of this paper, we assume that G satisfies the hypotheses of the theorem. That is, $G = U \rtimes A$ is a standard solvable group such that $H_2(\mathfrak{u})_0 = 0$, $\mathrm{Kill}_2(\mathfrak{u})_0 = 0$, and G does not surject onto a group of SOL type.*

Notation. Let $H = *_{C\in\mathcal{C}} G_C$ and $H_U[\mathcal{P}] = *_{C\in\mathcal{C}} U_C(\mathcal{P})$ for any commutative $\mathbb{R}$ algebra $\mathcal{P}$, and let $i_C\colon U_C(\mathcal{P}) \to H_U[\mathcal{P}]$ denote inclusion. We will write H_U for $H_U[\mathbb{R}]$. Let $\mathcal{S}_A$ be a compact generating set for A. For $C \in \mathcal{C}$, let $\mathcal{S}_C$ be a compact generating set for U_C. Let $\mathcal{S}_U = \bigcup_{C\in\mathcal{C}} \mathcal{S}_C$; by Theorem 4.12 this is a compact generating set for U. Let $\mathcal{S} = \mathcal{S}_A \cup \mathcal{S}_U$; this is a compact generating set for G. Let $\mathcal{S}_H$ be a generating set for H, which is equal to the union of compact generating sets for G_C as C ranges over $\mathcal{C}$, and let $\mathcal{S}_{H_U} \subset H_U$ be the union of the $\mathcal{S}_C$; this is a generating set for H_U. Given $C \in \mathcal{C}$ and $x \in U_C$, let $\overline{x} \in (\mathcal{S}_A \cup \mathcal{S}_C)^*$ be a minimal length word representing x. Let $\phi_A\colon G \to A$ be projection. Let the set theoretic map $\phi_U\colon G \to U$ be defined by $\phi_U(g) = \phi_A(g)^{-1}g$, so that $g = \phi_A(g)\phi_U(g)$.

Lemma 5.2 *Under our standing assumptions, there exists a finite sequence $C_1 \cdots C_k$ of conic subsets and a normal form $\omega\colon G \to \mathcal{S}^*$ such that ω has the following properties.*

(i) *For any $g \in G$, $\omega(g) = \omega(\phi_A(g))\omega(\phi_U(g))$.*
(ii) *For $a \in A$, $\omega(a) \in \mathcal{S}_A^*$ is a minimal length word representing a.*
(iii) *For $u \in U$, $\omega(u)$ has the form $\overline{x_1} \cdots \overline{x_k}$, where $x_i \in U_{C_i}$.*

Proof This follows from [4, Proposition 6.B.2], but we will now give a different proof in order to introduce a trick that will be used later.

5.2 The Cornulier–Tessera Trick

For a set Y, define the commutative $\mathbb{R}$-algebra $\mathcal{P}_Y$ as the collection of all functions $f\colon Y\times[1,\infty)\to\mathbb{R}$ such that there is some $\beta\in\mathbb{N}$ with $|f(y,t)|<(1+t)^\beta$. Note that an element of $U(\mathcal{P}_Y)$ can be identified with a function from $Y\times[1,\infty)$ to U. Alternatively, one can think of an element of $U(\mathcal{P}_Y)$ as a family of functions $[1,\infty)\to U$, indexed by Y with matrix coefficients uniformly bounded by some polynomial $(1+t)^\beta$.

Choose Y to have at least continuum cardinality, and let $\widetilde{g}\in U(\mathcal{P}_Y)$ be such that for every $g\in U$, there is some $y\in Y$ and $t=O(|g|_{\mathcal{S}_U})$ such that $\widetilde{g}(y,t)=g$. It is certainly possible to do this; for instance, one might take $Y=U$ and set $\widetilde{g}(y,t)$ to be 1_U for $t<|y|_{\mathcal{S}_U}$ and y for $t\geq|y|_{\mathcal{S}_U}$. In claiming that $\widetilde{g}\in U(\mathcal{P}_Y)$, we have used the fact that the matrix coefficients of $g\in U$ are at most polynomial in $|g|_{\mathcal{S}_U}$.

Since, by Theorem 4.12, $U(\mathcal{P}_Y)$ is generated by the union of the $U_C(\mathcal{P}_Y)$, we can write $\widetilde{g}=\widetilde{x}_1\cdots\widetilde{x}_k$, where each $\widetilde{x}_i$ is an element of some $U_{C_i}(\mathcal{P}_Y)$. Observe that there exist some $\alpha\in\mathbb{N}$ such that $|\widetilde{x}_i(y,t)|_{\mathcal{S}_{C_i}}\leq(1+t)^\alpha$, for all $i=1,\dots,k$, $y\in Y$, and $t\geq1$.

Now let g be an element of U. By definition of $\widetilde{g}$, there exist $y\in Y$ and $t=O(|g|_{\mathcal{S}_U})$ such that $\widetilde{g}(y,t)=g$. For $i=1,\dots,k$, let $x_i=\widetilde{x}_i(y,t)$. We have that

$$g=\widetilde{g}(y,t)=\widetilde{x}_1(y,t)\cdots\widetilde{x}_k(y,t)=x_1\cdots x_k,$$

and $x_i\in U_{C_i}(\mathcal{P}_Y)$ with $|x_i|_{\mathcal{S}_{C_i}}=O(t^\alpha)=O(|g|^\alpha_{\mathcal{S}_U})$. Take $\omega(g)$ to be $\overline{x_1}\cdots\overline{x_k}$, and note that $|\omega(g)|_{\mathcal{S}}=O(\log|g|_{\mathcal{S}_U})$ because $|\overline{x_i}|_{\mathcal{S}_A\cup\mathcal{S}_{C_i}}=O(\log|x_i|_{\mathcal{S}_{C_i}})=O(\log|g|_{\mathcal{S}_U})$ by Proposition 4.10.

We have thus defined ω on elements of U, with the desired properties. Define ω on A by taking $\omega(a)$ to be the shortest word in $\mathcal{S}_A^*$ representing $a\in A$. Extend ω to all of g by setting $\omega(g)=\omega(\phi_A(g))\omega(\phi_U(g))$. We must show that ω is a normal form, *i.e.*, that, for $g\in G$, $\ell(\omega(g))=O(|g|_{\mathcal{S}})$.

We have $\ell(\omega(\phi_A(g)))=|\phi_A(g)|_{\mathcal{S}}=O(|g|_{\mathcal{S}})$, so it suffices to show that

$$\ell(\omega(\phi_U(g)))=O(|g|_{\mathcal{S}}).$$

If $w\in\mathcal{S}^*$ is a minimal length word representing some $g\in G$, note that $\omega(\phi_A(g))^{-1}w$ represents $\phi_U(g)$. By Proposition 4.10, there is some constant $C>1$ not depending on g such that $|\omega(\phi_A(g))^{-1}w|_{\mathcal{S}}\geq\frac{1}{C}\log|\phi_U(g)|_{\mathcal{S}_U}$. Thus, since $|\omega(\phi_U(g))|_{\mathcal{S}}=O(\log|\phi_U(g)|_{\mathcal{S}_U})$, we have that $\omega(\phi_U(g))=O(|\phi_A(g)|_{\mathcal{S}}+|w|_{\mathcal{S}})=O(|g|_{\mathcal{S}})$ as desired. ∎

5.3 Filling ω-triangles

We wish to show that we can fill ω triangles, where ω is a normal form produced by Lemma 5.2. Proposition 5.5 will allow us to homotope ω-triangles into relations of the form $\overline{x_i}\cdots\overline{x_K}$, where each $\overline{x_i}$ is a word in $\mathcal{S}_A\cup\mathcal{S}_{C_i}$, efficiently representing an element x_i of U_{C_i}, where $C_1,\dots,C_K$ is some fixed sequence of conic subsets. In order to fill such relations, recall from Theorem 4.12 that, under our standing assumptions, $U(\mathcal{P})\cong\widehat{U}(\mathcal{P})$ for any commutative $\mathbb{R}$-algebra $\mathcal{P}$. Consequently, the kernel of $H_U[\mathcal{P}]\to U(\mathcal{P})$ is normally generated by elements of the form $i_C(u)^{-1}i_{C'}(u)$, where

$u \in U_C(\mathcal{P}) \cap U_{C'}(\mathcal{P})$. In order to fill $\overline{x_1} \cdots \overline{x_K}$ in Corollary 5.4, we will need to factor $x_1 \cdots x_K$ in the free product H_U as a product of a bounded number of elements of the form $g^{-1} i_C(u)^{-1} i_{C'}(u) g$, where each $g \in H_U$ is a product of a bounded number of elements living in some factors U_C, with $|g|_{\mathcal{S}_U}$ and $|u|_{\mathcal{S}_U}$ controlled by some polynomial of $\sum_{j=1}^{K} |x_j|_{\mathcal{S}_{C_j}}$.

Lemma 5.3 ([4, Lemma 7.B.1]) *Suppose that our standing assumptions are satisfied. Given a sequence of conical subsets $C_1, \ldots, C_K$, there exist natural numbers N, μ, β such that, for any sequence $x_i \in U_{C_i}$ with $x_1 x_2 \cdots x_K =_U 1_U$, there is an equality of the form $x_1 \cdots x_K =_{H_U} (g_1 r_1 g_1^{-1}) \cdots (g_N r_N g_N^{-1})$ satisfying*

(i) *$g_j =_{H_U} g_{j1} \cdots g_{j\mu}$, where $g_{jk} \in U_{C_{jk}}$ for some $C_{jk} \in \mathcal{C}$;*
(ii) *each r_j is of the form $i_{C_j'}(u_j) i_{C_j''}(u_j)^{-1}$ for some conical subsets C_j', C_j'' and some $u_j \in U_{C_j'} \cap U_{C_j''}$;*
(iii) *$|g_{jk}|_{\mathcal{S}_{C_{jk}}} = O(\ell^\beta)$, $|u_j|_{\mathcal{S}_{C_j'}} = O(\ell^\beta)$, and $|u_j|_{\mathcal{S}_{C_j''}} = O(\ell^\beta)$, where $\ell = 1 + \sum_{i=1}^{K} |x_i|$.*

Proof This is a special case of [4, Lemma 7.B.1], but we will reprise most of the details here. We will use the same Cornulier and Tessera trick we used to prove Lemma 5.2. Take $\mathcal{P}_Y$ as in the proof of Lemma 5.2. Recall that $\widetilde{x} \in U_C(\mathcal{P}_Y)$ may be thought of as a function from $Y \times [0, \infty)$ to U_C. Let Y be a set with at least continuum cardinality, so that there exists $(\widetilde{x}_1, \ldots, \widetilde{x}_K) \in U_{C_1}(\mathcal{P}_Y) \times \cdots \times U_{C_K}(\mathcal{P}_Y)$ that has the following strong surjectivity property: for any $(x_1 \cdots x_K) \in U_{C_1}(\mathcal{P}_Y) \times \cdots \times U_{C_K}(\mathcal{P}_Y)$ with $x_1 \cdots x_K =_U 1$, there exists $y \in Y$ and $t = O(|x_1|_{\mathcal{S}_{C_1}} + \cdots + |x_K|_{\mathcal{S}_{C_K}})$ with $\widetilde{x}_i(y, t) = x_i$.

By Theorem 4.12, we know that there is some equality of the form

$$\widetilde{x}_1 \cdots \widetilde{x}_K =_{H_U[\mathcal{P}_Y]} (\widetilde{g}_1^{-1} \widetilde{r}_1 \widetilde{g}_1) \cdots (\widetilde{g}_N^{-1} \widetilde{r}_N \widetilde{g}_N),$$

where $\widetilde{g} \in {}^*U_C(\mathcal{P}_Y)$ and each $\widetilde{r}_j$ has the form $i_{C_j'}(\widetilde{u}_j)^{-1} i_{C_j''}(\widetilde{u}_j)$, for some conical subsets C_j', C_j'' and $\widetilde{u}_j \in U_{C_j'}(\mathcal{P}_Y) \cap U_{C_j''}(\mathcal{P}_Y)$. Since the $U_C(\mathcal{P}_Y)$ generate $H_U[\mathcal{P}_Y]$, there must be some μ such that for all $j = 1, \ldots, N$, $\widetilde{g}_j = \widetilde{g}_{j1} \cdots \widetilde{g}_{j\mu}$, where each $\widetilde{g}_{jk}$ lives in $U_{C_{jk}}$ for some $C_{jk} \in \mathcal{C}$. Note that by definition of $\mathcal{P}_Y$, there is some β such that all $|\widetilde{g}_{jk}(y, t)|_{\mathcal{S}_U}$ and $|\widetilde{u}_j(y, t)|_{\mathcal{S}_{C_j'}}$ are $O(t^\beta)$.

Given, $x_1, \ldots, x_K \in U_{C_1} \times \cdots \times U_{C_K}$, let $\ell = \sum_{i=1}^{K} |x_i|_{\mathcal{S}_{C_i}}$ and choose $y \in Y$ and $t = O(\ell)$ such that $\widetilde{x}_i(y, t) = x_i$, for $i = 1, \ldots, K$. For $j = 1, \ldots, N$ and $k = 1, \ldots, \mu$, let $g_j = \widetilde{g}_j(y, t)$, $g_{jk} = \widetilde{g}_{jk}(y, t)$, $u_j = \widetilde{u}_j(y, t)$, and $r_j = i_{C'}(u_j)^{-1} i_{C''}(u_j)$. It follows that $x_1 \cdots x_K =_{H_U} (g_1 r_1 g_1^{-1}) \cdots (g_N r_N g_N^{-1})$, and the g_j and r_j satisfy the desired conditions. ∎

Corollary 5.4 *Suppose that our standing assumptions are satisfied. Given a sequence of conical subsets $C_1, \ldots, C_K$, we have in G that $\overline{x_1} \cdots \overline{x_K} \rightsquigarrow \varepsilon$. for any sequence $x_i \in U_{C_i}$ with $x_1 x_2 \cdots x_K =_U 1_U$.*

Proof By Lemma 5.3, we have (for β, N, μ independent of the x_j),

$$x_1 \cdots x_K =_{H_U} (g_1 r_1 g_1^{-1}) \cdots (g_N r_N g_N^{-1}),$$

where $g_j =_{H_U} g_{j1} \cdots g_{j\mu}$ for $g_{jk} \in U_{C_{jk}}$, each r_j is of the form $i'_C(u_j)^{-1} i_{C''}(u_j)$ for some conical subsets C'_j, C''_j, and some $u_j \in U_{C'_j} \cap U_{C''_j}$, and $|g_{jk}|_{\mathcal{S}_{C_{jk}}}, |u_j|_{\mathcal{S}_{C'_j}} = O(\ell^\beta)$, where $\ell = 1 + \sum_{i=1}^K |x_i|$.

For each $j = 1, \ldots, N$, let $\overline{g_j} \in \mathcal{S}^*$ be $\overline{g_{j1}} \cdots \overline{g_{j\mu}}$. Let $\overline{r_j} \in \mathcal{S}^*$ be equal to $(\overline{u}'_j)^{-1} \overline{u}''_j$, where $\overline{u}'_j \in \mathcal{S}_A \cup \mathcal{S}_{C'_j}$ is a minimal length word representing u_j in $G_{C'_j}$ and $\overline{u}''_j \in \mathcal{S}_A \cup \mathcal{S}_{C''_j}$ is a minimal length word representing u_j in $G_{C''_j}$. This implies that $\overline{r_j}$ represents r_j in H.

First we show that $\overline{r_j} \rightsquigarrow \varepsilon$. Note that $C'_j \cap C''_j$ is itself a conic subset, so there is some $\overline{u_j} \in (\mathcal{S}_A \cup \mathcal{S}_{C'_j \cap C''_j})^*$ that represents u_j in $G_{C'_j \cap C''_j}$ with $\ell(\overline{u_j}) = O(\ell(\overline{r_j}))$. Thus, by Proposition 4.5, we have $\overline{r_j} = (\overline{u}'_j)^{-1} \overline{u}''_j \rightsquigarrow \overline{u_j}^{-1} \overline{u_j} \rightsquigarrow \varepsilon$.

Next, observe that

$$\ell(\overline{g_j}) = \sum_{k=1}^{\mu} \ell(\overline{g_{jk}}) = \sum_{k=1}^{\mu} O(\log |g_{jk}|_{\mathcal{S}_{C_{jk}}}) = O\Big(\beta \log\Big(1 + \sum_{i=1}^{K} |x_i|\Big)\Big) = O(\ell(\overline{x_1} \cdots \overline{x_K})),$$

by Proposition 4.10, and $\ell(\overline{r_i}) = O(\ell(\overline{x_1} \cdots \overline{x_K}))$ similarly.

Because

$$\overline{x_1} \cdots \overline{x_K} =_H (\overline{g_1 r_1 g_1}^{-1}) \cdots (\overline{g_N r_N g_N}^{-1}),$$
$$\ell((\overline{g_1 r_1 g_1}^{-1}) \cdots (\overline{g_N r_N g_N}^{-1})) = O(\ell(\overline{x_1} \cdots \overline{x_K})),$$

we have by Lemma 3.6 and Lemma 4.9 that

$$\overline{x_1} \cdots \overline{x_K} \rightsquigarrow (\overline{g_1 r_1 g_1}^{-1}) \cdots (\overline{g_N r_N g_N}^{-1}) \rightsquigarrow (\overline{g_1}\, \overline{g_1}^{-1}) \cdots (\overline{g_N}\, \overline{g_N}^{-1}) \rightsquigarrow \varepsilon$$

because $\overline{r_j} \rightsquigarrow \varepsilon$ as noted above. ■

We need one more proposition before we can fill ω-triangles. Given $a, x \in G$, let ${}^a x$ denote axa^{-1}.

Proposition 5.5 *Fix a sequence of conical subsets $C_1, \ldots, C_K \in \mathcal{C}$. We have that $\omega(a)\overline{x_1} \cdots \overline{x_K} \rightsquigarrow \overline{{}^a x_1} \cdots \overline{{}^a x_2}\omega(a)$ for all $a \in A$ and $x_i \in U_{C_i}$.*

Proof By Proposition 4.5,

$$\omega(a)\overline{x_1} \cdots \overline{x_K} \rightsquigarrow (\omega(a)\overline{x_1}\omega(a)^{-1})(\omega(a)\overline{x_2}\omega(a)^{-1}) \cdots (\omega(a)\overline{x_K}\omega(a)^{-1})\omega(a)$$
$$\rightsquigarrow \overline{{}^a x_1} \ldots \overline{{}^a x_K}\omega(a), \quad ■$$

We now conclude the proof of our main theorem by showing that we can fill ω-triangles.

Lemma 5.6 *Under our standing assumptions, if $g_1, g_2, g_3 \in G$ with $g_1 g_2 g_3 =_G 1_G$, we have $\omega(g_1)\omega(g_2)\omega(g_3) \rightsquigarrow \varepsilon$.*

Proof Recall that $\omega(g)$ has the form $\omega(a)\overline{x_1} \cdots \overline{x_k}$, where $\overline{u}_i \in \mathcal{S}_A \cup \mathcal{S}_{C_i}$ for some fixed sequence $C_1, \ldots, C_k$ of conical subsets. Let $g_1, g_2, g_3 \in G$ with $g_1 g_2 g_3 = 1_G$, and let $a_i = \phi_A(g_i)$, for $i = 1, 2, 3$. Let $\omega(\phi_U(g_1)) = \overline{x_1} \cdots \overline{x_k}$, $\omega(\phi_U(g_2)) = \overline{x'_1} \cdots \overline{x'_k}$ and

$\omega(\phi_U(g_3)) = \overline{x_1''} \cdots \overline{x_k''}$. Expanding and applying Proposition 5.5 repeatedly, we slide the a-words to the right to see that

$$\begin{aligned}\omega(g_1)\omega(g_2)\omega(g_3) &= \omega(a_1)\overline{x_1} \cdots \overline{x_k}\omega(a_2)\overline{x_1'} \cdots \overline{x_k'}\omega(a_3)\overline{x_1''} \cdots \overline{x_k''} \\ &\rightsquigarrow \overline{{}^{a_1}x_1} \cdots \overline{{}^{a_1}x_k}\,\overline{{}^{a_2a_1}x'_1} \cdots \overline{{}^{a_2a_1}x_k'}\,\overline{{}^{a_3a_2a_1}x''_1} \cdots \overline{{}^{a_3a_2a_1}x_k''}\omega(a_1)\omega(a_2)\omega(a_3) \\ &\rightsquigarrow \overline{{}^{a_1}x_1} \cdots \overline{{}^{a_1}x_k}\,\overline{{}^{a_2a_1}x'_1} \cdots \overline{{}^{a_2a_1}x_k'}\,\overline{{}^{a_3a_2a_1}x''_1} \cdots \overline{{}^{a_3a_2a_1}x_k''}.\end{aligned}$$

This resulting word admits a Lipschitz filling by Corollary 5.4. ∎

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Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
e-mail: davidbrucecohen@gmail.com