On certain Quadric Hypersurfaces in Riemannian Space

By C. E. Weatherburn (Western Australia).

(*Received 8th October*, 1934. *Read 2nd November*, 1934.)

1. The Hypersurfaces Defined.

The use of geodesic polar coordinates in the intrinsic geometry of a surface leads to the concept of a geodesic circle, *i.e.* the locus of points at a constant distance from the pole O along the geodesics through O. A geodesic hypersphere is the obvious generalisation of this for a Riemannian V_n. We propose to consider more general central quadric hypersurfaces of V_n, which we define as follows. Let x^i $(i = 1, 2, \ldots, n)$ be a system of coordinates in V_n, whose metric is $g_{ij}\, dx^i\, dx^j$, and let a_{ij} be the components in the x's of a symmetric covariant tensor of the second order, evaluated at the point O, which is taken as pole. If s is the arc-length of a geodesic through O, the quantities ξ^i defined by

$$\xi^i = (dx^i/ds)_0$$

are the contravariant components of the unit vector in the direction of the geodesic at O, the suffix zero indicating that the derivative is to be evaluated at the pole. If s is measured from O along the geodesic to the current point P, the variables y^i defined by

$$y^i = \xi^i s$$

are the Riemannian coordinates of P relative to the pole O.

The quadric hypersurface defined by the equation

$$y^i\, a_{ij}\, y^j = 1 \qquad (1)$$

is clearly a central quadric. For the equation may be expressed

$$\xi^i\, a_{ij}\, \xi^j = 1/s^2 \qquad (2)$$

showing that, on a given geodesic through O, there are two points of the hypersurface, in opposite directions along the geodesic, and at equal geodesic distances from the pole. The positive value of s given by (2) may be called the *geodesic radius* of the quadric (1) for the direction ξ^i at O. The particular case of a hypersphere, of geodesic radius c, corresponds to

$$a_{ij} = g_{ij}/c^2.$$

For, if this value of a_{ij} be substituted in (2), we obtain $s^2 = c^2$, as required.

Further, we deduce immediately from (2) that

The sum of the inverse squares of the geodesic radii for n mutually orthogonal directions at O is an invariant, equal to $a_{ij}\, g^{ij}$.

For, if $e^i_{h|}$ $(h = 1, \ldots, n)$ are the contravariant components of the unit tangents at O to the curves of an orthogonal ennuple in V_n, it follows from (2) that the sum of the inverse squares of the geodesic radii for these directions is given by

$$\sum_h (s_{h|})^{-2} = \sum_h e^i_{h|}\, a_{ij}\, e^j_{h|} = a_{ij}\, g^{ij}$$

as stated.

Let y^i be the Riemannian coordinates of a point P, not necessarily on the quadric (1). Then the equation

$$Y^i\, a_{ij}\, y^j = 1 \qquad (3)$$

defines a hypersurface of V_n, the quantities Y^i being Riemannian coordinates of the current point on the hypersurface. We shall call this the *polar hypersurface* of P relative to the quadric (1). If P lies on the quadric it also lies on its polar hypersurface. It is easy to see that

The geodesic through P and the centre of the quadric is divided harmonically by the quadric, the point P and its polar hypersurface.

For if ξ^i is the unit tangent to this geodesic at O, s' the geodesic distance of P from O, and s, s'' those of the points B and Q in which the geodesic cuts the quadric and the polar hypersurface, the Riemannian coordinates of Q are $\xi^i s''$; and since this point lies on (3) we have

$$s'\, s''\, (\xi^i\, a_{ij}\, \xi^j) = 1,$$

and therefore, in virtue of (2),

$$s'\, s'' = s^2 \qquad (4)$$

as required.

Because the tensor a_{ij} is symmetric, it follows from (3) that if the polar hypersurface of P passes through a point Q, then that of Q passes through P.

2. RECIPROCAL QUADRICS.

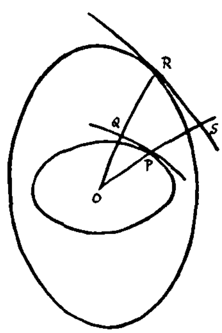
Let a^{ij} be the symmetric contravariant tensor at O reciprocal to a_{ij}, so that

$$a^{ij} a_{jk} = \delta^i_k. \tag{5}$$

Let η_i be the covariant components of the unit tangent to a geodesic at O, and t the distance along this geodesic to a point R. If we write

$$z_i = \eta_i t \tag{6}$$

the quantities z_i are covariant components of a vector at O,



determining the point R by its geodesic distance from the pole and the direction of the geodesic at O. Points R which satisfy the relation

$$z_i a^{ij} z_j = 1 \tag{7}$$

also lie on a central quadric with centre at O. We shall call this the *reciprocal* quadric to (1).

Let P be a point on (1) whose Riemannian coordinates y^i are $\xi^i s$. Then the point whose coordinates z_i are equal to $a_{ij} y^j$ lies on the reciprocal quadric. For, if these quantities are substituted for z_i in (7), the equation is satisfied in virtue of (5). Let R be this point on (7). The relation is reciprocal; for

$$a^{ij} z_j = a^{ij} a_{jk} y^k = \delta^i_k y^k = y^i,$$

and it follows from (7) that

$$z_i y^i = 1 \tag{8}$$

so that, if θ is the inclination at O of the central geodesics to P and R,

$$st \cos \theta = 1. \tag{8'}$$

https://doi.org/10.1017/S001309150000804X Published online by Cambridge University Press

Let Q be the point in which the geodesic OR cuts the polar hypersurface of P with respect to (1), and t' the geodesic distance OQ. Then the Riemannian coordinates of Q are $t'\eta^i$; and since this point lies on the polar hypersurface (3), we have

$$t'\eta^i a_{ij} y^j = 1,$$

that is

$$t'\eta^i z_i = 1,$$

and therefore, in virtue of (6),

$$tt' = 1, \tag{9}$$

since η^i and η_i are components of a unit vector. Thus the geodesic distances OQ and OR are reciprocal. Similarly if the geodesic OP cuts the polar hypersurface of R with respect to (7) in the point S at a geodesic distance s' from O, the (covariant) coordinates of S are $\xi_i s'$; and since this point lies on the polar hypersurface

$$Z_i a^{ij} z_j = 1,$$

it follows that

$$s'\xi_i y^i = 1,$$

which may be expressed

$$ss'\xi_i \xi^i = 1.$$

Consequently

$$ss' = 1 \tag{10}$$

and the geodesic distances OP and OS are reciprocal. Also from (8′), (9) and (10) it follows that

$$s\cos\theta = t' \tag{11}$$

and

$$t\cos\theta = s'. \tag{11'}$$

It is easy to show that the hypersurface $Y^i\eta_i = q$ is the polar hypersurface of a point on (1) with respect to that quadric provided that

$$\eta_i a^{ij}\eta_j = q^2;$$

and hence that the polar hypersurfaces of n points $y^i_{h|}$ on (1), for which the vectors $a_{ij} y^j_{h|}$ are mutually orthogonal, meet on the geodesic hypersphere $s^2 = a^{ij} g_{ij}$.

3. Conjugate Geodesic Radii.

Before considering conjugate directions and radii of the quadric (1), we introduce the symmetric covariant tensor A_{ij}, evaluated at O and defined by the relation

$$A_{ik} g^{kl} A_{lj} = a_{ij}. \tag{12}$$

https://doi.org/10.1017/S001309150000804X Published online by Cambridge University Press

Then, if y^i are the Riemannian coordinates of the point P on the quadric (1), this quadric is given by

$$y^i A_{ik} g^{kl} A_{lj} y^j = 1.$$

Consequently

$$E_k g^{kl} E_l = 1, \tag{13}$$

where we have written

$$E_k = y^i A_{ik}. \tag{14}$$

From (13) it is evident that these quantities E_k are covariant components of a *unit* vector, whose contravariant components E^i are therefore given by

$$E^i = g^{ij} E_j = g^{ij} A_{jk} y^k.$$

The components z_i of the covariant vector defining the corresponding point R on the reciprocal quadric are

$$z_i = a_{ij} y^j = A_{ik} g^{kl} A_{lj} y^j = A_{ik} E^k. \tag{15}$$

If A^{ij} is the reciprocal contravariant tensor to A_{ij}, the relations (14) and (15) are equivalent to

$$y^i = A^{ik} E_k \tag{14'}$$

and

$$E^i = A^{ij} z_j. \tag{15'}$$

And it is easily verified that, in terms of this tensor,

$$A^{ik} g_{kl} A^{lj} = a^{ij}. \tag{16}$$

The directions of the vectors u^i and v^i at O will be said to be *conjugate* with respect to the quadric (1) when they satisfy the relation

$$u^i a_{ij} v^j = 0 \tag{17}$$

and the geodesics which pass through O in these directions will be called conjugate geodesic diameters of (1). If $y^i_{h|}$ and $y^i_{k|}$ are the Riemannian coordinates of the extremities of conjugate geodesic diameters, we deduce from (17) and (14) that

$$y^i_{h|} A_{il} g^{lp} A_{pj} y^j_{k|} = 0;$$

that is

$$E_{h|l} g^{lp} E_{k|p} = 0, \tag{18}$$

showing that the corresponding vectors $E^i_{h|}$ and $E^i_{k|}$ are orthogonal. It is easily verified that, if the conditions (18) are satisfied, the corresponding points $z_{h|i}$ on (7) are the extremities of mutually conjugate geodesic radii of that quadric.

We may now establish the theorem:

https://doi.org/10.1017/S001309150000804X Published online by Cambridge University Press

The sum of the squares of n mutually conjugate geodesic radii of the quadric (1) *is an invariant, equal to* $a^{ij} g_{ij}$.

Let $y^i_{h|}$ $(h = 1, \ldots, n)$ be the Riemannian coordinates of the extremities of the n mutually conjugate geodesic radii, and $E_{h|i}$ the corresponding unit vectors (14). The conjugate relations are expressed by (18), where $h \neq k$. Further, if $s_{h|}$ is the length of the geodesic radius to $y^i_{h|}$, and $\xi^i_{h|}$ its direction at O, it follows from (14′) that

$$s_{h|}\, \xi^i_{h|} = A^{ij} E_{h|j}.$$

Taking the square of the length of each member, and summing for h from 1 to n, we have

$$\begin{aligned}\sum_h (s_{h|})^2 &= \sum_h E_{h|j} A^{ji} g_{ip} A^{pq} E_{h|q} \\ &= \sum_h E_{h|j} a^{jq} E_{h|q} = a^{jq} g_{jq},\end{aligned}$$

since the n vectors $E_{h|i}$ are mutually orthogonal unit vectors.

We may also observe in passing that

The polar hypersurfaces of the extremities of n mutually conjugate geodesic radii of the quadric (1) *meet on a similar quadric.*

For, in virtue of (3), (12) and (14), the polar hypersurface of the extremity $y^i_{h|}$ is given by

$$Y^i A_{il} g^{lk} E_{h|k} = 1.$$

Squaring both members, and summing for h from 1 to n we obtain

$$\sum_h (Y^i A_{il} g^{lk} E_{h|k})(E_{h|q} g^{qp} A_{pj} Y^j) = n,$$

and therefore

$$Y^i A_{il} g^{lk} g_{kq} q^{qp} A_{pj} Y^j = n,$$

which, in virtue of (12), reduces to

$$Y^i a_{ij} Y^j = n. \tag{19}$$

Thus the locus of the intersection Y^i of the n polar hypersurfaces is a similar quadric.

4. APPLICATION.

Let the V_n considered above be a hypersurface of an enveloping Riemannian V_{n+1}, with $\Omega_{ij}\, dx^i\, dx^j$ as the second fundamental form of V_n. It is well known that the quantities Ω_{ij} are components of a symmetric covariant tensor[1], and that the normal curvature κ_n of the hypersurface for the direction of the unit vector ξ^i is given by

$$\kappa_n = \xi^i \, \Omega_{ij} \, \xi^j. \tag{20}$$

It follows that, with the same notation as above, the normal curvature is equal to the inverse square of the geodesic radius of the quadric

$$y^i \, \Omega_{ij} \, y^j = 1 \tag{21}$$

in the direction ξ^i at O. And from the first theorem of § 1 we then deduce that

The sum of the normal curvatures of the hypersurface V_n for n mutually orthogonal directions at a point is invariant, and equal to $\Omega_{ij} \, g^{ij}$.

Similarly from the first theorem of § 3 it follows that, if Ω^{ij} is the reciprocal tensor to Ω_{ij},

The sum of the reciprocals of the normal curvatures of the hypersurface V_n for n mutually conjugate directions at a point of it is invariant, and equal to $\Omega^{ij} \, g_{ij}$.

The quadric (21) corresponds to Dupin's indicatrix for a surface.

[1] *Cf.* Eisenhart, *Riemannian Geometry*, §§ 43, 44.