ORBITAL ARC LENGTH AS A UNIVERSAL INDEPENDENT VARIABLE

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Abstract. A uniform treatment of the two-body problem leads to a differential time transformation to introduce the arc length along the orbit as the independent variable. The transformation is integrated in terms of the classical anomalies.

1. Introduction

For *analytical step regulation* in numerical integration of highly eccentric elliptic orbits, Brumberg (1992) used the *length of orbital arc* as the independent variable, and made his construction to fit into a pattern resembling that of two-parameter time transformations (Ferrándiz *et al.*, 1987; Floría, 1996). His pseudo-time does not belong to the class of elliptic anomalies of Ferrándiz *et al.*, and this argument is not obtained from the pattern due to these authors, but it is related to t by an equation similar to theirs.

Although Brumberg (1992 p. 325) stated that his transformation *is applicable to any kind of Keplerian orbit*, neither a proof nor a hint was given in this sense: his derivation was based on elliptic motion. *A justification and rigorous extension* of his approach is pertinent for applications, specially in an analytical perturbation theory for highly-eccentric orbits. In studying bounded motion, highly eccentric orbits are *close to the bifurcation case represented by parabolic motion*, and *the type of orbit is occasionally changed by perturbing forces acting during a finite interval of time* (Stiefel and Scheifele, 1971 p. 42). Future analytical developments in working out perturbation theories should benefit from the generality of a universal treatment.

Since this approach is not restricted to elliptic motion, we study a *systematic derivation* of this time-parameter – within a *universal formulation* (Stumpff, 1959 Chapter V; Stiefel and Scheifele, 1971 §11; Battin, 1987 §4.5 and §4.6) of two-body problems – to *adapt* his elliptic motion treatment

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and allow for *other cases of Keplerian orbits*, recovering his formulae under this *universal* treatment. Accordingly, the motivation for our study was the question *whether* (and *how*) Brumberg's (1992) considerations could be rendered applicable to the orbital arc length in cases of *non-elliptic* Keplerian motion. *These questions are answered on the stage of a universal formulation of two-body motion*, on the basis of which we *analytically integrate* the time transformation, in closed form, by means of elliptic integrals. This integration takes into account the main types of Keplerian orbits.

Here we continue previous research (Floría, 1996) concerning a universal function approach to differential changes of time variable for the treatment of Keplerian-like systems. *Universal-like functions* (see below) provide a tool for the study of orbital motion, particularly for a *compact* representation of analytical solutions of the two–body problem. Universal functions and some identities between them, along with a change of integration variable, will allow us to reduce the integration of the reparametrizing transformation to that of some *algebraic expressions*, which leads to incomplete elliptic integrals of the first and second kind whose modulus depends on the orbit eccentricity. We have in mind the universal-variable formulation and analytical treatment of *perturbed* Keplerian systems (e. g. perturbed highly eccentric elliptic motion of artificial satellites), and *transitions between reference orbits of different nature while performing perturbation studies*, especially when a universal independent argument is put in the place of time.

2. Universal Functions, Auxiliary Formulae and Notations

The following ideas and results are found in, or easily derived from, Stiefel and Scheifele (1971), Stumpff (1959), Battin (1987). The *Stumpff* $c_n(z)$ *functions* and *Battin universal* $U_n(s,\, \varrho)$ *functions* (Stiefel and Scheifele 1971, §11; Battin 1987, §4.5), with $z = \varrho\, s^2$, are transcendental functions defined by certain series that are absolutely convergent for all complex values of z (whence the series converge for all s regardless of ϱ). In applications, ϱ is a real parameter related to the energy of the system [Formula (4), below]. A relation between these functions is

$$U_n(s, \varrho) \equiv s^n c_n(\varrho s^2) = \sum_{k=0}^{\infty} \frac{(-\varrho)^k s^{2k+n}}{(2k+n)!}, \; n = 0, 1, 2, \ldots \tag{1}$$

For future reference, we record some useful properties and identities:

$$dU_n/ds = U_{n-1}, \quad n = 1, 2, 3, \ldots ; \tag{2}$$

$$1 = U_0^2 + \varrho\, U_1^2; \quad U_1^2 = 2\, U_2 - \varrho\, U_2^2 . \tag{3}$$

If $\mu \equiv$ gravitational parameter, $e \equiv$ eccentricity, $q \equiv$ *distance of the pericentre*, the negative of the *energy* of a Kepler problem is

$$L = \mu(1-e)/2q, \quad \text{and} \quad \varrho = 2L. \tag{4}$$

In terms of s as the argument of universal functions, the two-body problem admits a *uniform, closed-form* analytical *solution*. Using Cartesian coordinates (x, y) in the orbital plane, and the modulus r of the radius vector:

$$x = q - \mu U_2(s, 2L), \quad y = \sqrt{\mu q(1+e)}\, U_1(s, 2L); \tag{5}$$

$$r = q + \mu e U_2(s, 2L); \quad U_2(s, 2L) = (r-q)/(\mu e); \tag{6}$$

$$t = qs + \mu e U_3(s, 2L) \quad \textit{(Kepler's equation)}. \tag{7}$$

The dependence of s on t is given by (7), and s is introduced through

$$dt = r\,ds \quad \textit{(Sundman's transformation)}, \tag{8}$$

where $s = 0$ at a reference time which usually corresponds to the pericentre. For application in Section 4, we give some auxiliary formulae and notations. From the polar equation of a conic-section in the orbital plane:

$$r = q(1+e)/(1+e\cos f), \; r - q = qe(1-\cos f)/(1+e\cos f), \tag{9}$$

with the true anomaly f as the polar angle. From (3), (6), (4) and (9):

$$U_1^2 = 2\frac{r-q}{\mu e} - (2L)\left[\frac{r-q}{\mu e}\right]^2 = \frac{q(1+e)\sin^2 f}{\mu(1+e\cos f)^2}. \tag{10}$$

We show preference to the true anomaly f and the related expressions (9) and (10), due to their *universal nature*.

3. Extension of Brumberg's Transformation to Universal Form

Let $d\sigma$ be the *arc element* along a two-body orbit. Starting from Formulae (5), after differentiation with respect to s and use of (2) and (3), we obtain

$$(d\sigma/ds)^2 = (dx/ds)^2 + (dy/ds)^2 = \mu^2 U_1^2(s, 2L) + \mu q(1+e) - \mu^2(1-e^2)\, U_1^2(s, 2L),$$

and so, the relationship between the parameters σ and s is given by

$$d\sigma = \sqrt{\mu q(1+e) + \mu^2 e^2 U_1^2(s, 2L)}\; ds. \tag{11}$$

Application of the chain rule, along with Formulae (8) and (11), yields

$$\frac{dt}{d\sigma} = \frac{(dt/ds)}{(d\sigma/ds)} = \frac{r(s)}{\sqrt{\mu q(1+e) + \mu^2 e^2 U_1^2(s, 2L)}}. \tag{12}$$

To obtain another expression for the reparametrizing function occurring in (12), more easily interpretable within the framework of *two-parameter transformations of the independent variable* (Ferrándiz *et al.*, 1987; Flória, 1996), we perform the sequence of substitutions $U_1 \to U_2 \to r$, namely, we express U_1 in terms of U_2 [Formulae (3)], and then, by virtue of Formula (6), we replace U_2 by r. Thus, the function under the radical sign in (12) is converted into a simple polynomial in r, say: $\mu r \{2q - (1-e)r\} / q$, and then the right-hand side of (12) becomes an algebraic function of r:

$$\frac{dt}{d\sigma} = \frac{r}{\sqrt{\mu r/q}\,\sqrt{2q - (1-e)r}} = \sqrt{\frac{q}{\mu}}\,\frac{r^{1/2}}{\sqrt{2q - (1-e)r}}\,. \quad (13)$$

Thus, *under a universal treatment of two-body motion*, we have recovered the expression given by Brumberg [1992 p.325, Formulae (1) and (14)]:

$$dt = Q\,d\sigma\,, \qquad Q = r^{1/2}\,[2\mu - (2L)\,r]^{-1/2}\,. \quad (14)$$

Comparison with the transformations derived by Ferrándiz *et al.* (1987) shows that the time argument proposed by Brumberg is not obtained from the specific formulations leading to the class of the generalized elliptic anomalies introduced by those authors, although it can be viewed as a special case of a more general two-parameter time transformation.

4. Integration of the Time Transformation in Terms of Anomalies

The arc length σ of a Kepler problem, reckoned from the pericentre (at which $s = 0$), can be determined by our *universal* Formula (11). To integrate that transformation, we perform a *change of integration variable* $s \to v$,

$$U_1(s, 2L) = v \Rightarrow dv = U_0(s, 2L)\,ds = \sqrt{1 - (2L)\,v^2}\,ds, \quad (15)$$

with $v(0) = U_1(0, 2L) = 0$, taking into account (2) and (3), and the integration of (11) reduces to that of an *algebraic integrand* over v:

$$\sigma = \sqrt{q}\int_0^v \sqrt{(\mu q\,(1+e) + \mu^2 e^2 v^2) / (q - \mu\,(1-e)\,v^2)}\,dv, \quad (16)$$

where (4) has also been employed. To carry out the quadrature we separate the calculation into cases, distinguishing the functional form of the integrand according to the main types of Keplerian orbits, which is reflected in the different values of the eccentricity (or the value and sign of the parameter L, related to the energy). In what follows $\mathcal{E}$ and $\mathcal{F}$ denote, respectively, the elliptic and hyperbolic eccentric anomaly of Keplerian motion, while **F** and **E** refer to the incomplete elliptic integrals of the first and second kind.

4.1. ELLIPTIC MOTION: $L > 0$ (BRUMBERG, 1992)

The differential time transformation (16) is governed by the expression

$$\sigma_E = e\sqrt{\mu q/(1-e)}\, J_E\,, \quad J_E = \int_0^v \sqrt{(A+v^2)/(B-v^2)}\, dv, \tag{17}$$
$$A = q(1+e)/(\mu e^2)\,, \quad B = 1/(2L)\,, \quad A\,, B > 0\,.$$

From Gradshteyn and Ryzhik (1980 p.276, Formula 3.169.3), or Byrd and Friedman (1971, Formulae 214.11 and 315.02, and function relations)

$$J_E = \frac{1}{\sqrt{2L}}\left[\frac{1}{e}\,\mathbf{E}(\gamma, e) - e\sin\gamma\,\frac{e+\cos f}{1+e\cos f}\right], \tag{18}$$

$$\sin\gamma = \frac{\sin f}{\sqrt{1+e^2+2e\cos f}} = \frac{\sin\mathcal{E}}{\sqrt{1-e^2\cos^2\mathcal{E}}}\,. \tag{19}$$

Consequently, the integrated time transformation for the length of arc reads

$$\sigma_E = a\left[\mathbf{E}(\gamma, e) - e^2\sin\gamma\,\frac{e+\cos f}{1+e\cos f}\right], \quad a = \frac{\mu}{2L}\,, \tag{20}$$

and we recover a Brumberg's result [1992 p.325, Formulae (9) and (11)] in which the factor a, the semi-major axis, must be taken into account.

4.2. HYPERBOLIC MOTION: $L < 0$

The change (16) of time variable takes on the form

$$\sigma_H = e\sqrt{\mu q/(e-1)}\, J_H\,, \quad J_H = \int_0^v \sqrt{(A+v^2)/(B+v^2)}\, dv, \tag{21}$$
$$A = q(1+e)/(\mu e^2)\,, \quad B = 1/(-2L)\,, \quad 0 < A < B\,.$$

To calculate J_H we apply Gradshteyn and Ryzhik (1980), p.276, Formula 3.169.2 or Byrd and Friedman (1971), Formulae 221.04 and 313.02, and properties of elliptic functions; see also Byrd and Friedman, p.3, Example I:

$$J_H = \frac{A}{\sqrt{B}}\,\mathbf{F}\left(\alpha, \frac{1}{e}\right) - \sqrt{B}\,\mathbf{E}\left(\alpha, \frac{1}{e}\right) + v\sqrt{\frac{B+v^2}{A+v^2}}\,, \tag{22}$$

$$\tan\alpha = \frac{v}{\sqrt{A}} = \frac{e\sin f}{1+e\cos f} = \frac{e}{\sqrt{e^2-1}}\sinh\mathcal{F}\,, \tag{23}$$

$$\sin\alpha = \frac{e\sin f}{\sqrt{1+e^2+2e\cos f}} = \frac{e\sinh\mathcal{F}}{\sqrt{e^2\cosh^2\mathcal{F}-1}}\,, \tag{24}$$

$$v\sqrt{\frac{B+v^2}{A+v^2}} = \frac{\sin\alpha}{\sqrt{-2L}}\,\frac{e+\cos f}{1+e\cos f} = \frac{\sin\alpha}{\sqrt{-2L}}\cosh\mathcal{F}\,. \tag{25}$$

The final expression for the reparametrization, with $a = \mu/(-2L)$, is

$$\sigma_H = ae\left[\frac{e^2-1}{e^2}\,\mathbf{F}\left(\alpha,\frac{1}{e}\right) - \mathbf{E}\left(\alpha,\frac{1}{e}\right) + \sin\alpha\,\frac{e+\cos f}{1+e\cos f}\right] \quad (26)$$

4.3. PARABOLIC MOTION: $L = 0$

Now the expression for (16) reads:

$$\sigma_P = \mu J_P\,, \quad J_P = \int_0^v \mathcal{R}(v)\,dv\,, \quad \mathcal{R} = \sqrt{(2q/\mu)+v^2}\,. \quad (27)$$

In view of Gradshteyn and Ryzhik (1980), p.86, Formula 2.271.3,

$$J_P = \left[\frac{v}{2}\mathcal{R} + \frac{q}{\mu}\ln(v+\mathcal{R})\right]_0^v = \frac{v}{2}\mathcal{R} + \frac{q}{\mu}\ln\left[\frac{v+\mathcal{R}}{\sqrt{2q/\mu}}\right], \quad (28)$$

or, by virtue of Formula (10) for v, in terms of the true anomaly,

$$J_P = (q/\mu)\left[\tan(f/2)\,\sec(f/2) + \ln\{\tan(f/2)+\sec(f/2)\}\right] \quad (29)$$

And we conclude that, for parabolic Keplerian orbits,

$$\sigma_P = q\left[\tan(f/2)\,\sec(f/2) + \ln\{\tan(f/2)+\sec(f/2)\}\right]. \quad (30)$$

Remember that, in particular, $\mathbf{F}(\vartheta,1) = \ln(\tan\vartheta + \sec\vartheta)$.

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