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Pure Infiniteness of the Crossed Product of an AH-Algebra by an Endomorphism

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Abstract. It is shown that simplicity of the crossed product of a unital AH-algebra with slow dimension growth by an endomorphism implies that the algebra is also purely infinite, provided only that the endomorphism leaves no trace state invariant and takes the unit to a full projection.

1 Introduction

It has been shown by Deaconu [De] and Anantharaman-Delaroche [An] that the C^*-algebra of a local homeomorphism is the crossed product by an endomorphism of another C^*-algebra. As observed in [De] this implies that such an algebra is often infinite, and Anantharaman-Delaroche described in [An] a sufficient condition for the algebra to be purely infinite. Recall that a simple C^*-algebra is said to be purely infinite when all its non-zero hereditary C^*-subalgebras contain an infinite projection. Thanks to the classification results for purely infinite simple C^*-algebras, the crossed product description obtained by Deaconu and Anantharaman-Delaroche implies that the simple and purely infinite C^*-algebras that arise from local homeomorphisms are classified by their K-theory groups, and it therefore becomes an important question to decide when the algebra of a local homeomorphism is simple and purely infinite.

In [R1] Rørdam proved that the crossed product by a full corner endomorphism of a simple unital C^*-algebra of real rank zero with comparability of projections is simple and purely infinite. In particular, the crossed product of a simple unital AF-algebra by such an endomorphism is simple and purely infinite. In the same paper Rørdam also initiated the classification of purely infinite simple C^*-algebras that was subsequently completed, mutatis mutandis, by the classification results of Kirchberg and Phillips mentioned above. Rørdam's result on the crossed product by an endomorphism has been extended and used by several other mathematicians, but in most of these results the initial algebra, the one with the endomorphism, hereafter called the *core*, is assumed to be simple and to have various other properties. The simplicity of the crossed product, as well as its pure infiniteness, is then a consequence. The work of Dykema and Rørdam in [DR] is an exception, but they assume some rather special properties of the endomorphism that are not easy to establish.

For the application to the C^*-algebras of a local homeomorphism it is a nuisance to have to assume simplicity of the core. When the algebra of a local homeomorphism is simple, the core may or may not be simple, and hence the existing results,

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such as the one of Rørdam, on crossed products by endomorphisms can generally only be used by imposing additional assumptions. It is the purpose of the present paper to obtain a result regarding the pure infiniteness of a crossed product by an endomorphism in which simplicity is assumed of the crossed product rather than of the core, and which is general enough to cover the C^*-algebra of a local homeomorphism; assuming only that it is simple. The following, which is the main result of the paper, is such a theorem. The definition of a *unital AH-algebra with slow dimension growth* will be given in the next section.

Theorem 1.1 *Let A be a unital AH-algebra with slow dimension growth. Let $\beta\colon A \to A$ be an injective endomorphism such that*

(i) *$\beta(1)$ is a full projection in A (*i.e., $\overline{A\beta(1)A} = A$*), and*
(ii) *there is no trace state ω of A such that $\omega \circ \beta = \omega$.*

If the crossed product $A \times_\beta \mathbb{N}$ is simple, it is also purely infinite.

Applications of this result to the C^*-algebras of local homeomorphisms and locally injective surjections are given in [CT].

It must be observed that the crossed product $A \times_\beta \mathbb{N}$ in the theorem is not the same as the one introduced by W. Paschke and used by Rørdam in [R1] where it is assumed that β maps *onto* the corner $\beta(1)A\beta(1)$. In order to also cover the crossed products by endomorphisms arising from a locally injective surjection, which may not be open and hence not a local homeomorphism (cf. [Th1]), we use instead the crossed product introduced by Stacey in [St]. It can be defined as the universal C^*-algebra generated by a copy of A and an isometry v with the property that $vav^* = \beta(a)$ ([BKR]), and hence it agrees with the one used by Dykema and Rørdam in [DR]. In contrast to the crossed product of Paschke, it is not required that $v^*Av \subseteq A$. When β maps onto $\beta(1)A\beta(1)$, as is the case when the situation arises from a local homeomorphism in [An], the two crossed products coincide.

The main strategy of the proof is due to Rørdam. In [R2] he proved that the crossed product of a C^*-algebra A by an automorphism is (simple and) purely infinite when A is

(1) exact, finite and separable,
(2) simple,
(3) approximately divisible,

and has no densely defined non-zero trace that is invariant under the given automorphism. Although this is a result about an automorphism, it has bearing on crossed products by endomorphisms, since they can be realised as a corner in a crossed product by an automorphism.

The last condition, about the absence of invariant traces, is of course necessary. Conditions (1) are harmless and satisfied when the crossed product arises from one of the locally injective surjections we have in mind. As we explained above, we aim to move the simplicity assumption from the core to the crossed product, while approximate divisibility is a property that is hard to establish and about which we know next to nothing when the algebra comes from a local homeomorphism and the core is not simple. It is therefore interesting to observe that an important step in the following proof of Theorem 1.1 will be to show that a much weaker version of divisibility is automatic for unital AH-algebras with slow dimension growth.

2 Tracial Almost Divisibility for AH-algebras with Slow Dimension Growth

Let M_l denote the C^*-algebra of complex $l \times l$-matrices. In the following a *homogeneous C^*-algebra* will be a C^*-algebra A isomorphic to a C^*-algebra of the form

$$eC(X, M_l)e,$$

where X is a compact metric space and e is a projection in $C(X, M_l)$ such that $e(x) \neq 0$ for all $x \in X$. The *dimension ratio* $r(A)$ of A is then defined to be the number

$$r(A) = \max_{x \in X} \frac{\operatorname{Dim} X + 1}{\operatorname{Rank} e(x)}.$$

Definition 2.1 A unital C^*-algebra A is an *AH-algebra* when there is an increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ of unital C^*-subalgebras of A such that $A = \overline{\bigcup_n A_n}$ and each A_n is a homogeneous C^*-algebra. We say that A has *slow dimension growth* when there is such a sequence with the additional property that $\lim_{n\to\infty} r(A_n) = 0$.

There seems to be slightly varying definitions of slow dimension growth for AH-algebras, and it should therefore be observed that with the above definition we insist that the rank of the projections increase without bounds even when all the involved topological spaces are zero-dimensional.

Let A be a C^*-algebra and a, b two positive elements of A. Recall (see, for example, [T]) that a is *Cuntz subequivalent* to b when there is a sequence $\{z_n\}$ in A such that $a = \lim_{n\to\infty} z_n b z_n^*$. We write $a \preceq b$ when this holds. This notion extends the well-known subequivalence in the sense of Murray–von Neumann used for projections.

In the following we denote by $T(A)$ the convex set of trace states of a unital C^*-algebra A. The next definition is inspired by [W, Definition 2.5(ii)].

Definition 2.2 A unital C^*-algebra A is *tracially almost divisible* when the following holds. For any positive contraction h in A and any given $m \in \mathbb{N}$ there is a $\delta > 0$ with the property that for all $\epsilon > 0$, there are mutually orthogonal positive contractions $h_1, h_2, \ldots, h_m$ in A such that

$$h_1 + h_2 + \cdots + h_m \preceq h \quad \text{and} \quad \tau(h_i) \geq \delta\tau(h) - \epsilon$$

for all i and all $\tau \in T(A)$.

As an important step towards the main result of the paper, we first prove the following proposition.

Proposition 2.3 *Let A be a unital AH-algebra with slow dimension growth. Then A is tracially almost divisible.*

We will actually prove a slightly stronger result; namely that the δ of Definition 2.2 can be chosen to be $\frac{1}{4m}$, independently of h. However, the proof of the main result will not require this strengthening of the conclusion.

The main tools for the proof of Proposition 2.3 are methods and results of A. Toms [T] about Cuntz subequivalence in a homogeneous C^*-algebra. When A is a unital C^*-algebra and $\tau \in T(A)$, there is associated with τ a "dimension function" d_τ defined on positive contractions of A as

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) = \sup_{n\in\mathbb{N}} \tau(a^{\frac{1}{n}}).$$

By [T, Corollary 5.2] we have the following theorem.

Theorem 2.4 (A. Toms) *Let $A \simeq eC(X, M_l)e$ be a homogeneous C^*-algebra. Let $a, b \in A$ be positive contractions such that*

$$d_\tau(a) + \max_{x\in X} \frac{\operatorname{Dim} X}{2 \operatorname{Rank} e(x)} \leq d_\tau(b)$$

for all $\tau \in T(A)$. It follows that $a \preceq b$.

Actually, the result in [T] is slightly stronger, but the above theorem suffices for our purposes.

Lemma 2.5 *Let $\epsilon \in]0, \frac{1}{8}[$ and $m \in \mathbb{N}$. Let $eC(X, M_l)e$ be a homogeneous C^*-algebra such that* Rank $e(x) = M$ *is constant and*

$$\frac{\operatorname{Dim} X + 1}{M} < \frac{\epsilon}{8m}.$$

It follows that for every positive contraction $h \in eC(X, M_l)e$, there are m mutually orthogonal positive contractions $h_1, h_2, \dots, h_m$ in $eC(X, M_l)e$ such that

$$h_1 + h_2 + \cdots + h_m \preceq h$$

and

$$\tau(h_i) \geq \frac{1}{4m}\tau(h) - 2\epsilon \tag{2.1}$$

for all i and all trace states $\tau \in T(eC(X, M_l)e)$.

Proof Let $j \in \mathbb{N}$, $j \geq 2$, and set $d = \operatorname{Dim} X + 1$. Since $\frac{d}{M} < \frac{\epsilon}{4}$, we find that

$$\frac{(j-\frac{1}{2})\epsilon}{m} - \frac{d}{2Mm} - \frac{(j+\frac{1}{2})\epsilon}{2m} > \frac{(j-\frac{1}{2})\epsilon}{m} - \frac{\epsilon}{8m} - \frac{(j+\frac{1}{2})\epsilon}{2m} = \frac{(j-\frac{7}{4})\epsilon}{2m} \geq \frac{\epsilon}{8m}.$$

As $\frac{1}{M} < \frac{\epsilon}{8m}$, these estimates show that there is a natural number α_j such that

$$\frac{(j+\frac{1}{2})\epsilon}{2} \leq \frac{m\alpha_j}{M} < \left(j - \frac{1}{2}\right)\epsilon - \frac{d}{2M}.$$

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Note that we can arrange that $\alpha_j \leq \alpha_{j+1}$. Let J be the least natural number such that $\left(J + \frac{1}{2}\right)\epsilon \geq \frac{1}{2}$. (The condition $\epsilon < \frac{1}{8}$ ensures that $J \geq 4$.) Then

$$\frac{m\alpha_j}{M} \leq \frac{m\alpha_J}{M} \leq \left(J - \frac{1}{2}\right)\epsilon - \frac{d}{2M} < 1$$

for $2 \leq j \leq J$. We can therefore choose mutually orthogonal trivial projections $p_1^j, p_2^j, \dots, p_m^j$ in $C(X, M_l)$ for each $2 \leq j \leq J$ such that $\operatorname{Rank} p_i^j = \alpha_j, i = 1, 2, \dots, m$, and such that

$$p_i^j \leq p_i^{j+1},\ i = 1, 2, \dots, m,\ 2 \leq j \leq J - 1.$$

Then

$$\operatorname{Rank}\left(\sum_{i=1}^{m} p_i^J\right) + \frac{d}{2} = m\alpha_J + \frac{d}{2} \leq \left(J - \frac{1}{2}\right)\epsilon M \leq \frac{M}{2} \leq M = \operatorname{Rank} e,$$

so the projection $\sum_{i=1}^m p_i^J$ is Murray–von Neumann equivalent to a subprojection of e; this is a classical fact about vector bundles, but it also follows from Theorem 2.4. Consequently we may assume that $p_i^j \in eC(X, M_l)e$ for all i, j, with the reduction that they may no longer be trivial projections. Set $p_i^0 = p_i^1 = 0$ for $i = 1, 2, \dots, m$.

For each $0 \leq j \leq J-1$ we choose a continuous function $g_j \colon [(j+\frac{1}{2})\epsilon, (j+\frac{3}{2})\epsilon] \to [0, 1]$ such that $g_j((j+\frac{1}{2})\epsilon) = 1$ and $g_j((j+\frac{3}{2})\epsilon) = 0$. For each $i = 1, 2, \dots, m$, define a continuous function

$$H_i \colon [0, 1] \times X \to M_l$$

such that

$$H_i(t, x) = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\epsilon\right], \\ g_j(t)p_i^j(x) + (1 - g_j(t))p_i^{j+1}(x), & t \in \left[(j + \frac{1}{2})\epsilon, (j + \frac{3}{2})\epsilon\right], \\ & \quad 0 \leq j \leq J - 1, \\ p_i^J(x), & t \geq (J + \frac{1}{2})\epsilon. \end{cases}$$

Then $H_i(t, x)$ is a positive contraction and $H_i(t, x)H_{i'}(t, x) = 0$, $i \neq i'$, for all t, x.

For each $x \in X$ we consider the extremal trace state τ_x of $eC(X, M_l)e$ defined as $\tau_x(f) = \operatorname{tr}(f(x))$, where tr is the trace state of $e(x)M_le(x)$. For each $i = 1, 2, \dots, m$, we define $h_i' \in C(X, M_l)$ such that $h_i'(x) = H_i(\tau_x(h), x)$. Note that the h_i''s are mutually orthogonal positive contractions and that $h_i' \in eC(X, M_l)e$, since

$$e(x)H_i(t, x)e(x) = H_i(t, x), \quad \forall t, x.$$

Set

$$U = \left\{x \in X : \tau_x(h) > \left(2 + \frac{1}{2}\right)\epsilon\right\}$$

and consider an $x \in \overline{U}$. Then $\tau_x(h) \in \left[(j+\frac{1}{2})\epsilon, (j+\frac{3}{2})\epsilon\right]$ for some $2 \leq j \leq J-1$ or $\tau_x(h) \geq (J+\frac{1}{2})\epsilon$. In the first case we find that

$$\begin{aligned}\tau_x(h_i') &= g_j\big(\tau_x(h)\big)\frac{\alpha_j}{M} + \big(1-g_j\big(\tau_x(h)\big)\big)\frac{\alpha_{j+1}}{M}\\ &\geq g_j\big(\tau_x(h)\big)\frac{(j+\frac{1}{2})\epsilon}{2m} + \big(1-g_j\big(\tau_x(h)\big)\big)\frac{(j+\frac{3}{2})\epsilon}{2m} \geq \frac{\tau_x(h)}{2m} - \frac{\epsilon}{2m},\end{aligned}$$

while

$$d_{\tau_x}\left(\sum_{i=1}^m h_i'\right) \leq \frac{m\alpha_{j+1}}{M} \leq \left(j+\frac{1}{2}\right)\epsilon - \frac{d}{2M} \leq \tau_x(h) - \frac{d}{2M}.$$

When $\tau_x(h) \geq (J+\frac{1}{2})\epsilon$ we find that

$$\tau_x(h_i') = \frac{\alpha_J}{M} \geq \frac{(J+\frac{1}{2})\epsilon}{2m} \geq \frac{1}{4m} \geq \frac{1}{4m}\tau_x(h),$$

while

$$d_{\tau_x}\left(\sum_{i=1}^m h_i'\right) = \frac{m\alpha_J}{M} \leq \left(J-\frac{1}{2}\right)\epsilon - \frac{d}{2M} \leq \tau_x(h) - \frac{d}{2M}.$$

All in all we conclude that

$$d_{\tau_x}\left(\sum_{i=1}^m h_i'\right) \leq \tau_x(h) - \frac{d}{2M} \quad \text{and} \quad \tau_x(h_k') \geq \frac{1}{4m}\tau_x(h) - \frac{\epsilon}{2}$$

for all $k = 1, 2, \dots, m$ and all $x \in \overline{U}$.

If $U = \varnothing$, we set $h_1 = h_2 = \cdots = h_m = 0$. Since $0 \preceq h$ and $\frac{1}{4m}\tau_x(h) \leq 2\epsilon$ for all x, this will prove the lemma in this case. Assume therefore that $U \neq \varnothing$. Recall that $T(eC(\overline{U}, M_l)e)$ is the closed convex hull of $\{\tau_x : x \in \overline{U}\}$. Since $d_{\tau_x}\big(\sum_{i=1}^m h_i'\big) \leq \tau_x(h) - \frac{d}{2M}$ for all $x \in \overline{U}$, and since

$$\tau \mapsto d_\tau\left(\sum_{i=1}^m h_i'|_{\overline{U}}\right) - \tau(h|_{\overline{U}}) + \frac{d}{2M}$$

is affine and lower semi-continuous on $T\big(eC(\overline{U}, M_l)e\big)$, we find that

$$d_\tau\left(\sum_{i=1}^m h_i'|_{\overline{U}}\right) + \frac{d}{2M} \leq \tau(h|_{\overline{U}}) \leq d_\tau(h|_{\overline{U}})$$

for all $\tau \in T\big(eC(\overline{U}, M_l)e\big)$. Since $\operatorname{Dim}\overline{U} \leq \operatorname{Dim} X$, it follows from Theorem 2.4 that there is a sequence $\{z_n\} \in eC(\overline{U}, M_l)e$ such that

$$\lim_{n\to\infty} z_n(x)h(x)z_n(x)^* = \sum_{i=1}^m h_i'(x)$$

uniformly in $x \in \overline{U}$. Set

$$K = \left\{ x \in X : \tau_x(h) \geq \left(3 + \frac{1}{2}\right) \epsilon \right\}$$

and let $\psi \colon X \to [0,1]$ be a continuous function such that $\psi(x) = 1, x \in K$, and $\operatorname{supp} \psi \subseteq U$. We consider ψ as a central element of $eC(X, M_l)e$ in the obvious way. Now let

$$h_i = \psi h_i', \; i = 1, 2, \dots, m,$$

and set $z_n' = z_n \sqrt{\psi} \in eC(X, M_l)e$. Then

$$\lim_{n \to \infty} z_n' h z_n'^{*} = \sum_{i=1}^{m} h_i$$

and $\tau_x(h_i) \geq \frac{1}{4m} \tau_x(h) - \epsilon$ for all $x \in K$. Since $\frac{1}{4m} \tau_x(h) - 2\epsilon \leq \frac{4\epsilon}{4m} - 2\epsilon < 0 \leq \tau_x(h_i)$ when $x \notin K$, we obtain (2.1). ■

Proof of Proposition 2.3 Consider a positive contraction $b \in A$. Let $m \in \mathbb{N}$ and $\epsilon \in \left]0, \frac{1}{8}\right[$ be given. We will complete the proof by showing that there are mutually orthogonal positive contractions $b_1, b_2, \dots, b_m$ in A such that $b_1 + b_2 + \cdots + b_m \preceq b$ and $\tau(b_i) \geq \frac{1}{4m} \tau(b) - 3\epsilon$ for all i.

Since A is an AH-algebra with slow dimension growth it follows from [KR, Lemma 2.5(ii)] that there is a unital homogeneous C^*-sub-algebra $B \subseteq A$ and a positive contraction $a \in B$ such that $r(B) < \frac{\epsilon}{8m}$, $a \preceq b$ and $\|a - b\| \leq \epsilon$. Note that B is isomorphic to a direct sum

$$B \simeq \oplus_{j=1}^{N} e_j C(X_j, M_l) e_j$$

of homogeneous C^*-algebras such that $\operatorname{Rank} e_j(x) = K_j$ is constant on X_j and

$$\frac{\operatorname{Dim} X_j + 1}{K_j} \leq r(B) < \frac{\epsilon}{8m}$$

for all j. We can therefore apply Lemma 2.5 to each summand and in this way obtain mutually orthogonal positive contractions $b_1, b_2, \dots, b_m$ in B such that $b_1 + b_2 + \cdots + b_m \preceq a$ and $\tau(b_i) \geq \frac{1}{4m} \tau(a) - 2\epsilon$ for all $\tau \in T(B)$. Since $\frac{1}{4m} \tau(a) \geq \frac{1}{4m} (\tau(b) - \epsilon) \geq \frac{1}{4m} \tau(b) - \epsilon$ for all $\tau \in T(A)$, we are done. ■

3 Proof of the Main Result

In this section we prove Theorem 1.1 by an elaboration of Rørdam's proof of [R2, Theorem 2.1]. For this purpose we isolate the following lemmas. In the statement of the first we use the (standard) notation $M_\infty(B)$ for the union $\bigcup_n M_n(B)$. Recall that a projection $p \in B$ is *full* when $\overline{BpB} = B$.

Lemma 3.1 *Let Z be a compact metric space of dimension $\dim Z \leq d$ and let $p \leq e \in C(Z, M_l)$ be projections. Assume that there is a natural number $N \in \mathbb{N}$ such that*

$$\operatorname{Rank} p(z) > (N+1)(N+2)\left[\frac{d}{2}\right]$$

for all $z \in Z$, where $[d/2]$ is the least natural number larger or equal to $d/2$. It follows that there is a projection $p' \in M_\infty(eC(Z, M_l)e)$ such that

$$N[p'] \leq [p] \leq (N+3)[p']$$

in $K_0(eC(Z, M_l)e)$

Proof Let $Z = Z_1 \sqcup Z_2 \sqcup \cdots \sqcup Z_k$ be a partition of Z by clopen sets such that $\operatorname{Rank} p$ is constant on each Z_j. Fix j and set $d_j = \operatorname{Dim} Z_j$. Note that $d_j \leq d$. Write $\operatorname{Rank} p = l(N+1)[d_j/2] + r$ where $l, r \in \mathbb{N}$ and $1 \leq r \leq (N+1)[d_j/2]$. Then $l \geq N+2$ by assumption, and hence

$$Nl\left[\frac{d_j}{2}\right] + l\left[\frac{d_j}{2}\right] < \operatorname{Rank} p \leq Nl\left[\frac{d_j}{2}\right] + 2l\left[\frac{d_j}{2}\right] \tag{3.1}$$

on Z_j. Let q_j be a trivial projection on Z_j of constant rank $l[d_j/2]$. Since $e|_{Z_j}$ is a full projection, q_j is equivalent to a projection p'_j in $M_d(eC(Z_j, M_l)e)$ for some d. Since $l[d_j/2] \geq d_j/2$, it follows from (3.1) and the theory of vector bundles (or Theorem 2.4), that $N[p'_j] \leq [p|_{Z_j}] \leq (N+3)[p'_j]$ in $K_0(eC(Z_j, M_l)e)$. Set $p' = \sum_j p'_j$. ∎

Lemma 3.2 *Let A be a unital AH-algebra with slow dimension growth and let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be a sequence of homogeneous C^*-sub-algebras,*

$$A_n \simeq e_n C(X_n, M_{m_n}) e_n$$

such that $1 \in A_1$, $A = \overline{\bigcup_n A_n}$ and $\lim_{n\to\infty} r(A_n) = 0$. Let p be a full projection in A and let $K \in \mathbb{N}$ be given. It follows that there is an $n \in \mathbb{N}$ and a projection $q \in A_n$ such that q is unitarily equivalent to p in A and

$$\operatorname{Rank} q(x) \geq K(\operatorname{Dim} X_n + 1)$$

for all $x \in X_n$.

Proof A standard argument shows that p is unitarily equivalent to a projection q in A_m for some m. Since p is full, we can assume, by increasing m, that q is full in A_m. There is then a $k \in \mathbb{N}$ such that $\operatorname{Rank} q(x) \geq \frac{1}{k} \operatorname{Rank} e_n(x)$ for all $x \in X_n$, $n \geq m$. Therefore the desired inequality will hold for all sufficiently large n thanks to the slow dimension growth condition. ∎

Let $a, b \in K_0(A)$. In the following we write $a \prec b$ when there is a full projection q in $M_n(A)$ for some n such that $b - a = [q]$.

Lemma 3.3 *Let A be an AH-algebra with slow dimension growth. Let $e, f \in A$ be projections such that $N[e] \prec N[f]$ in $K_0(A)$ for some $N \in \mathbb{N}$. It follows that $[e] \prec [f]$ in $K_0(A)$ and $e \preceq f$ in A.*

Proof It follows from Lemma 3.2 that the difference $N[f] - N[e]$ is represented by a projection p in a homogeneous C^*-algebra, containing also projections e' and f' unitarily equivalent to e and f, respectively, such that $\inf_x \operatorname{Rank} p(x)$ is greater than $N + 1$ times the dimension of the spectrum. Then well-known facts about vector bundles, or Theorem 2.4, show that in this algebra e' is equivalent to a subprojection e'' of f' such that $f' - e''$ is full. The lemma follows. ■

Lemma 3.4 *Let B be a C^*-algebra with the property that $B = \overline{\bigcup_n B_n}$, where $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$ are C^*-subalgebras of B each of which is a unital AH-algebra with slow dimension growth. Furthermore, assume that the unit of B_n is a full projection in B_{n+1} for each n. Let α be an automorphism of B such that $\omega \circ \alpha \neq \omega$ for all non-zero densely defined lower semi-continuous traces ω on B. Assume that $B \times_\alpha \mathbb{Z}$ is simple. It follows that every full projection in B is infinite in $B \times_\alpha \mathbb{Z}$.*

Proof We elaborate on Rørdam's proof of [R2, Lemma 2.5]. Let e be a full projection in B.

(a) The first step is to show that there is an element $x \in K_0(B)$ such that $x \geq \alpha_*(x)$ and $x \neq \alpha_*(x)$. As in [R2] this follows from the absence of α-invariant traces, by use of results of Blackadar–Rørdam and Goodearl–Handelman. We refer to [R2] for the details of the argument.

(b) Let $I \subseteq B$ be a non-zero ideal such that $\alpha(I) \subseteq I$. It follows that $I = B$. Indeed

$$J = \overline{\bigcup_{n \geq 0} \alpha^{-n}(I)}$$

is an ideal in B such that $\alpha(J) = J$. Since $B \times_\alpha \mathbb{Z}$ is simple, it follows that $J = B$. In particular there is an $n \in \mathbb{N}$ and an element $b \in I$ such that $\|\alpha^n(e) - b\| = \|e - \alpha^{-n}(b)\| < \frac{1}{3}$. As is well known this implies that I contains a projection equivalent to $\alpha^n(e)$. This projection is full in A, since $\alpha^n(e)$ is, whence $I = B$.

In the following we extend α to $M_n(B)$ for all n in the canonical way.

(c) Let $p \in M_n(B)$ be a projection such that $[p] = x - \alpha_*(x)$, where $x \in K_0(B)$ is the element from (a). Since

$$\overline{\bigcup_k \left\{ a_0 p b_0 + a_1 \alpha(p) b_1 + \cdots + a_k \alpha^k(p) b_k : \ a_i, b_i \in M_n(B), \ i = 0, 1, \dots, k \right\}}$$

is a non-zero ideal I in $M_n(B)$ such that $\alpha(I) \subseteq I$, it follows from (b) that $I = M_n(B)$ and hence it contains a full projection. By definition of I this implies that there is a k such that $[p] + \alpha_*[p] + \alpha_*^2[p] + \cdots + \alpha_*^k[p]$ is an order-unit in $K_0(B)$. Set $y = x + \alpha_*(x) + \alpha_*^2(x) + \cdots + \alpha_*^k(x)$. Then $y - \alpha_*(y) = [p] + \alpha_*[p] + \alpha_*^2[p] + \cdots + \alpha_*^k[p]$. By exchanging y for x we may therefore assume that $x - \alpha_*(x) = [p]$ for some full projection p of $M_\infty(B)$, *i.e.*, $\alpha_*(x) \prec x$.

(d) Write $x = g_1 - g_2$, where $g_1, g_2 \in K_0(B)^+$. We may assume that $g_i \succ 0, i = 1, 2$. There is an $N \in \mathbb{N}$ such that

$$3g_1 + 3\alpha_*(g_2) \prec N\big(x - \alpha_*(x)\big) \quad \text{and} \quad g_1 + \alpha_*(g_2) \prec N[e].$$

Note that since $g_i \geq 0$, there are $L, n \in \mathbb{N}$ such that $g_i = [p_i]$ for some projection p_i in $M_L(B_n)$. Since in fact $g_i \succ 0$, it follows from the assumption about the unit of B_n being full in B_{n+1} for each n, that we can assume that p_i is full in $M_L(B_n)$. It follows then from Lemma 3.2 that we can realise the projections $p_i, i = 1, 2$, in a homogeneous C^*-subalgebra of $M_L(B_n)$ such that the assumptions of Lemma 3.1 hold for both. In this way we get elements $f_1, f_2 \in K_0(B)^+$ such that $Nf_j \leq g_j \leq (N+3)f_j$, $j = 1, 2$, and we set $f = f_1 - f_2$. Then

$$N[p] \succ 3g_1 + 3\alpha_*(g_2) \geq 3N\big(f_1 + \alpha_*(f_2)\big),$$

which implies that $[p] \succ 3(f_1 + \alpha_*(f_2))$ by Lemma 3.3. It follows first that

$$\begin{aligned} Nf &= x - (g_1 - Nf_1) + (g_2 - Nf_2) \geq \alpha_*(x) + [p] - 3f_1 \\ &= N\alpha_*(f) + \alpha_*\big((g_1 - Nf_1) - (g_2 - Nf_2)\big) + [p] - 3f_1 \\ &\geq N\alpha_*(f) + [p] - 3\big(f_1 + \alpha_*(f_2)\big) \succ N\alpha_*(f), \end{aligned}$$

and then from Lemma 3.3 that

$$f \succ \alpha_*(f). \tag{3.2}$$

Since $N[e] \succ g_1 + \alpha_*(g_2) \geq N(f_1 + \alpha_*(f_2))$, we have also that $[e] \succ f_1 + \alpha_*(f_2)$. It follows then from Lemma 3.3 that there are projections $p, q \in B$ such that $[p] = f_1$, $[q] = f_2$ and $p + \alpha(q) \leq e$. Then $[\alpha(p)] + [q] \prec [p] + [\alpha(q)]$ by (3.2), and another application of Lemma 3.3 implies that there is a partial isometry $t \in M_2(B)$ such that

$$t\begin{pmatrix} \alpha(p) & 0 \\ 0 & q \end{pmatrix} t^* \leq \begin{pmatrix} p + \alpha(q) & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$t\begin{pmatrix} \alpha(p) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix},$$

where $w, v \in B$ are partial isometries such that $v^*v = \alpha(p)$, $w^*w = q$, and $v\alpha(p)v^* + wqw^* < p + \alpha(q)$. Let u be the canonical unitary in the multiplier algebra of $B \times_\alpha \mathbb{Z}$ which implements α on B. Set $s = vup + wqu^* + (e - p - \alpha(q))$ and note that $s^*s = p + \alpha(q) + (e - p - \alpha(q)) = e$, while $ss^* = v\alpha(p)v^* + wqw^* + (e - p - \alpha(q)) < e$. ■

Let $0 < \epsilon < \frac{1}{2}$ and define $f_1^\epsilon, f_0^\epsilon : [0,1] \to [0,1]$ such that

$$f_1^\epsilon(t) = \begin{cases} 0, & 0 \le t \le \frac{\epsilon}{2} \\ 1, & t \in [\epsilon, 1] \\ \text{linear}, & \text{else} \end{cases}$$

and $f_0^\epsilon(t) = \max\{0, t - \epsilon\}$.

Lemma 3.5 *There is a $\delta > 0$ such that for all $\epsilon \in]0, \frac{1}{2}[$ the following holds: When b, b' are positive contractions in a C^*-algebra B such that*

$$\| f_1^\epsilon(b) - f_1^\epsilon(b') \| \le \delta \tag{3.3}$$

and $\overline{f_0^\epsilon(b')Bf_0^\epsilon(b')}$ contains a projection p, then $\overline{f_1^\epsilon(b)Bf_1^\epsilon(b)}$ contains a projection that is Murray–von Neumann equivalent to p.

Proof Let $\delta > 0$ be so small that $\overline{yBy}$ contains a projection Murray–von Neumann equivalent to q whenever x, y, q are positive contractions in a C^*-algebra B such that q is a projection and $\|xyx - q\| \le \delta$. This δ will work, because $f_1^\epsilon(b')p = p$, and it follows therefore from (3.3) that $\|pf_1^\epsilon(b)p - p\| \le \delta$. ■

Proof of Theorem 1.1 The general setup for the proof is the following. Let A_∞ be the inductive limit of the sequence

$$A \xrightarrow{\ \beta\ } A \xrightarrow{\ \beta\ } A \xrightarrow{\ \beta\ } \cdots \tag{3.4}$$

We can then define an automorphism α of A_∞ such that $\alpha \circ \rho_{\infty,n} = \rho_{\infty,n} \circ \beta$, where $\rho_{\infty,n} \colon A \to A_\infty$ is the canonical $*$-homomorphism from the n-th level in the sequence (3.4) into the inductive limit algebra. In this notation the inverse of α is defined such that $\alpha^{-1} \circ \rho_{\infty,n} = \rho_{\infty,n+1}$. Let $e \in A_\infty$ be the projection $e = \rho_{\infty,1}(1)$, which is a full projection of A_∞ by assumption (i) and hence also a full projection of the crossed product $A_\infty \times_\alpha \mathbb{Z}$. By a result of Stacey [St] there is an isomorphism $A \times_\beta \mathbb{N} \to e(A_\infty \times_\alpha \mathbb{Z})e$ sending $a \in A$ to $\rho_{\infty,1}(a)$ and the canonical isometry $v \in A \times_\beta \mathbb{N}$ to eue, where u is the canonical unitary in the multiplier algebra of $A_\infty \times_\alpha \mathbb{Z}$. Note that $A_\infty \times_\alpha \mathbb{Z}$ is stably isomorphic to $A \times_\beta \mathbb{N}$ and hence simple by assumption. Thanks to condition (i) the unit of $\rho_{\infty,k}(A)$ is full in $\rho_{\infty,k+1}(A)$ so that the sequence $B_k = \rho_{\infty,k}(A)$, $k = 1, 2, \dots$, will have properties required in Lemma 3.4. Furthermore, it follows from condition ii) that there can not be any non-zero densely defined lower semi-continuous α-invariant trace on A_∞; because if there was it would have to be non-zero on some B_k and it would then give rise to a β-invariant trace state on A. In this way it follows from Lemma 3.4 that every full projection of A_∞ is infinite in $A_\infty \times_\alpha \mathbb{Z}$. In fact, the same argument shows that a full projection in $M_k(A_\infty)$ is infinite in $M_k(A_\infty \times_\alpha \mathbb{Z})$ for any $k \in \mathbb{N}$.

We make now the following

Assertion 3.6 Let $h \in A_\infty \setminus \{0\}$ be a positive contraction. It follows that $\overline{h(A_\infty \times_\alpha \mathbb{Z})h}$ contains an infinite projection.

Assuming that Assertion 3.6 holds, the proof of Theorem 1.1 is completed as follows. Let $b \in (A_\infty \times_\alpha \mathbb{Z}) \setminus \{0\}$ be a positive contraction. Let $E\colon A_\infty \times_\alpha \mathbb{Z} \to A_\infty$ be the canonical conditional expectation. Let $\epsilon > 0$. As in the proof of [R2, Lemma 2.4] we can find positive elements $a, x \in A_\infty$ such that $\|a\| \geq 1 - \epsilon$, $\|x\| \leq 1$, and $\| \|E(b)\|^{-1} xbx - a\| \leq \epsilon$. The only change we have to make to Rørdam's argument is to replace the lemma used by Kishomoto with [OP2, Lemma 7.1]. Some backtracking through the work of Olesen and Pedersen is needed to verify that [OP2, Lemma 7.1] applies. What is needed is to show that the simplicity of $A_\infty \times_\alpha \mathbb{Z}$ forces all the automorphisms $\alpha^n, n \in \mathbb{Z}\setminus\{0\}$, to be properly outer, since this is the assumption in [OP2, Lemma 7.1]. This follows from the implication (i) $\Rightarrow$ (vi) of [OP2, Theorem 10.4], since the Connes spectrum $\Gamma(\alpha)$ is the whole circle by [OP1, Proposition 6.3].

Having the element a, set $a' = f(a)$, where $f\colon [0,1] \to [0,1]$ is a continuous function such that $f(t) = 1, t \in [1-2\epsilon, 1]$ and $|f(t) - t| \leq 2\epsilon$ for all $t \in [0,1]$. Then $\|a' - a\| \leq 2\epsilon$ and spectral theory gives us a positive element $h \in A_\infty$ such that $\|h\| = 1$ and $a'h = h$. It follows now from Assertion 3.6 that $\overline{h(A_\infty \times_\alpha \mathbb{Z})h}$ contains an infinite projection p. Since $\| \|E(b)\|^{-1} xbx - a'\| \leq 3\epsilon$ and $a'p = p$, we find that $\| \|E(b)\|^{-1} pxbxp - p\| \leq 3\epsilon$. Thus, if only ϵ is small enough $\|E(b)\|^{-1} \sqrt{b} xpx \sqrt{b}$ will be close to a projection in $\overline{b\,(A_\infty \times_\alpha \mathbb{Z})\,b}$ that is Murray–von Neumann equivalent to p and hence infinite. This shows that $A_\infty \times_\alpha \mathbb{Z}$ is purely infinite, as is $A \times_\beta \mathbb{N}$ since it is stably isomorphic to $A_\infty \times_\alpha \mathbb{Z}$ (*cf.* [PS, Proposition 5.5]).

It remains to prove Assertion 3.6. Since $A_\infty = \overline{\bigcup_n \rho_{\infty,n}(A)}$, an approximation argument based on Lemma 3.5 shows that we may assume that $h \in \rho_{\infty,n_0}(A)$ for some $n_0 \in \mathbb{N}$. Set $A' = \rho_{\infty,n_0}(1) A_\infty \rho_{\infty,n_0}(1)$ and note that $uA'u^* \subseteq A'$.

Deviating slightly from the notation used so far, let $T(A_\infty)$ denote the set of densely defined, lower, semi-continuous traces ω on A_∞ such that $\omega(e) = 1$. This is a compact space in a topology described before [Th2, Lemma 3], which is the same topology it gets through the identification of $T(A_\infty)$ with the tracial state space $T(eA_\infty e)$. Since $h \leq \rho_{\infty,n_0}(1)$, it follows that $\omega \mapsto \omega(u^k h u^{*k})$ is continuous on $T(A_\infty)$ for all k. We claim that there is an $m \in \mathbb{N}$ such that

$$\omega\big(h + uhu^* + u^2hu^{*2} + u^3hu^{*3} + \cdots + u^mhu^{*m}\big) > 0 \tag{3.5}$$

for all $\omega \in T(A_\infty)$. Indeed, if not, there is for each $n \in \mathbb{N}$ a trace $\omega_n \in T(A_\infty)$ such that $\omega_n(h + uhu^* + u^2hu^{*2} + u^3hu^{*3} + \cdots + u^nhu^{*n}) = 0$. A condensation point of $\{\omega_n\}$ in $T(A_\infty)$ will be a densely defined, lower, semi-continuous trace ω such that $\omega(u^nhu^{*n}) = 0$ for all n. Then $\{x \in A_\infty : \omega(u^n x^* x u^{*n}) = 0 \forall n\}$ is a non-zero, closed, two-sided ideal I in A_∞ such that $\alpha(I) \subseteq I$. As in (b) from the proof of Lemma 3.4, this implies that $I = A_\infty$, which is impossible, since $e \notin I$. This proves the claim.

Let $T(A')$ be the tracial state space of A'. Since $\beta(1)$ is full in A it follows that $\rho_{\infty,n_0}(1)$ is a full projection in A_∞, so any $\omega \in T(A')$ is the restriction to A' of a densely defined lower semi-continuous trace on A_∞ (*cf.* [Pe, Theorem 5.2.7]). It follows therefore from (3.5) that $\sum_{j=0}^m \omega\left(u^j h u^{*j}\right) > 0$ for all $\omega \in T(A')$. By compactness of $T(A')$, there is a $\delta > 0$ such that

$$\sum_{j=0}^{m} \omega(u^j h u^{*j}) \geq \delta$$

for all $\omega \in T(A')$. Since $A \simeq \rho_{\infty,n_0}(A)$ is tracially almost divisible by Proposition 2.3 and since $\rho_{\infty,n_0}(A)$ is a unital C^*-subalgebra of A' that contains h, there is a $\delta' > 0$ with the property that for any $\epsilon_1 > 0$ there are orthogonal positive elements $h_0, h_1, \ldots, h_m$ in A' such that $h_0 + h_1 + \cdots + h_m \preceq h$ in A' and $\tau(h_i) \geq \delta' \tau(h) - \epsilon_1$ for all i and all $\tau \in T(A')$. Let $\tau' \in T\left(M_{m+1}(A')\right)$ be the tracial state space of $M_{m+1}(A')$. Then $\tau' = \tau \otimes \mathrm{tr}$ for some $\tau \in T(A')$, where tr is the trace state of M_{m+1}. It follows that

$$\begin{aligned}
&\tau' \begin{pmatrix} h_0 & 0 & \ldots & 0 \\ 0 & uh_1u^* & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u^m h_m u^{*m} \end{pmatrix} \\
&\quad = \frac{1}{m+1} \sum_{j=0}^{m} \tau(u^j h_j u^{*j}) \\
&\quad \geq \frac{1}{m+1} \sum_{j=0}^{m} \Big(\delta' \tau\big(u^j h u^{*j}\big) - \tau\big(\rho_{\infty,n_0}(\beta^j(1))\big)\epsilon_1 \Big) \\
&\quad \geq \frac{\delta\delta'}{m+1} - \epsilon_1 \left(\sum_{j=0}^{m} \tau\Big(\rho_{\infty,n_0}\big(\beta^j(1)\big)\Big) \right).
\end{aligned}$$

Choose $\epsilon_1 > 0$ such that

$$\delta_1 = \frac{\delta\delta'}{m+1} - \epsilon_1 \sup_{\omega \in T(A')} \left(\sum_{j=0}^{m} \omega\Big(\rho_{\infty,n_0}\big(\beta^j(1)\big)\Big) \right) > 0.$$

Set

$$H = \begin{pmatrix} h_0 & 0 & \ldots & 0 \\ 0 & uh_1u^* & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u^m h_m u^m \end{pmatrix}$$

and notice that $\tau(H) \geq \delta_1$ for all $\tau \in T(M_{m+1}(A'))$. Let $\epsilon_0 \in \left]0, \frac{1}{4}\right[$ be so small that

$$\tau\big(f_0^{\epsilon_0}(H)\big) \geq \frac{\delta_1}{2} \tag{3.6}$$

for all $\tau \in T(M_{m+1}(A'))$. It may or may not be the case that A' is an AH-algebra with slow dimension growth, but since A has these properties and since

$$A' = \overline{\bigcup_{k \geq n_0} \rho_{\infty,n_0}(1)\rho_{\infty,k}(A)\rho_{\infty,n_0}(1)},$$

we can pick an increasing sequence $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ of finite subsets with dense union in A' and write $A' = \overline{\bigcup_l A_l}$ such that each A_l is a homogeneous C^*-algebra with $\rho_{\infty,n_0}(1) \in A_l$ and $F_l \subseteq_{\frac{1}{l}} A_l$, meaning that every element of F_l has distance less than $\frac{1}{l}$ to an element of A_l, and such that $\lim_{l\to\infty} r(A_l) = 0$. Let $\epsilon > 0$. We can then find $n_\epsilon \in \mathbb{N}$ and for each $l \geq n_\epsilon$ a positive contraction $k_l \in M_{m+1}(A_l)$ such that

$$\big\| f_1^{\epsilon_0}(H) - f_1^{\epsilon_0}(k_l)\big\| \leq \epsilon \quad \text{and} \quad \big\| f_0^{\epsilon_0}(H) - f_0^{\epsilon_0}(k_l)\big\| \leq \frac{\delta_1}{6}. \tag{3.7}$$

In particular, it follows from the last condition and (3.6) that there is an $n'_\epsilon \geq n_\epsilon$ such that

$$\tau\big(f_0^{\epsilon_0}(k_l)\big) \geq \frac{\delta_1}{4} \tag{3.8}$$

for all $\tau \in T(M_{m+1}(A_l))$ and all $l \geq n'_\epsilon$. (This is proved by contradiction. If there are arbitrary large n_i for which $T(M_{m+1}(A_{n_i}))$ contains an element τ_{n_i} with $\tau_{n_i}\left(f_0^{\epsilon_0}\left(k_{n_i}\right)\right) < \frac{\delta_1}{4}$, consider a state extension $\tilde{\tau_{n_i}}$ of τ_{n_i} to $M_{m+1}(A')$. A weak* condensation point of $\{\tilde{\tau_{n_i}}\}$ will be an element of $T(M_{m+1}(A'))$ for which (3.6) fails.)

Consider an $l \geq n'_\epsilon$. Let $d_\tau \colon M_{m+1}(A_l)^+ \to \mathbb{R}^+$ denote the dimension function corresponding to $\tau \in T(M_{m+1}(A_l))$, *i.e.*, $d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}})$. It follows then from (3.8) that

$$d_\tau\big(f_0^{\epsilon_0}(k_l)\big) \geq \frac{\delta_1}{4}.$$

Since $\lim_{l\to\infty} r(M_{m+1}(A_l)) = 0$, it follows from well-known properties of vector bundles, or from Theorem 2.4, that for all large l there is a projection $p_l \in M_{m+1}(A_l)$ with constant rank 1 over the spectrum of A_l. Then $d_\tau\left(p_l\right) \leq r(M_{m+1}(A_l))$, and hence

$$d_\tau\big(f_0^{\epsilon_0}(k_l)\big) \geq \frac{\delta_1}{4} > \frac{r(M_{m+1}(A_l))}{2} + d_\tau(p_l)$$

for all $\tau \in T(M_{m+1}(A_l))$ when l is large enough. Fix such an l. Theorem 2.4 now gives us a sequence $\{x_n\}$ in $M_{m+1}(A_l)$ such that

$$\lim_{n\to\infty} x_n f_0^{\epsilon_0}(k_l)\, x_n^* = p_l.$$

Note that p_l is a full projection in $M_{m+1}(A_\infty)$. As pointed out in the beginning of the proof p_l is then an infinite projection in $M_{m+1}(A_\infty \times_\alpha \mathbb{Z})$. Note also that $\|x_n\sqrt{f_0^{\epsilon_0}(k_l)}\| \leq 2$, which, combined with (3.7), implies that

$$\left\| x_n\sqrt{f_0^{\epsilon_0}(k_l)} f_1^{\epsilon_0}(H)\sqrt{f_0^{\epsilon_0}(k_l)}x_n^* - x_n\sqrt{f_0^{\epsilon_0}(k_l)} f_1^{\epsilon_0}(k_l)\sqrt{f_0^{\epsilon_0}(k_l)}x_n^*\right\| \leq 4\epsilon$$

for all large n. Since $f_1^{\epsilon_0} f_0^{\epsilon_0} = f_0^{\epsilon_0}$, we conclude that

$$\left\| x_n\sqrt{f_0^{\epsilon_0}(k_l)} f_1^{\epsilon_0}(H)\sqrt{f_0^{\epsilon_0}(k_l)}x_n^* - p_l\right\| \leq 5\epsilon$$

for all large n. Let $X \in M_{m+1}(A_\infty \times_\alpha \mathbb{Z})$ be the matrix

$$X = \begin{pmatrix} \sqrt{f_1^{\epsilon_0}(h_0)} & 0 & \dots & 0 \\ u\sqrt{f_1^{\epsilon_0}(h_1)} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u^m\sqrt{f_1^{\epsilon_0}(h_m)} & 0 & \dots & 0 \end{pmatrix}.$$

Then $XX^* = f_1^{\epsilon_0}(H)$, since the h_i's are mutually orthogonal and

$$X^*X = \operatorname{diag}\Big(\sum_{j=0}^{m} f_1^{\epsilon_0}(h_j), 0, 0, \dots, 0\Big),$$

i.e., $f_1^{\epsilon_0}(H) \sim X^*X$ in the sense of [KR], which implies that $f_1^{\epsilon_0}(H) \preceq X^*X$. Since $\sum_{j=0}^m f_1^{\epsilon_0}(h_j) = f_1^{\epsilon_0}\big(\sum_{j=0}^m h_j\big)$ and $\sum_{j=0}^m h_j \preceq h$, it follows from [W, Proposition 1.11] that there is an element d_0 in the hereditary C^*-subalgebra of A' generated by h such that $\sum_{j=0}^m f_1^{\epsilon_0}(h_j) \preceq d_0$. Thus $f_1^{\epsilon_0}(H) \preceq d_0'$, where $d_0' = \operatorname{diag}(d_0, 0, 0, \dots, 0)$, *i.e.*, there is a sequence $\{z_n\}$ in $M_{m+1}(A_\infty \times_\alpha \mathbb{Z})$ such that $\lim_n z_n d_0' z_n^* = f_1^{\epsilon_0}(H)$. Then

$$\Big\| x_n \sqrt{f_0^{\epsilon_0}(k_l)} z_{n'} d_0' z_{n'}^* \sqrt{f_0^{\epsilon_0}(k_l)} x_n^* - p_l \Big\| \leq 6\epsilon$$

when n' and n are sufficiently large, and it follows that

$$\sqrt{d_0'} z_{n'}^* \sqrt{f_0^{\epsilon_0}(k_l)} x_n^* x_n \sqrt{f_0^{\epsilon_0}(k_l)} z_{n'}^* \sqrt{d_0'}$$

will be close to a projection in $\overline{d_0' M_{m+1}(A_\infty \times_\alpha \mathbb{Z})\, d_0'}$ that is equivalent to p_l. Since p_l is infinite, this gives us the desired projection, completing the proof of Assertion 3.6 and hence the proof of the theorem. ■

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