13

Non-Abelian X-ray Transforms

In this chapter we introduce the non-Abelian X-ray transform and we study some of its basic properties. At first we discuss the theory in a fairly general setting for matrix-valued attenuations defined in the whole unit sphere bundle and then we discuss injectivity results when the attenuation is given by a connection plus a matrix field (a Higgs field) on the surface. The main result in this chapter is scattering rigidity up to the natural gauge when the connection and the matrix field take values in skew-Hermitian matrices. In order to show this, we establish an injectivity result for the geodesic X-ray transform with attenuation given by a skew-Hermitian connection and Higgs field. Using the ideas involved in the proof we also give an alternative proof of the tensor tomography problem. The skew-Hermitian assumption will be removed in Chapter 14, which gives a solution of the scattering rigidity problem when the connection and the matrix field take values in an arbitrary Lie algebra.

13.1 Scattering Data

Let (M, g) be a compact non-trapping manifold of dimension $d \geq 2$ with strictly convex boundary ∂M. Consider a matrix attenuation $\mathcal{A}$ as in Section 5.3, namely, let $\mathcal{A}\colon SM \to \mathbb{C}^{n \times n}$ be a smooth function. The notation deviates slightly from previous chapters: in this chapter we write $d = \dim M$, and the attenuation is an $n \times n$ matrix function.

Consider (M, g) isometrically embedded in a closed manifold (N, g) and extend $\mathcal{A}$ smoothly to SN. Under these assumptions, we have seen in Section 5.3 that $\mathcal{A}$ on SN defines a *smooth* cocycle over the geodesic flow φ_t of (N, g). Recall that the cocycle takes values in the group $GL(n, \mathbb{C})$ and is determined by the following matrix ODE along the orbits of the geodesic flow:

$$\frac{d}{dt}C(x, v, t) + \mathcal{A}(\varphi_t(x, v))C(x, v, t) = 0, \qquad C(x, v, 0) = \mathrm{Id}.$$

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In Lemma 5.3.2 we have seen that the function

$$U_+(x, v) := [C(x, v, \tau(x, v)]^{-1}$$

is smooth in SM and solves

$$\begin{cases} XU_+ + \mathcal{A}U_+ = 0, \\ U_+|_{\partial_- SM} = \mathrm{Id}. \end{cases} \tag{13.1}$$

Definition 13.1.1 The *scattering data* of $\mathcal{A}$ is the map

$$C_{\mathcal{A}} = C_{\mathcal{A},+} \colon \partial_+ SM \to GL(n, \mathbb{C}),$$

given by

$$C_{\mathcal{A},+} := U_+|_{\partial_+ SM}.$$

We shall also call $C_{\mathcal{A},+}$ *the non-Abelian X-ray transform* of $\mathcal{A}$.

Remark 13.1.2 Note that for $n = 1$ we may explicitly write

$$C_{\mathcal{A},+} = \exp(I(\mathcal{A})),$$

where $I(\mathcal{A})$ is the geodesic X-ray transform of $\mathcal{A}$. Thus having information on $C_{\mathcal{A},+}$ is equivalent to having information on $I(\mathcal{A})$. However, for $n \geq 2$ such a formula is no longer available due to non-commutativity of matrices and hence we use the name *non-Abelian X-ray transform*.

Note that $C_{\mathcal{A},+} \in C^\infty(\partial_+ SM, GL(n, \mathbb{C}))$. We can also consider the unique solution of

$$\begin{cases} XU_- + \mathcal{A}U_- = 0, \\ U_-|_{\partial_+ SM} = \mathrm{Id}, \end{cases} \tag{13.2}$$

and define scattering data $C_{\mathcal{A},-} \colon \partial_- SM \to GL(n, \mathbb{C})$ by setting

$$C_{\mathcal{A},-} := U_-|_{\partial_- SM}.$$

Both quantities are related by

$$C_{\mathcal{A},-} = [C_{\mathcal{A},+}]^{-1} \circ \alpha. \tag{13.3}$$

Exercise 13.1.3 Prove (13.3).

Remark 13.1.4 We can interpret the scattering data $C_{\mathcal{A},-}$ as follows. Let $(x, v) \in \partial_+ SM$ and let b be a vector in $\mathbb{C}^n$. Suppose that $b(t)$ solves the ODE

$$\dot{b}(t) + \mathcal{A}(\varphi_t(x, v))b(t) = 0, \qquad b(0) = b.$$

We consider an experiment where we send a vector b from a boundary point x in direction v and then we measure the vector $b(\tau(x, v))$ on the boundary when $b(t)$ exits M. Since $(X + \mathcal{A})(U_{-}b) = 0$, the measurement is given by $b(\tau(x,v)) = U_{-}(\alpha(x,v))b$. Thus knowing $C_{\mathcal{A},-}$ is equivalent to knowing how vectors evolve under the attenuation $\mathcal{A}$ when they travel through M along geodesics. This interpretation is particularly relevant when $\mathcal{A}$ corresponds to a connection, since then $b(t)$ is just the parallel transport of b with respect to this connection (see (13.7)).

By (13.3), if the metric g (and hence α) is known then $C_{\mathcal{A},+}$ and $C_{\mathcal{A},-}$ are equivalent information. From now on we shall only work with $C_{\mathcal{A},+}$ and we shall drop the subscript $+$ from the notation.

We conclude this section by describing some motivation for studying the non-Abelian X-ray transform. We will consider the special case where the attenuation is given by

$$\mathcal{A}(x,v) = A_x(v) + \Phi(x),$$

where A is an $n \times n$ matrix of smooth 1-forms in M, and Φ is a smooth $n \times n$ matrix function on M. We say that A is a *connection* and Φ is a *Higgs field*, and we write the scattering data as $C_{A,\Phi} := C_{\mathcal{A}}$. See Section 13.3 for more information on connections. Note that one has $\mathcal{A} \in \Omega_1 \oplus \Omega_0 \oplus \Omega_{-1}$, which is similar to Chapter 12 where we studied the scalar attenuated X-ray transform.

The map $(A,\Phi) \mapsto C_{A,\Phi}$ appears naturally in several contexts. For instance, when $\Phi = 0$, $C_{A,0}$ represents the parallel transport of the connection A along geodesics connecting boundary points. Then the injectivity question for the non-Abelian X-ray transform reduces to the question of recovering a connection up to gauge from its parallel transport along a distinguished set of curves, i.e. the geodesics of the metric g. We may also consider the twisted or connection Laplacian $d_A^* d_A$, where $d_A = d + A$. Egorov's theorem for the connection Laplacian naturally produces the parallel transport of A along geodesics of g as a high energy limit, cf. Jakobson and Strohmaier (2007, Proposition 3.3). This data can also be obtained from the corresponding wave equation following Oksanen et al. (2020); Uhlmann (2004).

When $A = 0$ and $\Phi \in C^\infty(M, \mathfrak{so}(3))$, the non-Abelian X-ray transform $\Phi \mapsto C_{0,\Phi}$ arises in Polarimetric Neutron Tomography (Desai et al., 2020; Hilger et al., 2018), a new tomographic method designed to detect magnetic fields inside materials by probing them with neutron beams. The case of pairs (A,Φ) arises in the literature on solitons, mostly in the context of the Bogomolny equations in $2+1$ dimensions (Manakov and Zakharov, 1981; Ward, 1988). Applications to coherent quantum tomography are given in Ilmavirta (2016). We refer to Novikov (2019) for a recent survey on the non-Abelian X-ray transform and its applications.

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13.2 Pseudo-linearization Identity

Given two $\mathcal{A}, \mathcal{B} \in C^\infty(SM, \mathbb{C}^{n\times n})$ we would like to have a formula that relates $C_\mathcal{A}$ and $C_\mathcal{B}$ with a certain attenuated X-ray transform. We first introduce the map $E(\mathcal{A}, \mathcal{B})\colon SM \to \mathrm{End}(\mathbb{C}^{n\times n})$ given by

$$E(\mathcal{A}, \mathcal{B})U := \mathcal{A}U - U\mathcal{B}.$$

Here, $\mathrm{End}(\mathbb{C}^{n\times n})$ denotes the linear endomorphisms of $\mathbb{C}^{n\times n}$.

Proposition 13.2.1 *Let (M, g) be a non-trapping manifold with strictly convex boundary. Given $\mathcal{A}, \mathcal{B} \in C^\infty(SM, \mathbb{C}^{n\times n})$, we have*

$$C_\mathcal{A} C_\mathcal{B}^{-1} = \mathrm{Id} + I_{E(\mathcal{A},\mathcal{B})}(\mathcal{A} - \mathcal{B}), \tag{13.4}$$

where $I_{E(\mathcal{A},\mathcal{B})}$ denotes the attenuated X-ray transform with attenuation $E(\mathcal{A}, \mathcal{B})$ as defined in Definition 5.3.3.

Proof Consider the fundamental solutions for both $\mathcal{A}$ and $\mathcal{B}$, namely

$$\begin{cases} XU_\mathcal{A} + \mathcal{A}U_\mathcal{A} = 0, \\ U_\mathcal{A}|_{\partial_- SM} = \mathrm{Id}, \end{cases}$$

and

$$\begin{cases} XU_\mathcal{B} + \mathcal{B}U_\mathcal{B} = 0, \\ U_\mathcal{B}|_{\partial_- SM} = \mathrm{Id}. \end{cases}$$

Let $W := U_\mathcal{A} U_\mathcal{B}^{-1} - \mathrm{Id}$. A direct computation shows that

$$\begin{cases} XW + \mathcal{A}W - W\mathcal{B} = -(\mathcal{A} - \mathcal{B}), \\ W|_{\partial_- SM} = 0. \end{cases}$$

By definition of $I_{E(\mathcal{A},\mathcal{B})}$ we have

$$I_{E(\mathcal{A},\mathcal{B})}(\mathcal{A} - \mathcal{B}) = W|_{\partial_+ SM}.$$

Since by construction $W|_{\partial_+ SM} = C_\mathcal{A} C_\mathcal{B}^{-1} - \mathrm{Id}$, the proposition follows. □

Remark 13.2.2 Note that the function $U := U_\mathcal{A} U_\mathcal{B}^{-1}$ satisfies

$$\begin{cases} \mathcal{B} = U^{-1}XU + U^{-1}\mathcal{A}U, \\ U|_{\partial_- SM} = \mathrm{Id}. \end{cases}$$

The identity (13.4) is called a *pseudo-linearization identity*, since it reduces the non-linear inverse problem of determining $\mathcal{A}$ (up to gauge) from $C_\mathcal{A}$ into the linear inverse problem of inverting the X-ray transform $I_{E(\mathcal{A},\mathcal{B})}$ (up to a

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natural kernel), where the attenuation $E(\mathcal{A},\mathcal{B})$ depends on $\mathcal{A}$ and $\mathcal{B}$. Namely, $C_{\mathcal{A}} = C_{\mathcal{B}}$ if and only if

$$I_{E(\mathcal{A},\mathcal{B})}(\mathcal{A}-\mathcal{B}) = 0.$$

We can also phrase this result in terms of a transport equation problem.

Proposition 13.2.3 *Let (M,g) be a non-trapping manifold with strictly convex boundary. Given $\mathcal{A},\mathcal{B} \in C^\infty(SM,\mathbb{C}^{n\times n})$, we have $C_{\mathcal{A}} = C_{\mathcal{B}}$ if and only if there exists a smooth $U\colon SM \to GL(n,\mathbb{C})$ with $U|_{\partial SM} = \mathrm{Id}$ and such that*

$$\mathcal{B} = U^{-1}XU + U^{-1}\mathcal{A}U.$$

Proof If such a smooth function U exists, then the function $V = UU_{\mathcal{B}}$ satisfies $XV + \mathcal{A}V = 0$ and $V|_{\partial_- SM} = \mathrm{Id}$. Therefore $V = U_{\mathcal{A}}$ and consequently $C_{\mathcal{A}} = C_{\mathcal{B}}$. Conversely, if the non-Abelian X-ray transforms agree, the function W in the proof of Proposition 13.2.1 has zero boundary value and by Theorem 5.3.6 it must be smooth. Hence $U = W + \mathrm{Id}$ is smooth and by Remark 13.2.2 it satisfies the required equation. □

Exercise 13.2.4 Consider the Hermitian inner product on the set of $n \times n$ matrices $\mathbb{C}^{n\times n}$ given by $(U,V) = \mathrm{trace}(UV^*)$ where V^* denotes the conjugate transpose of V. Show that the adjoint of $E(\mathcal{A},\mathcal{B})$ with respect to this inner product is

$$[E(\mathcal{A},\mathcal{B})]^*U = E(\mathcal{A}^*,\mathcal{B}^*)U.$$

Conclude that if both $\mathcal{A}$ and $\mathcal{B}$ are skew-Hermitian, i.e. $\mathcal{A}^* = -\mathcal{A}$ and $\mathcal{B}^* = -\mathcal{B}$, then $E^* = -E$ as well.

13.3 Elementary Background on Connections

To make further progress in the study of the non-Abelian X-ray transform on surfaces we would like to consider attenuations $\mathcal{A}$ of a special type, namely those with Fourier expansion in $\Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. It turns out that this is equivalent to giving a *connection* (corresponding to the Fourier modes in $\Omega_{-1} \oplus \Omega_1$) and a matrix-valued *Higgs field* (corresponding to the Fourier mode in Ω_0). In this section we make a brief interlude to give some background on connections in a way that is suitable for our setting.

Consider the trivial bundle $M \times \mathbb{C}^n$. For us a connection A will be a complex $n \times n$ matrix whose entries are smooth 1-forms on M. Another way to think of A is to regard it as a smooth map $A\colon TM \to \mathbb{C}^{n\times n}$ that is linear in $v \in T_xM$ for each $x \in M$.

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Very often in physics and geometry one considers *unitary* or *Hermitian* connections. This means that the range of A is restricted to skew-Hermitian matrices. In other words, if we denote by $\mathfrak{u}(n)$ the Lie algebra of the unitary group $U(n)$, we have a smooth map $A\colon TM \to \mathfrak{u}(n)$ that is linear in the velocities. There is yet another equivalent way to phrase this. The connection A induces a covariant derivative d_A on sections $s \in C^\infty(M,\mathbb{C}^n)$ by setting $d_A s = ds + As$. Then A being Hermitian or unitary is equivalent to requiring compatibility with the standard Hermitian inner product of $\mathbb{C}^n$ in the sense that

$$d\langle s_1, s_2\rangle = \langle d_A s_1, s_2\rangle + \langle s_1, d_A s_2\rangle,$$

for any pair of functions s_1, s_2. The set of all smooth unitary connections is denoted by $\Omega^1(M,\mathfrak{u}(n))$.

Given two unitary connections A and B we shall say that A and B are *gauge equivalent* if there exists a smooth map $u\colon M \to U(n)$ such that

$$B = u^{-1}du + u^{-1}Au. \tag{13.5}$$

In terms of the derivative d_A acting on sections, gauge equivalence means just that

$$d_A(us) = u(d_B s), \qquad s \in C^\infty(M,\mathbb{C}^n). \tag{13.6}$$

The *curvature* of the connection is the operator $F_A = d_A \circ d_A$ acting on sections, written more precisely as

$$F_A s = (d + A\wedge)(ds + As) = (dA + A \wedge A)s,$$

where we used the properties of the exterior derivative d. Thus F_A is in fact a 2-form with values in $\mathfrak{u}(n)$ given by

$$F_A := dA + A \wedge A.$$

This can be written elementwise: if $A = (A_{jk})_{j,k=1}^n$ where each A_{jk} is a scalar 1-form, then

$$F_A = \left(dA_{jk} + \sum_{l=1}^d A_{jl} \wedge A_{lk}\right)_{j,k=1}^n.$$

If A and B are gauge equivalent as in (13.5), then by (13.6) one has $F_B = d_B \circ d_B = u^{-1}d_A u \circ u^{-1}d_A u$. This shows that the curvatures of gauge equivalent connections satisfy

$$F_B = u^{-1}\, F_A\, u.$$

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Given a smooth curve $\gamma\colon [a,b] \to M$, the *parallel transport* of a vector $w \in \mathbb{C}^n$ along γ with respect to the connection A is obtained by solving the following linear differential equation:

$$\begin{cases} \dot{s} + A(\gamma(t), \dot{\gamma}(t))s = 0, \\ s(a) = w. \end{cases} \tag{13.7}$$

The parallel transport operator $P_A(\gamma)\colon \mathbb{C}^n \to \mathbb{C}^n$ is defined as

$$P_A(\gamma)(w) := s(b).$$

It is an isometry since A is unitary. We also consider the fundamental unitary matrix solution $U\colon [a,b] \to U(n)$ of (13.7). It solves

$$\begin{cases} \dot{U} + A(\gamma(t), \dot{\gamma}(t))U = 0, \\ U(a) = \mathrm{Id}. \end{cases} \tag{13.8}$$

Clearly $P_A(\gamma)(w) = U(b)w$.

A connection A naturally gives rise to a matrix attenuation of special type, simply by setting $\mathcal{A}(x,v) := A(x,v)$. Note that since A is a matrix of 1-forms, it is completely determined by its values on SM. The scattering data $C_A\colon \partial_+ SM \to GL(n,\mathbb{C})$ encapsulates the parallel transport of A along geodesics running between boundary points.

In the next chapter we will be interested in connections taking values in an arbitrary Lie algebra $\mathfrak{g}$. We shall denote the space of such connections as $\Omega^1(M,\mathfrak{g})$.

13.4 Structure Equations Including a Connection

In this section we consider an oriented Riemannian surface (M,g) and a connection A on the trivial bundle $M \times \mathbb{C}^n$. We will regard A both as a matrix 1-form on M, and as a function $A\colon SM \to \mathbb{C}^{n\times n}$ with $A \in \Omega_{-1} \oplus \Omega_1$. Recall that the metric g induces a Hodge star operator $\star$ acting on forms. We claim that

$$\star A = -VA.$$

This follows from the computation

$$VA(x,v) = A(x,v^{\perp}) = -\star A(x,v),$$

where $v^{\perp}$ is the rotation of v by 90° counterclockwise.

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The main purpose of this section is to establish the following lemma that generalizes the basic commutator formulas in Lemma 3.5.5 to the case where X is replaced by $X + A$ and $X_\perp$ by $X_\perp + \star A$. Here we understand that A and $\star A$ act on functions by multiplication.

Lemma 13.4.1 *The following equations hold:*

$$\begin{aligned}
[V, X + A] &= -(X_\perp + \star A),\\
[V, X_\perp + \star A] &= X + A,\\
[X + A, X_\perp + \star A] &= -KV - \star F_A.
\end{aligned}$$

Proof Let us recall the standard bracket relations from Lemma 3.5.5:

$$\begin{aligned}
[V, X] &= -X_\perp,\\
[V, X_\perp] &= X,\\
[X, X_\perp] &= -KV.
\end{aligned}$$

Hence the first two bracket relations in the lemma follow from $[V, A] = V(A) = -\star A$ and $[V, \star A] = -V^2(A) = A$. To check the third bracket it suffices to prove that

$$\star F_A = X_\perp(A) - X(\star A) + [\star A, A]. \tag{13.9}$$

Given a unit vector $v \in T_x M$, $(v, v^\perp)$ is a positively oriented orthonormal basis. Thus

$$\begin{aligned}
\star F_A(x) = F_A(v, v^\perp) &= dA(v, v^\perp) + (A \wedge A)(v, v^\perp)\\
&= dA(v, v^\perp) + [A(v), A(v^\perp)].
\end{aligned}$$

But $\star A(x, v) = -A(v^\perp)$ and hence $[\star A, A](x, v) = [-A(v^\perp), A(v)]$. Thus to complete the proof of (13.9) we just have to show that

$$X_\perp(A)(x, v) - X(\star A)(x, v) = dA(v, v^\perp).$$

Let $\pi : SM \to M$ be the canonical projection. Recall that $d\pi(X(x, v)) = v$ and $d\pi(X_\perp(x, v)) = -v^\perp$. Consider $\pi^* A$ and note (using the standard formula for d applied to $\pi^* A$) that

$$d(\pi^* A)(X, X_\perp) = X(\pi^* A(X_\perp)) - X_\perp(\pi^* A(X)) - \pi^* A([X, X_\perp]).$$

By the structure equations, the term $[X, X_\perp]$ is purely vertical, hence it is killed by $\pi^* A$. Next note that $(\pi^* A(X_\perp))(x, v) = A(-v^\perp) = (\star A)(v)$ and $\pi^* A(X) = A(v)$. This shows that

$$d(\pi^* A)(X, X_\perp) = X(\star A) - X_\perp(A).$$

Finally, note that

$$d(\pi^* A)(X, X_\perp) = (\pi^* dA)(X, X_\perp) = dA(d\pi(X), d\pi(X_\perp))$$
$$= -dA(v, v^\perp).$$

This concludes the proof. □

Given a connection $A \in \Omega_{-1} \oplus \Omega_1$ we write it as $A = A_{-1} + A_1$ with $A_{\pm 1} \in \Omega_{\pm 1}$. Next we consider the Guillemin–Kazhdan operators $\eta_\pm$ from Definition 6.1.4 in the presence of a connection.

Definition 13.4.2 If (M, g) is a Riemann surface and A is a connection, define

$$\mu_\pm := \eta_\pm + A_{\pm 1}.$$

Clearly $X + A = \mu_+ + \mu_-$. These operators also satisfy nice bracket relations.

Lemma 13.4.3 *The following bracket relations hold:*

$$[\mu_\pm, iV] = \pm\mu_\pm, \quad [\mu_+, \mu_-] = \frac{i}{2}\,(KV + \star F_A)\,.$$

Moreover

$$\mu_+ \colon \Omega_k \to \Omega_{k+1}, \quad \mu_- \colon \Omega_k \to \Omega_{k-1}.$$

If A is unitary, one has $(\mu_\pm)^* = -\mu_\mp$.

Proof We only prove the relation $[\mu_+, \mu_-] = \frac{i}{2}(KV + \star F_A)$, the rest is left as an exercise. First we note that

$$\mu_\pm = \frac{(X + A) \pm i(X_\perp + \star A)}{2}.$$

Hence

$$[\mu_+, \mu_-] = \frac{i}{2}[X_\perp + \star A, X + A],$$

and the desired relation follows from Lemma 13.4.1. □

Exercise 13.4.4 Complete the details in the proof of Lemma 13.4.3.

Exercise 13.4.5 Show that $X + A$ maps even functions to odd functions and odd functions to even functions.

Exercise 13.4.6 Let A be a connection and let $\Phi \in C^\infty(M, \mathbb{C}^{n\times n})$. If H denotes the Hilbert transform, show that for any smooth function $u \in C^\infty(SM, \mathbb{C}^n)$ one has

$$[H, X + A + \Phi]u = (X_\perp + \star A)(u_0) + ((X_\perp + \star A)(u))_0.$$

13.5 Scattering Rigidity and Injectivity for Connections

In this section we would like to consider the following geometric inverse problem: is a connection A determined by C_A?

We first observe that the problem has a gauge. Let A and B be two gauge equivalent connections, so that (as functions on SM)

$$B = u^{-1}Xu + u^{-1}Au,$$

where $u\colon M \to GL(n,\mathbb{C})$ is a smooth map with $u|_{\partial M} = \mathrm{Id}$. If U_A solves $XU_A + AU_A = 0$ with $U_A|_{\partial_- SM} = \mathrm{Id}$, then

$$(X+B)(u^{-1}U_A) = -u^{-1}(Xu)u^{-1}U_A + u^{-1}XU_A + Bu^{-1}U_A = 0,$$

and $u^{-1}U_A|_{\partial_- SM} = \mathrm{Id}$. It follows that $u^{-1}U_A = U_B$ and hence

$$C_{u^{-1}du+u^{-1}Au} = C_A.$$

Our main goal will be to show the following result.

Theorem 13.5.1 *Let (M,g) be a simple surface and let A and B be two unitary connections with $C_A = C_B$. Then there exists a smooth $u\colon M \to U(n)$ with $u|_{\partial M} = \mathrm{Id}$ such that $B = u^{-1}du + u^{-1}Au$.*

From Proposition 13.2.3 we know that $C_A = C_B$ means that there exists a smooth $U\colon SM \to U(n)$ such that $U|_{\partial SM} = \mathrm{Id}$ and

$$B = U^{-1}XU + U^{-1}AU.$$

Notice the similarity of this equation with our goal, which is to show that

$$B = u^{-1}du + u^{-1}Au.$$

In fact if U only had dependence on x and not on v, then $U = u$, $XU(x,v) = du|_x(v)$ and we would be done. We will accomplish this for a simple surface.

We start by rephrasing our problem in terms of an attenuated X-ray transform. Showing that U depends only on x is equivalent to showing that $W = U - \mathrm{Id}$ depends only on x. But as we have seen, if $C_A = C_B$ then W satisfies the equation

$$XW + AW - WB = -(A-B) \text{ in } SM, \qquad W|_{\partial SM} = 0.$$

This means that the attenuated X-ray transform $I_{E(A,B)}(A-B)$ vanishes. Note that $A - B \in \Omega_{-1} \oplus \Omega_1$.

Hence, making the choice to ignore the specific form of the connection $E(A,B)$ but noting that it is unitary by Exercise 13.2.4, the proof of

Theorem 13.5.1 reduces to showing the following important injectivity result for the attenuated X-ray transform with a connection.

Theorem 13.5.2 *Let (M, g) be a simple surface and let A be a unitary connection. Suppose that $u \in C^\infty(SM, \mathbb{C}^n)$ satisfies*

$$\begin{cases} Xu + Au = f \in \Omega_{-1} \oplus \Omega_1, \\ u|_{\partial SM} = 0. \end{cases}$$

Then $u = u_0$ and $f = d_A u_0 = du_0 + Au_0$ with $u_0|_{\partial M} = 0$.

The first key ingredient in the proof of Theorem 13.5.2 is an energy identity that generalizes the standard Pestov identity from Proposition 4.3.2 to the case when a connection is present. Recall that the curvature F_A of the connection A is defined as $F_A = dA + A \wedge A$ and $\star F_A$ is a function $\star F_A : M \to \mathfrak{u}(n)$.

Lemma 13.5.3 (Pestov identity with connection) *Suppose that (M, g) is a compact surface with boundary, and let A be a unitary connection. If $u : SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial SM} = 0$, then*

$$\begin{aligned} &\|V(X + A)u\|^2 \\ &\qquad = \|(X + A)Vu\|^2 - (K\, Vu, Vu) - (\star F_A u, Vu) + \|(X + A)u\|^2. \end{aligned}$$

Proof We adopt the same approach as in the proof of Proposition 4.3.2 and define $P = V(X + A)$. Since A is a unitary connection, $A^* = -A$ and hence $P^* = (X + A)V$. Let us compute using the structure equations from Lemma 13.4.1:

$$\begin{aligned} [P^*, P] &= (X + A)VV(X + A) - V(X + A)(X + A)V \\ &= V(X + A)V(X + A) + (X_\perp + \star A)V(X + A) \\ &\quad - V(X + A)V(X + A) - V(X + A)(X_\perp + \star A) \\ &= V[X_\perp + \star A, X + A] - (X + A)^2 = -(X + A)^2 + VKV + \star F_A V. \end{aligned}$$

The identity in the lemma now follows from this bracket calculation and

$$\|Pu\|^2 = \|P^* u\|^2 + ([P^*, P]u, u)$$

for a smooth u with $u|_{\partial SM} = 0$. □

In order to use the Pestov identity with a connection, we need to control the signs of various terms. The first easy observation is the following.

Lemma 13.5.4 *Assume $(X + A)u = f_{-1} + f_0 + f_1 \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. Then*

$$\|(X + A)u\|^2 = \|V(X + A)u\|^2 + \|f_0\|^2.$$

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Proof It suffices to note the identities

$$\|V(X+A)u\|^2 = \|V(f_{-1}+f_1)\|^2 = \|-if_{-1}+if_1\|^2 = \|f_{-1}\|^2 + \|f_1\|^2,$$
$$\|(X+A)u\|^2 = \|f_{-1}\|^2 + \|f_1\|^2 + \|f_0\|^2. \qquad \square$$

Next we have the following lemma due to the absence of conjugate points on simple surfaces (compare with Proposition 4.4.3).

Lemma 13.5.5 *Let M be a compact simple surface. If $u\colon SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial SM} = 0$, then*

$$\|(X+A)Vu\|^2 - (K\,Vu, Vu) \geq 0.$$

Proof Consider a smooth function $a\colon SM \to \mathbb{R}$ that solves the Riccati equation $Xa + a^2 + K = 0$. These exist by the absence of conjugate points, see Proposition 4.6.1. Set for simplicity $\psi = V(u)$. Clearly $\psi|_{\partial SM} = 0$.

Let us compute using that A is skew-Hermitian:

$$\begin{aligned}
&|(X+A)(\psi) - a\psi|^2_{\mathbb{C}^n}\\
&\quad = |(X+A)(\psi)|^2_{\mathbb{C}^n} - 2\,\mathrm{Re}\,\langle (X+A)(\psi), a\psi\rangle_{\mathbb{C}^n} + a^2|\psi|^2_{\mathbb{C}^n}\\
&\quad = |(X+A)(\psi)|^2_{\mathbb{C}^n} - 2a\,\mathrm{Re}\,\langle X(\psi), \psi\rangle_{\mathbb{C}^n} + a^2|\psi|^2_{\mathbb{C}^n}.
\end{aligned}$$

Using the Riccati equation we have

$$X\big(a|\psi|^2_{\mathbb{C}^n}\big) = \big(-a^2 - K\big)|\psi|^2_{\mathbb{C}^n} + 2a\,\mathrm{Re}\,\langle X(\psi), \psi\rangle_{\mathbb{C}^n}.$$

Thus

$$|(X+A)(\psi) - a\psi|^2_{\mathbb{C}^n} = |(X+A)(\psi)|^2_{\mathbb{C}^n} - K|\psi|^2_{\mathbb{C}^n} - X\big(a|\psi|^2_{\mathbb{C}^n}\big).$$

Integrating this equality over SM with respect to $d\Sigma^3$ and using that ψ vanishes on ∂SM we obtain

$$\|(X+A)(\psi)\|^2 - (K\,\psi, \psi) = \|(X+A)(\psi) - a\psi\|^2 \geq 0. \qquad \square$$

We now show an analogue of Proposition 10.2.6 in the presence of a connection.

Theorem 13.5.6 *Let $f\colon SM \to \mathbb{C}^n$ be a smooth function. Suppose $u\colon SM \to \mathbb{C}^n$ satisfies*

$$\begin{cases} Xu + Au = f,\\ u|_{\partial SM} = 0.\end{cases}$$

Then if $f_k = 0$ for all $k \leq -2$ and $i \star F_A(x)$ is a negative definite Hermitian matrix for all $x \in M$, the function u must be holomorphic. Moreover, if $f_k = 0$ for all $k \geq 2$ and $i \star F_A(x)$ is a positive definite Hermitian matrix for all $x \in M$, the function u must be anti-holomorphic.

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Proof Let us assume that $f_k = 0$ for $k \le -2$ and $i \star F_A$ is a negative definite Hermitian matrix; the proof of the other claim is similar.

Let $q := \sum_{-\infty}^{-1} u_k$. We need to show that $q = 0$. Since $A = A_{-1} + A_1$ and $f_k = 0$ for $k \le -2$, we see that $(X + A)q \in \Omega_{-1} \oplus \Omega_0$. Now we are in good shape to use the Pestov identity from Lemma 13.5.3. We will apply it to q, noting that $q|_{\partial SM} = 0$. We know from Lemma 13.5.4 that

$$\|(X + A)q\|^2 = \|V(X + A)q\|^2 + \|h_0\|^2,$$

for some $h_0 \in \Omega_0$. Using Lemma 13.5.5 in the Pestov identity implies that

$$0 = \|(X + A)Vq\|^2 - (K\, Vq, Vq) - (\star F_A q, Vq) + \|h_0\|^2 \ge -(\star F_A q, Vq).$$

Thus

$$(\star F_A q, Vq) \ge 0.$$

But on the other hand

$$(\star F_A q, Vq) = -\sum_{k=-\infty}^{-1} k(i \star F_A u_k, u_k),$$

and since $i \star F_A$ is negative definite this forces $u_k = 0$ for all $k < 0$. □

Note that Theorem 13.5.6 allows us to control the negative Fourier coefficients of u if $f_k = 0$ for $k \le -2$ and if the matrix $i \star F_A$ is negative definite. Thus if we start with a solution of $(X + A)u = f$ as in Theorem 13.5.2, we would like to apply a holomorphic integrating factor to end up with an equation like

$$(X + A_s)\tilde{u} = \tilde{f},$$

where $\tilde{f}_k = 0$ for $k \le -2$ and $i \star F_{A_s}$ is negative definite. We can achieve this by choosing a holomorphic integrating factor related to the area form of g. This idea, which corresponds to twisting the trivial bundle $M \times \mathbb{C}^n$ so that its curvature becomes negative, was introduced in Paternain et al. (2012) and it also appears in the proof of the Kodaira vanishing theorem in complex geometry.

We are now ready to complete the proof of Theorem 13.5.2.

Proof Consider the area form ω_g of the metric g (in earlier notation we had $\omega_g = dV^2$). Since M is simply connected, there exists a smooth real-valued 1-form φ such that $\omega_g = d\varphi$. Given $s \in \mathbb{R}$, consider the Hermitian connection

$$A_s := A - is\varphi\,\mathrm{Id}.$$

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Clearly its curvature is given by

$$F_{A_s} = F_A - is\omega_g \mathrm{Id}.$$

Therefore

$$i \star F_{A_s} = i \star F_A + s\mathrm{Id},$$

from which we see that there exists $s_0 > 0$ such that for $s > s_0$, $i \star F_{A_s}$ is positive definite and for $s < -s_0$, $i \star F_{A_s}$ is negative definite.

Let e^{sw} be an integrating factor of $-is\varphi$. In other words $w\colon SM \to \mathbb{C}$ satisfies $X(w) = i\varphi$. By Proposition 10.1.2 we know we can choose w to be holomorphic or anti-holomorphic. Observe now that $u_s := e^{sw}u$ satisfies $u_s|_{\partial SM} = 0$ and solves

$$(X + A_s)(u_s) = e^{sw} f.$$

Choose w to be holomorphic. Since $f \in \Omega_{-1} \oplus \Omega_1$, the function $e^{sw} f$ has the property that its Fourier coefficients $(e^{sw} f)_k$ vanish for $k \leq -2$. Choose s such that $s < -s_0$ so that $i \star F_{A_s}$ is negative definite. Then Theorem 13.5.6 implies that u_s is holomorphic and thus $u = e^{-sw}u_s$ is also holomorphic.

Choosing w anti-holomorphic and $s > s_0$ we show similarly that u is anti-holomorphic. This implies that $u = u_0$. Together with the fact that $(X+A)u = f$, this gives $du_0 + Au_0 = f$. □

13.6 An Alternative Proof of Tensor Tomography

In this section we shall use the ideas from Section 13.5 to give an alternative proof of Corollary 10.2.7 for the case where (M, g) is a simple surface.

Corollary 10.2.7 is an immediate consequence of the next result, which is a special case of Proposition 10.2.6. Recall that Proposition 10.2.6 was proved by applying a holomorphic integrating factor for the connection $A = r^{-1}Xr$ where $r = e^{-im\theta}$. The proof below will use a connection related to the area form instead, together with a Beurling contraction type argument similar to the one in Theorem 7.1.2. Both of these proofs were given in Paternain et al. (2013).

Proposition 13.6.1 *Let (M, g) be a simple surface, and assume that $u \in C^\infty(SM)$ satisfies $Xu = -f$ in SM with $u|_{\partial SM} = 0$. If $m \geq 0$ and if $f \in C^\infty(SM)$ is such that $f_k = 0$ for $k \leq -m-1$, then $u_k = 0$ for $k \leq -m$. Similarly, if $m \geq 0$ and if $f \in C^\infty(SM)$ is such that $f_k = 0$ for $k \geq m+1$, then $u_k = 0$ for $k \geq m$.*

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Below we will use the operators $\mu_\pm$ introduced in Section 13.4. Recall that when A is unitary, one has

$$(\mu_\pm u, v) = -(u, \mu_\mp v) \tag{13.10}$$

for $u, v \in C^\infty(SM)$ with $u|_{\partial SM} = v|_{\partial SM} = 0$. We also recall the commutator formula from Lemma 13.4.3:

$$[\mu_+, \mu_-]u = \frac{i}{2}(KVu + (\star F_A)u). \tag{13.11}$$

Proof of Proposition 13.6.1 We will only prove the first claim in Proposition 13.6.1, the proof of the second claim being completely analogous. Assume that f is even, m is even, and u is odd. Let ω_g be the area form of (M, g) and choose a real-valued 1-form φ with $d\varphi = \omega_g$. Consider the unitary connection

$$A(x, v) := is\varphi_x(v),$$

where $s > 0$ is a fixed number to be chosen later. Then $i \star F_A = -s$. By Proposition 10.1.2, there exists a holomorphic $w \in C^\infty(SM)$ satisfying $Xw = -i\varphi$. We may assume that w is even. The functions $\tilde{u} := e^{sw}u$ and $\tilde{f} := e^{sw}f$ then satisfy

$$(X + A)\tilde{u} = -\tilde{f} \text{ in } SM, \quad \tilde{u}|_{\partial SM} = 0.$$

Using that e^{sw} is holomorphic, we have $\tilde{f}_k = 0$ for $k \leq -m-1$. Also, since e^{sw} is even, $\tilde{f}$ is even and $\tilde{u}$ is odd. We now define

$$v := \sum_{k=-\infty}^{-m-1} \tilde{u}_k.$$

Then $v \in C^\infty(SM)$, $v|_{\partial SM} = 0$, and v is odd. Also, $((X+A)v)_k = \mu_+ v_{k-1} + \mu_- v_{k+1}$. If $k \leq -m-2$ one has $((X+A)v)_k = ((X+A)\tilde{u})_k = 0$, and if $k \geq -m+1$ then $((X+A)v)_k = 0$ since $v_j = 0$ for $j \geq -m$. Also $((X+A)v)_{-m-1} = 0$ because v is odd. Therefore the only nonzero Fourier coefficient is $((X+A)v)_{-m}$, and

$$(X + A)v = \mu_+ v_{-m-1} \text{ in } SM, \quad v|_{\partial SM} = 0.$$

We apply the Pestov identity in Lemma 13.5.3 with attenuation A to v, so that

$$\|V(X+A)v\|^2 = \|(X+A)Vv\|^2 - (KVv, Vv) + (\star F_A Vv, v) + \|(X+A)v\|^2.$$

We know from Lemma 13.5.5 that if (M, g) is simple and $v|_{\partial SM} = 0$, then

$$\|(X + A)Vv\|^2 - (KVv, Vv) \geq 0. \tag{13.12}$$

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We also have

$$(\star F_A V v, v) = -\sum_{k=-\infty}^{-m-1} i|k|(\star F_A v_k, v_k) = s \sum_{k=-\infty}^{-m-1} |k| \|v_k\|^2. \tag{13.13}$$

For the remaining two terms, we compute

$$\|(X + A)v\|^2 - \|V(X + A)v\|^2 = \|\mu_+ v_{-m-1}\|^2 - m^2 \|\mu_+ v_{-m-1}\|^2.$$

If $m = 0$, then this expression is non-negative and we obtain from the Pestov identity that $v = 0$. Assume from now on that $m \geq 2$. Using (13.10), (13.11), and the fact that $v_k|_{\partial SM} = 0$ for all k, we have

$$\begin{aligned}\|\mu_+ v_k\|^2 &= \|\mu_- v_k\|^2 + \frac{i}{2}(K V v_k + (\star F_A)v_k, v_k) \\ &= \|\mu_- v_k\|^2 - \frac{s}{2}\|v_k\|^2 - \frac{k}{2}(K v_k, v_k).\end{aligned}$$

If $k \leq -m - 1$ we also have

$$\mu_+ v_{k-1} + \mu_- v_{k+1} = ((X + A)v)_k = 0.$$

We thus obtain

$$\begin{aligned}&\|(X + A)v\|^2 - \|V(X + A)v\|^2 \\ &\quad = -(m^2 - 1)\|\mu_+ v_{-m-1}\|^2 \\ &\quad = -(m^2 - 1)\left[\|\mu_- v_{-m-1}\|^2 - \frac{s}{2}\|v_{-m-1}\|^2 + \frac{m+1}{2}(K v_{-m-1}, v_{-m-1})\right] \\ &\quad = -(m^2 - 1)\left[\|\mu_+ v_{-m-3}\|^2 - \frac{s}{2}\|v_{-m-1}\|^2 + \frac{m+1}{2}(K v_{-m-1}, v_{-m-1})\right] \\ &\quad = -(m^2 - 1)\Big[\|\mu_- v_{-m-3}\|^2 - \frac{s}{2}(\|v_{-m-1}\|^2 + \|v_{-m-3}\|^2) \\ &\qquad\qquad + \frac{m+1}{2}(K v_{-m-1}, v_{-m-1}) + \frac{m+3}{2}(K v_{-m-3}, v_{-m-3})\Big].\end{aligned}$$

Continuing this process, and noting that $\mu_- v_k \to 0$ in $L^2(SM)$ as $k \to -\infty$ (which follows since $\mu_- v \in L^2(SM)$), we obtain

$$\begin{aligned}&\|(X + A)v\|^2 - \|V(X + A)v\|^2 \\ &\qquad = \frac{m^2 - 1}{2} s \sum \|v_k\|^2 - \frac{m^2 - 1}{2} \sum |k|(K v_k, v_k).\end{aligned} \tag{13.14}$$

Collecting (13.12)–(13.14) and using them in the Pestov identity implies that

$$0 \geq \frac{m^2 - 1}{2} s \sum \|v_k\|^2 + \left(s - \frac{m^2 - 1}{2} \sup_M K\right) \sum |k| \|v_k\|^2.$$

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If we choose $s > \frac{m^2-1}{2} \sup_M K$, then both terms above are non-negative and therefore have to be zero. It follows that $v = 0$, so $\tilde{u}_k = 0$ for $k \leq -m-1$ and also $u_k = 0$ for $k \leq -m-1$ since $u = e^{-sw}\tilde{u}$ where e^{-sw} is holomorphic. $\square$

13.7 General Skew-Hermitian Attenuations

Remarkably, many aspects of the arguments done in the previous sections work for general attenuations $\mathcal{A}\colon SM \to \mathbb{C}^{n\times n}$ as long as $\mathcal{A}^* = -\mathcal{A}$. In the next section we will use these extensions to include a matrix field. We begin with the Pestov identity. Define

$$F_{\mathcal{A}} := XV(\mathcal{A}) + X_\perp(\mathcal{A}) + [\mathcal{A}, V(\mathcal{A})], \tag{13.15}$$

$$\varphi(\mathcal{A}) := -V^2(\mathcal{A}) - \mathcal{A}. \tag{13.16}$$

Note that if $\mathcal{A} = A \in \Omega_1 + \Omega_{-1}$, one has $F_{\mathcal{A}} = \star F_A$ by (13.9) and $\varphi(\mathcal{A}) = 0$.

Lemma 13.7.1 (Pestov identity) *Let (M, g) be a compact oriented Riemannian surface with boundary. Assume $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ is skew-Hermitian, i.e. $\mathcal{A}^* = -\mathcal{A}$. If $u\colon SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial SM} = 0$, then*

$$\begin{aligned}
&\|(X+\mathcal{A})Vu\|^2 - (K\,Vu, Vu) - (F_{\mathcal{A}}u, Vu) + ((X+\mathcal{A})u, \varphi(\mathcal{A})u)\\
&\quad = \|V(X+\mathcal{A})(u)\|^2 - \|(X+\mathcal{A})u\|^2.
\end{aligned}$$

Proof If we let $G := X + \mathcal{A}$, then routine calculations as in Lemma 13.4.1 show that

$$\begin{aligned}
[V, G] &= -(X_\perp - V(\mathcal{A})) := -G_\perp,\\
[V, G_\perp] &= G + \varphi(\mathcal{A}),\\
[G, G_\perp] &= -KV - F_{\mathcal{A}}.
\end{aligned}$$

We adopt the standard approach (as in the proof of Proposition 4.3.2) and define $P = VG$. Since $\mathcal{A}^* = -\mathcal{A}$ we have $P^* = GV$. Using the bracket relations above we compute that

$$\begin{aligned}
[P^*, P] &= GVVG - VGGV\\
&= VGVG + G_\perp VG - VGVG - VGG_\perp\\
&= V[G_\perp, G] - G^2 - \varphi(\mathcal{A})G = -G^2 - \varphi(\mathcal{A})G + VKV + VF_{\mathcal{A}}.
\end{aligned}$$

The identity in the lemma now follows from this bracket calculation and

$$\|Pu\|^2 = \|P^*u\|^2 + ([P^*, P]u, u)$$

for a smooth u with $u|_{\partial SM} = 0$. $\square$

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Lemma 13.7.2 *Let M be a compact simple surface and $\mathcal{A}\colon SM \to \mathbb{C}^{n\times n}$ such that $\mathcal{A}^* = -\mathcal{A}$. If $u\colon SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial SM} = 0$, then*

$$\|(X+\mathcal{A})Vu\|^2 - (K\,Vu, Vu) \geq 0.$$

The proof of this lemma is exactly the same as the proof of Lemma 13.5.5. Finally, in Lemma 13.5.4 we may replace A by $\mathcal{A}$ without trouble.

We can now interpret the quantities (13.15) and (13.16) as naturally appearing curvature terms of a suitable connection in SM. Consider the co-frame of 1-forms $\{\omega_1, \omega_2, \psi\}$ dual to the frame of vector fields $\{X, X_\perp, V\}$. The structure equations from Lemma 3.5.5 imply

$$d\omega_1 = -\psi \wedge \omega_2, \tag{13.17}$$

$$d\omega_2 = \psi \wedge \omega_1, \tag{13.18}$$

$$d\psi = K\omega_1 \wedge \omega_2. \tag{13.19}$$

Given $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ with $\mathcal{A}^* = -\mathcal{A}$, we define a unitary connection $\mathbb{A}$ on SM by setting

$$\mathbb{A} := \mathcal{A}\,\omega_1 - V(\mathcal{A})\,\omega_2.$$

Exercise 13.7.3 If A is a connection in M, show that

$$\pi^* A = A\omega_1 - V(A)\omega_2.$$

Lemma 13.7.4 *With $\mathbb{A}$ defined as above we have*

$$F_{\mathbb{A}} = -F_{\mathcal{A}}\,\omega_1 \wedge \omega_2 + \varphi(\mathcal{A})\psi \wedge \omega_2.$$

Proof Recall that $F_{\mathbb{A}} = d\mathbb{A} + \mathbb{A} \wedge \mathbb{A}$. We compute

$$\mathbb{A} \wedge \mathbb{A} = (\mathcal{A}\omega_1 - V(\mathcal{A})\omega_2) \wedge (\mathcal{A}\omega_1 - V(\mathcal{A})\omega_2) = -[\mathcal{A}, V(\mathcal{A})]\,\omega_1 \wedge \omega_2.$$

Next note that

$$\begin{aligned} d\mathbb{A} = {} & X_\perp(\mathcal{A})\omega_2 \wedge \omega_1 + V(\mathcal{A})\psi \wedge \omega_1 + \mathcal{A}d\omega_1 \\ & - XV(\mathcal{A})\omega_1 \wedge \omega_2 - V^2(\mathcal{A})\psi \wedge \omega_2 - V(\mathcal{A})d\omega_2. \end{aligned}$$

Using the structure equations (13.17) and (13.18) we see that

$$d\mathbb{A} = -(XV(\mathcal{A}) + X_\perp(\mathcal{A})\omega_1 \wedge \omega_2 - (V^2(\mathcal{A}) + \mathcal{A})\psi \wedge \omega_2,$$

and the lemma follows. $\square$

13.8 Injectivity for Connections and Higgs Fields

We now wish to extend the key Theorem 13.5.2 to include a *Higgs field*. For us this means an element $\Phi \in C^\infty(M, \mathbb{C}^{n\times n})$ and we may also refer to Φ simply as a matrix field. We will assume that Φ is skew-Hermitian, i.e. $\Phi^* = -\Phi$. The following result generalizes Theorem 13.5.2.

Theorem 13.8.1 *Let (M,g) be a simple surface, A a unitary connection and Φ a skew-Hermitian Higgs field. Suppose there is a smooth function $u\colon SM \to \mathbb{C}^n$ such that*

$$\begin{cases} Xu + (A+\Phi)u = f \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1, \\ u|_{\partial SM} = 0. \end{cases}$$

Then $u = u_0$ and $f = d_A u_0 + \Phi u_0 = du_0 + Au_0 + \Phi u_0$ with $u_0|_{\partial M} = 0$.

Proof We will prove that u is both holomorphic and anti-holomorphic. If this is the case then $u = u_0$ only depends on x and $u_0|_{\partial M} = 0$, and we have

$$du_0 + Au_0 = f_{-1} + f_1, \quad \Phi u_0 = f_0,$$

which proves the result.

The first step, as in the proof of Theorem 13.5.2, is to replace A by a connection whose curvature has a definite sign. We choose a real-valued 1-form φ such that $d\varphi = \omega_g$ where ω_g is the area form of (M,g), and let

$$A_s := A + is\varphi \mathrm{Id}.$$

Here $s > 0$ so that A_s is unitary and $i \star F_{A_s} = i \star F_A - s\mathrm{Id}$. We use Proposition 10.1.2 to find a holomorphic scalar function $w \in C^\infty(SM)$ satisfying $Xw = -i\varphi$. Then $u_s = e^{sw}u$ satisfies

$$(X + A_s + \Phi)u_s = -e^{sw} f.$$

Let $v := \sum_{-\infty}^{-1} (u_s)_k$. Since $(e^{sw} f)_k = 0$ for $k \le -2$, we have

$$(X + A_s + \Phi)v \in \Omega_{-1} \oplus \Omega_0.$$

Let $h := [(X + A_s + \Phi)v]_0$.

We apply the Pestov identity given in Lemma 13.7.1 with attenuation $\mathcal{A} := A_s + \Phi$ to the function v, which also satisfies $v|_{\partial SM} = 0$. Note that $\varphi(\mathcal{A}) = -\Phi$ and $F_{\mathcal{A}} = \star F_{A_s} + \star d_{A_s}\Phi$, where $d_{A_s}\Phi = d\Phi + [A_s, \Phi]$. Thus we obtain, after taking real parts, that

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$$\begin{aligned}
&\|(X + A_s + \Phi)(Vv)\|^2 - (K\,V(v), V(v)) \\
&\quad + \|(X + A_s + \Phi)v\|^2 - \|V[(X + A_s + \Phi)v]\|^2 \\
&\quad - (\star F_{A_s} v, V(v)) - \mathrm{Re}\,((\star d_{A_s}\Phi)v, V(v)) - \mathrm{Re}\,(\Phi v, (X + A_s + \Phi)v) \\
&= 0. \qquad (13.20)
\end{aligned}$$

It was proved in Lemmas 13.5.4 and 13.7.2 that

$$\|(X + A_s + \Phi)(Vv)\|^2 - (K\,V(v), V(v)) \geq 0, \qquad (13.21)$$

$$\|(X + A_s + \Phi)v\|^2 - \|V[(X + A_s + \Phi)v]\|^2 = \|h\|^2 \geq 0. \qquad (13.22)$$

The term involving the curvature of A_s satisfies

$$\begin{aligned}
-(\star F_{A_s} v, V(v)) &= \sum_{k=-\infty}^{-1} |k|(-i \star F_{A_s} v_k, v_k) \\
&\geq (s - \|F_A\|_{L^\infty(M)}) \sum_{k=-\infty}^{-1} |k|\|v_k\|^2. \qquad (13.23)
\end{aligned}$$

Here we can choose $s > 0$ large to obtain a positive term. For the next term in (13.20), we consider the Fourier expansion of $d_{A_s}\Phi = d_A\Phi = b_1 + b_{-1}$ where $b_{\pm 1} \in \Omega_{\pm 1}$. Note that $\star d_A\Phi = -V(d_A\Phi) = -ib_1 + ib_{-1}$. Then, since $v_k = 0$ for $k \geq 0$,

$$\begin{aligned}
((\star d_A\Phi)v, V(v)) &= \sum_{k=-\infty}^{-1} (-ib_1 v_{k-1} + ib_{-1} v_{k+1}), ikv_k) \\
&= \sum_{k=-\infty}^{-1} |k| \left[(b_1 v_{k-1}, v_k) - (b_{-1} v_{k+1}, v_k)\right].
\end{aligned}$$

Consequently, using that $v_0 = 0$, we have

$$\mathrm{Re}\,((\star d_A\Phi)v, V(v)) \leq C_{A,\Phi} \sum_{k=-\infty}^{-1} |k|\|v_k\|^2. \qquad (13.24)$$

We now study the last term in (13.20). We note that $v_k = 0$ for $k \geq 0$ and that $(X + A_s + \Phi)v \in \Omega_{-1} \oplus \Omega_0$. Therefore

$$(\Phi v, (X + A_s + \Phi)v) = (\Phi v_{-1}, ((X + A_s + \Phi)v)_{-1}).$$

Recall that we may write $X = \eta_+ + \eta_-$. Expand $A = A_1 + A_{-1}$ and $\varphi = \varphi_1 + \varphi_{-1}$ so that $A_s = (A_1 + is\varphi_1\mathrm{Id}) + (A_{-1} + is\varphi_{-1}\mathrm{Id}) =: a_1 + a_{-1}$ where $a_j \in \Omega_j$. Since A_s is unitary we have $a_{\pm 1}^* = -a_{\mp 1}$.

The fact that $(X + A_s + \Phi)v \in \Omega_{-1} \oplus \Omega_0$ implies that

$$\eta_+ v_{-2} + a_1 v_{-2} + \Phi v_{-1} = ((X + A_s + \Phi)v)_{-1},$$
$$\eta_+ v_{-k-1} + a_1 v_{-k-1} + \eta_- v_{-k+1} + a_{-1} v_{-k+1} + \Phi v_{-k} = 0, \qquad k \geq 2.$$

Note that $(\eta_\pm a, b) = -(a, \eta_\mp b)$ when $a|_{\partial SM} = 0$. Using this and the fact that Φ is skew-Hermitian, we have

$$\begin{aligned}
&\mathrm{Re}\,(\Phi v_{-1}, ((X + A_s + \Phi)v)_{-1})\\
&\quad = \mathrm{Re}(\Phi v_{-1}, \eta_+ v_{-2} + a_1 v_{-2} + \Phi v_{-1})\\
&\quad = \mathrm{Re}\left[(\eta_- v_{-1}, \Phi v_{-2}) - ((\eta_- \Phi)v_{-1}, v_{-2}) + (\Phi v_{-1}, a_1 v_{-2}) + \|\Phi v_{-1}\|^2\right].
\end{aligned}$$

We claim that for any $N \geq 1$ one has

$$\mathrm{Re}\,(\Phi v_{-1}, ((X + A_s + \Phi)v)_{-1}) = p_N + q_N,$$

where

$$\begin{aligned}
p_N &:= (-1)^{N-1}\mathrm{Re}\,(\eta_- v_{-N}, \Phi v_{-N-1}),\\
q_N &:= \mathrm{Re}\sum_{j=1}^{N}\big[(-1)^j((\eta_-\Phi)v_{-j}, v_{-j-1}) + (-1)^{j-1}(\Phi v_{-j}, a_1 v_{-j-1})\\
&\qquad + (-1)^{j-1}\|\Phi v_{-j}\|^2\big] + \mathrm{Re}\sum_{j=1}^{N-1}(-1)^j(a_{-1}v_{-j}, \Phi v_{-j-1}).
\end{aligned}$$

We have proved the claim when $N = 1$. If $N \geq 1$ we compute

$$\begin{aligned}
p_N &= (-1)^N\mathrm{Re}\,((\eta_+ + a_1)v_{-N-2} + a_{-1}v_{-N} + \Phi v_{-N-1}, \Phi v_{-N-1})\\
&= (-1)^N\mathrm{Re}\big[(\Phi v_{-N-2}, \eta_- v_{-N-1}) - (v_{-N-2}, (\eta_-\Phi)v_{-N-1})\\
&\qquad\qquad + (a_1 v_{-N-2} + a_{-1}v_{-N} + \Phi v_{-N-1}, \Phi v_{-N-1})\big]\\
&= p_{N+1} + q_{N+1} - q_N.
\end{aligned}$$

This proves the claim for any N.

Note that since $\|\eta_- v\|^2 = \sum\|\eta_- v_k\|^2$, we have $\eta_- v_k \to 0$ and similarly $v_k \to 0$ in $L^2(SM)$ as $k \to -\infty$. Therefore $p_N \to 0$ as $N \to \infty$. We also have

$$\|q_N\| \leq C_\Phi \sum\|v_k\|^2 + \left|\sum_{j=1}^{N}(-1)^j([a_{-1}, \Phi]v_{-j}, v_{-j-1})\right| \leq C_{A,\Phi}\sum\|v_k\|^2.$$

Here it was important that the term in a_{-1} involving s is a scalar, so it goes away when taking the commutator $[a_{-1}, \Phi]$ and thus the constant is independent of s. After taking a subsequence, (q_N) converges to some q having a similar bound. We finally obtain

$$\operatorname{Re}(\Phi v, (X + A_s + \Phi)v) = \lim_{N\to\infty}(p_N + q_N) \le C_{A,\Phi} \sum \|v_k\|^2. \quad (13.25)$$

Collecting the estimates (13.21)–(13.25) and using them in (13.20) shows that

$$0 \ge \|h\|^2 + (s - C_{A,\Phi}) \sum_{k=-\infty}^{-1} |k| \|v_k\|^2.$$

Choosing s large enough implies $v_k = 0$ for all k. This proves that u_s is holomorphic, and therefore $u = e^{-sw}u_s$ is holomorphic as required. □

We now rephrase Theorem 13.8.1 as an injectivity result for a matrix attenuated X-ray transform. We let $\mathcal{A}(x, v) := A_x(v) + \Phi(x)$ and we let $I_{A,\Phi} := I_{\mathcal{A}}$ be the associated attenuated X-ray transform.

Theorem 13.8.2 *Let M be a compact simple surface. Assume that $f : SM \to \mathbb{C}^n$ is a smooth function of the form $F(x) + \alpha_x(v)$, where $F: M \to \mathbb{C}^n$ is a smooth function and α is a $\mathbb{C}^n$-valued 1-form. Let also A be a unitary connection and Φ a skew-Hermitian matrix function. If $I_{A,\Phi}(f) = 0$, then $F = \Phi p$ and $\alpha = d_A p$, where $p: M \to \mathbb{C}^n$ is a smooth function with $p|_{\partial M} = 0$.*

Proof If $I_{A,\Phi}(f) = 0$, we know by Theorem 5.3.6 that there is a C^∞ function u satisfying

$$(X + A + \Phi)u = -f \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1,$$

with $u|_{\partial SM} = 0$. Thus by Theorem 13.8.1, u only depends on x and upon setting $p = -u_0$ the result follows. □

13.9 Scattering Rigidity for Connections and Higgs Fields

In this section we extend the scattering rigidity result for unitary connections in Theorem 13.5.1 to pairs (A, Φ), where A is a unitary connection and Φ is a skew-Hermitian matrix-valued function. We let $C_{A,\Phi} := C_{\mathcal{A}}$ be the scattering data that is associated with the attenuation $\mathcal{A}(x, v) = A_x(v) + \Phi(x)$.

Theorem 13.9.1 *Assume M is a compact simple surface, let A and B be two unitary connections, and let Φ and Ψ be two skew-Hermitian Higgs fields. Then $C_{A,\Phi} = C_{B,\Psi}$ implies that there exists a smooth $u: M \to U(n)$ such that $u|_{\partial M} = \mathrm{Id}$ and $B = u^{-1}du + u^{-1}Au$, $\Psi = u^{-1}\Phi u$.*

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Proof From Proposition 13.2.3 we know that $C_{A,\Phi} = C_{B,\Psi}$ means that there exists a smooth $U : SM \to U(n)$ such that $U|_{\partial SM} = \mathrm{Id}$ and

$$\mathcal{B} = U^{-1}XU + U^{-1}\mathcal{A}U, \tag{13.26}$$

where $\mathcal{B}(x,v) = B_x(v) + \Psi(x)$. We rephrase this information in terms of an attenuated X-ray transform. If we let $W = U - \mathrm{Id}$, then $W|_{\partial SM} = 0$ and

$$XW + \mathcal{A}W - W\mathcal{B} = -(\mathcal{A} - \mathcal{B}).$$

Hence the attenuated X-ray transform $I_{E(\mathcal{A},\mathcal{B})}(\mathcal{A} - \mathcal{B})$ vanishes. Note that $\mathcal{A} - \mathcal{B} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$.

Hence, making the choice to ignore the specific form $E(\mathcal{A},\mathcal{B})$ but noting that it is unitary by Exercise 13.2.4, we can apply Theorem 13.8.1 to deduce that W only depends on x. Hence U only depends on x and if we set $u(x) = U_0$, then (13.26) easily translates into $B = u^{-1}du + u^{-1}Au$ and $\Psi = u^{-1}\Phi u$ just by looking at the components of degree 0 and ± 1. $\square$

Remark 13.9.2 Note that the theorem implies in particular that scattering ridigity just for Higgs fields does not have a gauge. Indeed, if $C_\Phi = C_\Psi$ where Φ and Ψ are two skew-Hermitian matrix fields, Theorem 13.9.1 applied with $A = B = 0$ implies that $u = \mathrm{Id}$ and thus $\Phi = \Psi$.

13.10 Matrix Holomorphic Integrating Factors

Unfortunately, it is not possible to extend the proof of Theorem 13.8.1 to the case of attenuations that are not skew-Hermitian. The main issue is that the Pestov identity given in Lemma 13.7.1 has a particularly nice form when $\mathcal{A}$ is skew-Hermitian. While it is possible to derive a more general Pestov identity, new terms appear and there is a priori no clear way as to how to control them.

An alternative approach would be to try to prove the existence of certain *matrix holomorphic integrating factors*. Note that the proof of Theorem 13.8.1 uses the existence of *scalar* holomorphic integrating factors. In this section we try to explain the main difficulties with this approach and state some recent results.

We start with a general definition.

Definition 13.10.1 Let (M,g) be a compact oriented Riemann surface and let $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$. We say that $R \in C^\infty(SM, GL(n,\mathbb{C}))$ is a *matrix holomorphic integrating factor for* $\mathcal{A}$ if

(i) R solves $XR + \mathcal{A}R = 0$;
(ii) both R and R^{-1} are fibrewise holomorphic.

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There is an analogous definition for *anti-holomorphic* integrating factors. The existence of such integrating factors imposes conditions on $\mathcal{A}$. For $k \in \mathbb{Z}$ and $I \subset \mathbb{Z}$, we will use the notation $\oplus_{k\in I}\Omega_k$ to indicate the set of smooth functions $\mathcal{A}$ such that $\mathcal{A}_k = 0$ for $k \notin I$.

Lemma 13.10.2 *If $\mathcal{A}$ admits a holomorphic integrating factor then $\mathcal{A} \in \oplus_{k\geq -1}\Omega_k$. If $\mathcal{A}$ admits both holomorphic and anti-holomorphic integrating factors, then $\mathcal{A} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$.*

Proof This follows right away from writing $\mathcal{A} = -(XR)R^{-1}$, since R^{-1} is holomorphic and $X(R) \in \oplus_{k\geq -1}\Omega_k$ given the mapping property

$$X\colon\ \oplus_{k\geq 0}\,\Omega_k \to \oplus_{k\geq -1}\Omega_k.$$

The second statement in the lemma follows immediately. □

Thus if we wish to use holomorphic and anti-holomorphic integrating factors, the attenuation $\mathcal{A}$ must be of the form $\mathcal{A}(x,v) = A_x(v) + \Phi(x)$ where A is a connection and Φ a matrix-valued field. The relevance of these types of integrating factors can be seen in the following proposition.

Proposition 13.10.3 *Let (M,g) be a non-trapping surface with strictly convex boundary such that I_0 is injective and I_1 is solenoidal injective. Let (A,Φ) be a pair given by a connection A and a matrix-valued field Φ. If (A,Φ) admits holomorphic and anti-holomorphic integrating factors, then $I_{A,\Phi}$ has the same kernel as in Theorem 13.8.2.*

Proof Assume that $u \in C^\infty(SM,\mathbb{C}^n)$ satisfies $u|_{\partial SM} = 0$ and that one has $(X + A + \Phi)u = -f \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. We wish to show that $u = u_0$. For this it is enough to show that u is both holomorphic and anti-holomorphic.

Let R be a matrix holomorphic integrating factor for $A + \Phi$. Since R^{-1} solves $XR^{-1} - R^{-1}(A + \Phi) = 0$, a computation shows that

$$X(R^{-1}u) = -R^{-1}f.$$

Since R^{-1} is holomorphic, $(R^{-1}f)_k = 0$ for $k \leq -2$. Thus if we set $v = \sum_{-\infty}^{-1}(R^{-1}u)_k$, then $v|_{\partial SM} = 0$ and

$$Xv \in \Omega_{-1} \oplus \Omega_0.$$

Using the hypotheses on I_0 and I_1, we deduce that $v = 0$ and thus $R^{-1}u$ is holomorphic. It follows that $u = RR^{-1}u$ is also holomorphic since R is holomorphic.

An analogous argument using anti-holomorphic integrating factors shows that u is anti-holomorphic and hence $u = u_0$ as desired. □

We can now state the following question.

Question. Let (M, g) be a simple surface and let (A, Φ) be a pair, where A is a connection and Φ is a matrix field. Do holomorphic (antiholomorphic) integrating factors exist for (A, Φ)?

Note that Proposition 12.2.6 gives a positive answer to this question when $n = 1$. It suffices to take $R := e^{-w}$ where w is given by the proposition. In the non-Abelian case $n \geq 2$ we can no longer argue using an exponential. While we can certainly find a holomorphic matrix W such that $XW = A + \Phi$, the exponential of W might not solve the relevant transport problem since XW and W do not necessarily commute.

Exercise 13.10.4 Show that for any $W \in C^\infty(SM, \mathbb{C}^{n\times n})$ we have

$$e^W X\big(e^{-W}\big) = \int_0^1 e^{-sW}(XW)e^{sW}\, ds.$$

A positive answer to the question of existence of matrix holomorphic integrating factors has recently been given in Bohr and Paternain (2021). However, the answer is based on essentially knowing injectivity *first* for the general linear group of complex matrices, so we need an alternative way of establishing injectivity. We will do that in the next chapter using a factorization result from loop groups.

We conclude this section by studying the group of all smooth $R\colon SM \to GL(n, \mathbb{C})$ such that $XR = 0$ for (M, g) a simple surface. We start with an auxiliary lemma.

Lemma 13.10.5 *Let $F\colon M \to GL(n, \mathbb{C})$ be such that $\eta_- F = 0$. Then we can write F as*

$$F = F_1 \cdots F_r,$$

where each $F_j\colon M \to GL(n, \mathbb{C})$ has the property that $\eta_- F_j = 0$ and $|\mathrm{Id} - F_j(x)| < 1$ for all $x \in M$ and $1 \leq j \leq r$.

Proof The proof of this lemma is almost identical to the proof of (Gunning and Rossi, 1965, Lemma on p.194); we include a sketch for completeness.

The set G of all $F\colon M \to GL(n, \mathbb{C})$ with $\eta_- F = 0$ clearly forms a group. In fact it is a connected topological group with the supremum norm. Such groups are generated by any open neighbourhood of the identity. Considering a neighbourhood of the form

$$U = \{F \in G : \ \|F - \mathrm{Id}\|_{L^\infty} < 1\},$$

the result follows. □

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We now prove a certain matrix analogue of Theorem 8.2.2.

Theorem 13.10.6 *Let (M, g) be a simple surface and let $F\colon M \to GL(n, \mathbb{C})$ with $\eta_- F = 0$ be given. Then there exists a smooth $R\colon SM \to GL(n, \mathbb{C})$ such that*

(i) $XR = 0$ *and* $R_0 = F$*;*
(ii) *both* R *and* R^{-1} *are fibrewise holomorphic.*

Proof By Lemma 13.10.5 we may write $F = F_1 \cdots F_r$ where each $F_j\colon M \to GL(n, \mathbb{C})$ is such that $\eta_- F_j = 0$ and $|\mathrm{Id} - F_j(x)| < 1$ for all x. Hence we can write $F_j = e^{P_j}$, where $P_j\colon M \to \mathbb{C}^{n\times n}$ is such that $\eta_- P_j = 0$. By the surjectivity of I_0^*, there is a smooth W_j such that $XW_j = 0$, W_j is fibrewise holomorphic and $(W_j)_0 = P_j$. Now set

$$R := e^{W_1} \cdots e^{W_r}.$$

We claim that R has all the desired properties. Since each e^{W_j} is a first integral, so is R. By construction, each e^{W_j} is holomorphic, hence so is their product. Since

$$R^{-1} = e^{-W_r} \cdots e^{-W_1},$$

it follows that R^{-1} is also fibrewise holomorphic. It remains to prove that $R_0 = F$. But since R is holomorphic we must have

$$R_0 = \left(e^{W_1}\right)_0 \cdots \left(e^{W_r}\right)_0.$$

But for each j, $(e^{W_j})_0 = e^{(W_j)_0} = e^{P_j} = F_j$ and the theorem is proved. □

13.11 Stability Estimate

It is possible to derive a *quantitative* version of Theorem 13.9.1 and obtain a stability estimate for the scattering data. This has been carried out in the case of matrix fields where the inverse has no gauge. The result is as follows.

Theorem 13.11.1 (Monard et al., 2021a) *Let (M, g) be a simple surface. Given two matrix fields Φ and Ψ in $C^1(M, \mathfrak{u}(n))$ there exists a constant $c(\Phi, \Psi)$ such that*

$$\|\Phi - \Psi\|_{L^2(M)} \leq c(\Phi, \Psi)\|C_\Phi C_\Psi^{-1} - \mathrm{id}\|_{H^1(\partial_+ SM)},$$

where $c(\Phi, \Psi)$ is a continuous function of $\|\Phi\|_{C^1} \vee \|\Psi\|_{C^1}$, explicitly

$$c(\Phi, \Psi) = C_1 \left(1 + \left(\|\Phi\|_{C^1} \vee \|\Psi\|_{C^1}\right)\right) e^{C_2(\|\Phi\|_{C^1} \vee \|\Psi\|_{C^1})}, \tag{13.27}$$

and where the constants C_1, C_2 only depend on (M, g).

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The proof of Theorem 13.11.1 initially follows the approach for obtaining $L^2 \to H^1$ stability estimates for the geodesic X-ray transform I as presented in Theorem 4.6.4. The starting point is the pseudo-linearization formula (13.4)

$$C_\Phi C_\Psi^{-1} = \mathrm{Id} + I_{E(\Phi,\Psi)}(\Phi - \Psi).$$

To prove Theorem 13.11.1 it suffices to show that

$$\|\Phi - \Psi\|_{L^2(M)} \leq c(\Phi,\Psi)\|I_{E(\Phi,\Psi)}(\Psi - \Phi)\|_{H^1(\partial_+ SM)}.$$

To this end, one has to go carefully through the proof of Theorem 13.8.1 that uses holomorphic integrating factors to control additional terms in the Pestov identity due to the matrix fields. Taming the holomorphic integrating factors has a cost that is reflected in the constant $c(\Phi,\Psi)$ given in (13.27). The details of the proof are fairly involved and the reader is referred to Monard et al. (2021a). The overall strategy is similar to the proof of Novikov and Sharafutdinov (2007, Theorem 5.1) for polarization tomography; however, the main virtue of Theorem 13.11.1 is that there is no restriction on the size of the fields Φ and Ψ.

Theorem 13.11.1 paves the way for a statistical algorithm that allows one to recover Φ from noisy measurements of C_Φ, more precisely a frequentist consistency of reconstruction in the large sample limit. See Monard et al. (2021a) for details.