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LE HER WITH *s* SUITS AND *d* DENOMINATIONS

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Abstract

In 2002, Benjamin and Goldman gave a complete solution to a *variant* of the two-player card game Le Her. We extend their result by giving optimal strategies for the *authentic* version played with a deck consisting of arbitrary numbers of suits and denominations. Additionally, we show that the player who has the advantage in the game when *one* standard deck is used does not have the advantage if *nineteen* or more standard decks are used.

Keywords: Le Her; game theory

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1. Introduction

The two-player card game Le Her has been analyzed for centuries by mathematicians. It has an important place in the history of game theory, as it was the first game ever solved using game-theoretic methods. The earliest known solution to Le Her (played with one standard deck) was provided by James Waldegrave in 1713. (See [6] for more discussion of the history of Le Her.)

Le Her is played in the following way. Player 1 and player 2 are each given a card from the deck, neither of them knowing what card the other has. If player 2 has a king, the game immediately ends with victory for player 2. Otherwise, player 1 has the option of exchanging cards with player 2. Next, player 2 has the option of exchanging his card with the top card of the deck. If he does decide to exchange his card and the top card is a king, the exchange is void and player 2 must keep his card. If player 2 has the higher-ranking card *or* has a card of the same rank as player 1, he wins. (Cards are ranked ace, 2, 3, . . . , queen, king. In particular, aces have the lowest rank.) Otherwise, player 1 wins. A comment about the game: if player 1 decides to *exchange* his card, it is clear that player 2 will then know whether or not he has the higher-ranking card. He will exchange his card with the top card of the deck only if his original card was ranked higher than his new card.

In this paper, we will assume that the number of different denominations in the deck is an arbitrary number $d \geq 3$, instead of the standard 13, and that the number of times each of the d denominations appear in the deck is s. (The cases $d = 1$ and $d = 2$ are trivial.) The cards are ranked with d being the highest and 1 being the lowest. (Therefore, kings play the role of d and aces play the role of 1 in the standard version.) For each of the d cards, player 1 has to decide whether or not he wants to keep the card if he gets it. (Of course, he has to keep his card and lose if player 2 has card d.) As mentioned before, player 2 will know exactly what to do if player 1 exchanges his card. If player 1 does not exchange his card, player 2 has to make the same decision that player 1 had to make. It is intuitively clear (and can be proved rigorously)

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that if it is optimal for player 1 or player 2 to keep a certain card, he should also keep any card ranked higher. It is also clear that a player should stand if he has card d and exchange his card if he has card 1. Therefore, player 1 and player 2 have $d - 1$ pure strategies from which to choose: for each i, $1 \leq i \leq d - 1$, if a player gets a card that is ranked lower or has the same value as i, he should exchange his card; otherwise, he should stand.

As mentioned, the optimal mixed strategies for each player have been found when a standard deck is used. However, a 'quick' method for finding optimal strategies when an arbitrary number of suits and denominations are used has yet to be discovered. In order to find optimal strategies, we will employ techniques similar to those used by Benjamin and Goldman [2] to find optimal strategies for an N-card variant of Le Her. The N-card version differs from the one considered here in three respects:

1. The number of suits is 1 and not an arbitrary number s.
2. Player 1 *can* exchange his card with player 2 if player 2 has card d.
3. Player 2 *can* exchange his card with the top card if it is card d.

In the N-card version, it is not possible for player 1's card to have the same rank as player 2's, since there is only one card for each denomination.

2. Optimal strategies for $d < 7$

We begin by giving a polynomial expression for the probability that player 1 will win, as a function of the strategies he and player 2 decide to use. (We will use the label 'strategy i' to denote the strategy whereby player 1 exchanges his card if it has the same value as or is ranked lower than i and stands otherwise.)

Lemma 2.1. *Let B_{ij} be the event that player 1 wins if he uses strategy i and player 2 uses strategy j. Then, for $1 \leq i, j \leq d - 1$,*

$$\begin{aligned} \mathrm{P}(B_{ij}) = \frac{s^2}{6(sd)_3}[&-si^3 - 3si^2j - 3(sd+s-3)i^2 + 3(2sd-s-2)ij \\ &+ (3sd^2 - 2s - 6d + 3)i - 3(sd-2)(d-1)j \\ &+ 3(sd-2)d(d-1)] \quad \text{if } i \geq j, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathrm{P}(B_{ij}) = \frac{s^2}{6(sd)_3}[&-si^3 - 3si^2j - 3(2s-3)i^2 + 3sij + 3(sd-s-2)j^2 \\ &+ (3sd^2 - 3sd + s - 6d + 3)i - 3(sd^2 - 2sd + s - 2d + 2)j \\ &+ 3(sd-2)d(d-1)] \quad \text{if } i \leq j. \end{aligned} \tag{2.2}$$

(The notation $(sd)_3$ is used to represent $sd(sd-1)(sd-2)$.)

Proof. In [3, pp. 140–141], Ethier proved that $\mathrm{P}(B_{ij})$ is given by the following formula for the standard deck (with $s = 4$ and $d = 13$), where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and $\delta_{i,j}$ is the Kronecker delta:

$$\begin{aligned} \mathrm{P}(B_{ij}) = &\sum_{k=i+1}^{13} \sum_{l=1}^{j} \frac{4}{52} \cdot \frac{4 - \delta_{k,l}}{51} \cdot \frac{4(k-1) + (3 - \delta_{k,13})\,\mathbf{1}_{\{k>l\}}}{50} \\ &+ \sum_{k=1}^{i} \sum_{l=1}^{13} \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4l-1}{50} \mathbf{1}_{\{k<l<13\}} + \sum_{k=i+1}^{13} \sum_{l=j+1}^{13} \frac{4}{52} \cdot \frac{4}{51} \mathbf{1}_{\{k>l\}}\,. \end{aligned}$$

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We use this formula to generalize to the case in which the number of suits is s and the number of denominations is d:

$$\begin{aligned}
\mathrm{P}(B_{ij}) &= \sum_{k=1}^{i}\sum_{l=1}^{d}\frac{s}{sd}\cdot\frac{s}{sd-1}\cdot\frac{sl-1}{sd-2}\,\mathbf{1}_{\{k<l<d\}}\\
&\quad+\sum_{k=i+1}^{d}\sum_{l=1}^{j}\frac{s}{sd}\cdot\frac{s-\delta_{k,l}}{sd-1}\cdot\frac{s(k-1)+((s-1)-\delta_{k,d})\,\mathbf{1}_{\{k>l\}}}{sd-2}\\
&\quad+\sum_{k=i+1}^{d}\sum_{l=j+1}^{d}\frac{s}{sd}\cdot\frac{s}{sd-1}\,\mathbf{1}_{\{k>l\}}\\
&=: S_1+S_2+S_3.
\end{aligned}$$

We begin by evaluating S_1. Using the formulae for the sum of the first n integers and the sum of their squares, we obtain

$$\begin{aligned}
S_1 &= \frac{s^2}{(sd)_3}\sum_{k=1}^{i}\sum_{l=1}^{d}(sl-1)\,\mathbf{1}_{\{k<l<d\}}\\
&= \frac{s^2}{(sd)_3}\left[s\sum_{k=1}^{i}\sum_{l=k+1}^{d-1} l-\sum_{k=1}^{i}\sum_{l=k+1}^{d-1}1\right]\\
&= \frac{s^2}{6(sd)_3}[-si^3-(3s-3)i^2+(3sd^2-3sd-2s-6d+9)i].
\end{aligned}$$

Suppose that $i \geq j$. Then

$$\begin{aligned}
S_2 &= \frac{s^2}{(sd)_3}\sum_{k=i+1}^{d}\sum_{l=1}^{j}[s(k-1)+s-1-\delta_{k,d}] \quad (\text{since } \delta_{k,l}=0 \text{ and } \mathbf{1}_{\{k>l\}}=1)\\
&= \frac{s^2}{(sd)_3}\sum_{k=i+1}^{d}\sum_{l=1}^{j}(sk-1-\delta_{k,d})\\
&= \frac{s^2}{(sd)_3}\left[sj\sum_{k=i+1}^{d}k-(d-i)j-j\right]\\
&= \frac{s^2}{6(sd)_3}[-3si^2j-(3s-6)ij+(3sd^2+3sd-6d-6)j],\\
S_3 &= \frac{s^2}{(sd)_3}(sd-2)\sum_{k=i+1}^{d}\sum_{l=j+1}^{k-1}1\\
&= \frac{s^2}{6(sd)_3}\,[-(3sd-6)i^2+(6sd-12)ij+(3sd-6)i\\
&\qquad -(6sd^2-12d)j+3(sd-2)d(d-1)].
\end{aligned}$$

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Thus, (2.1) follows. Now suppose that $i \le j$. Then

$$S_2 = \frac{s}{(sd)_3} \sum_{k=i+1}^{d} \sum_{l=1}^{j} (s - \delta_{k,l})[s(k-1) + (s-1-\delta_{k,d})\,\mathbf{1}_{\{k>l\}}]$$
$$= \frac{s^2}{(sd)_3} \sum_{k=i+1}^{d} \sum_{l=1}^{j} [(s - \delta_{k,l})(k-1) + (s-1-\delta_{k,d})\,\mathbf{1}_{\{k>l\}}],$$

since $\delta_{k,l}\,\mathbf{1}_{\{k>l\}} = 0$. The latter double sum can be written as

$$s \sum_{k=i+1}^{d} \sum_{l=1}^{j} (k-1) - \sum_{k=i+1}^{d} \sum_{l=1}^{j} \delta_{k,l}(k-1)$$
$$+ (s-1) \sum_{k=i+1}^{d} \sum_{l=1}^{j} \mathbf{1}_{\{k>l\}} - \sum_{k=i+1}^{d} \sum_{l=1}^{j} \delta_{k,d}\,\mathbf{1}_{\{k>l\}}$$
$$=: sT_1 - T_2 + (s-1)T_3 - T_4,$$

where

$$T_1 = j \sum_{k=i+1}^{d} (k-1), \qquad T_2 = \sum_{k=i+1}^{d} (k-1)\,\mathbf{1}_{\{k\le j\}} = \sum_{k=i+1}^{j} (k-1),$$
$$T_3 = (d-i)j - \sum_{k=i+1}^{d} \sum_{l=1}^{j} \mathbf{1}_{\{k\le l\}} = (d-i)j - \sum_{k=i+1}^{j} \sum_{l=k}^{j} 1, \qquad T_4 = \sum_{l=1}^{j} \mathbf{1}_{\{d>l\}} = j.$$

We find that

$$S_2 = \frac{s^2}{6(sd)_3}[-3si^2 j - (3s-6)i^2 + 3sij - 3sj^2 + (3s-6)i + (3sd^2 + 3sd - 3s - 6d)j].$$

Continuing under the assumption that $i \le j$, we obtain

$$S_3 = \frac{s^2}{(sd)_3}(sd-2) \sum_{k=i+1}^{d} \sum_{l=j+1}^{d} \mathbf{1}_{\{k>l\}} = \frac{s^2}{(sd)_3}(sd-2) \sum_{k=j+1}^{d} \sum_{l=j+1}^{k-1} 1,$$

whence

$$S_3 = \frac{s^2}{6(sd)_3}[(3sd-6)j^2 - (6sd^2 - 3sd - 12d + 6)j + 3(sd-2)d(d-1)].$$

After summing S_1, S_2, and S_3, (2.2) follows.

We assume that the payoff for victory is 1 unit (of reward). Therefore, the payoff for player 1 if he uses strategy i and player 2 uses strategy j is

$$P(i,j) - (1 - P(i,j)) = 2P(i,j) - 1,$$

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where we write $P(i, j)$ in place of $\mathrm{P}(B_{ij})$ for ease of notation. As a result, the entries of the payoff matrix $\boldsymbol{A}$ are given by $a_{ij} = 2P(i, j) - 1$. For $d = 3$, the payoff matrix is

$$\begin{array}{c c} & \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 2P(1,1)-1 & 2P(1,2)-1 \\ 2P(2,1)-1 & 2P(2,2)-1 \end{pmatrix}. \end{array}$$

(The rows correspond to player 1's strategies and the columns to player 2's strategies.)

After substituting values for d, i, and j, the resulting payoff matrix (after adding 1 to each of the entries and dividing by $2s^2/(sd)_3$) is

$$\begin{array}{c c} & \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 10s-6 & 11s-7 \\ 8s-5 & 8s-5 \end{pmatrix}. \end{array}$$

For all values of s, row 1 dominates row 2, so using only strategy 1 is optimal for player 1. After row domination, column 1 dominates column 2, so using strategy 1 is also optimal for player 2. The payoff matrix for $d = 4$ is

$$\begin{array}{c c} & \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 26s-12 & 26s-12 & 29s-14 \\ 27s-12 & 26s-11 & 28s-13 \\ 20s-9 & 20s-9 & 20s-9 \end{pmatrix}. \end{array}$$

Here column 2 dominates columns 1 and 3, so using only strategy 2 is optimal for player 2. Since a_{22} is greater than both a_{12} and a_{32}, strategy 2 is optimal for player 1. Using similar techniques for $d = 5$, strategy 2 is found to be optimal for both players, and for $d = 6$ strategy 3 is found to be optimal.

For large values of d, it is impractical to go through the procedure used above to find optimal strategies. Therefore, a quicker method needs to be developed. In [2], for an N-card variant of Le Her, the 'convexity' of the payoff matrix was used for this purpose. In the next section, we adopt this method to find optimal strategies for each player for $d \geq 7$.

3. Optimal strategies for $d \geq 7$

We begin this section by using a special property of the payoff matrix to show that player 1 has a 'critical denomination'.

Lemma 3.1. *For each i, $1 \leq i \leq d-2$, $a_{i,j+1} - a_{ij}$ is nondecreasing in $j \geq 1$.*

Proof. Since a_{ij} is a linear function of $P(i, j)$, all we need to show is that $P(i, j+1) - P(i, j)$ is nondecreasing in $j \geq 1$. It suffices to show that

$$D := \frac{(sd)_3}{s^2}[(P(i, j+2) - P(i, j+1)) - (P(i, j+1) - P(i, j))] \geq 0.$$

Case 1: $i > j-1$. Using (2.1), let $P_1(i, j) = [(sd)_3/s^2]P(i, j)$. Then

$$D = (P_1(i, j+2) - P_1(i, j+1)) - (P_1(i, j+1) - P_1(i, j)) = 0 \geq 0.$$

Case 2: $i = j-1$. Using (2.2), let $P_2(i, j) = [(sd)_3/s^2]P(i, j)$. Then

$$D = (P_2(i, j+2) - P_2(i, j+1)) - (P_1(i, j+1) - P_1(i, j)) = s(d-1) - i - 1 \geq 0.$$

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Case 3: $i < j - 1$. Here we obtain

$$D = (P_2(i, j+2) - P_2(i, j+1)) - (P_2(i, j+1) - P_2(i, j)) = sd - s - 2 \geq 0.$$

Lemma 3.2. *Player 1's optimal strategy combines at most two of the pure strategies. If the strategy does combine two different pure strategies, they are of the form x and $x+1$, for some $x \leq d-3$.*

Proof. According to [5], if the rows of the payoff matrix are discrete convex, then player 1's optimal mixed strategy involves either one row or two consecutive rows. In Lemma 3.1 we have shown that the columns satisfy this condition. Therefore, the result follows.

Theorem 3.1. *There exists a denomination x that will serve as a critical denomination for player 1. In optimal play, player 1 will always exchange his card if it is ranked lower than x, will always stand if he has a card ranked higher than x, and will exchange his card with a certain probability if he has a card whose denomination is x.*

Proof. A player using strategy i will exchange his card if he has card i or any card ranked lower. Therefore, the only difference between strategies i and $i+1$ is what is done when card $i+1$ is received: under strategy i the player keeps this card, and under strategy $i+1$ the player exchanges the card. The theorem then follows since, by Lemma 3.2, the player will either use a pure strategy i or a mixture of strategies i and $i+1$, for some i.

Our next goal is to show that, when trying to determine his critical denomination, player 1 should only consider entries of the payoff matrix with $i \leq j$. We prove this in Lemma 3.7. Lemmas 3.3–3.6 are needed first. (In Lemmas 3.3–3.8, we will write $P(i, j)$ in place of $[s^2/(sd)_3]P(i, j)$, for simplicity.)

Lemma 3.3. *If $P(i,i) \geq P(i+1,i)$ then row i dominates row $i+1$. Additionally, row i dominates rows $i+2, \dots, d-1$.*

Proof. We can prove the first claim by showing that the following assertions hold.

(i) $i > j \Rightarrow P(i,i) - P(i+1,i) \leq P(i,j) - P(i+1,j)$. This holds because

$$(P_1(i,i) - P_1(i+1,i)) - (P_1(i,j) - P_1(i+1,j)) = (i-j)(s(i-(d-1))+1) \leq 0.$$

(ii) $i = j \Rightarrow P(i,j) - P(i+1,j) \leq P(i,i+1) - P(i+1,i+1)$. This holds because

$$(P_1(i,i) - P_1(i+1,i)) - (P_2(i,i+1) - P_2(i+1,i+1)) = (1-s)i \leq 0.$$

(iii) $i+1 < j \Rightarrow P(i,i+1) - P(i+1,i+1) \leq P(i,j) - P(i+1,j)$. This holds because

$$(P_2(i,i+1) - P_2(i+1,i+1)) - (P_2(i,j) - P_2(i+1,j)) = (i+1-j)si \leq 0.$$

To prove the second claim, it suffices to show that, for any j,

$$P(j+1, j+1) - P(j+2, j+1) \geq P(j, j) - P(j+1, j).$$

This holds because

$$(P_1(j+1, j+1) - P_1(j+2, j+1)) - (P_1(j, j) - P_1(j+1, j)) = 3sj + 4s - 2 \geq 0.$$

Lemma 3.4. *For all i and $j < i$, $P(i, i+1) - P(i, i) \le P(j, i+1) - P(j, i)$.*

Proof. The assertion follows because

$$(P_2(i, i+1) - P_2(i, i)) - (P_2(j, i+1) - P_2(j, i)) = \frac{s}{2}(j-i)(j+i-1) \le 0.$$

Lemma 3.5. *Suppose that $d \ge 7$ and $P(i, i+1) > P(i, i)$ for some i. Column $i+1$ is dominated by column i after rows $i+1, i+2, \ldots, d-1$ have been eliminated.*

Proof. If $i \ge (d-1)/2$ then $P(i, i) - P(i+1, i) \ge P(i, i+1) - P(i, i)$, because

$$(P_1(i, i) - P_1(i+1, i)) - (P_2(i, i+1) - P_2(i, i)) = s(2i+2)\left(i - \frac{d-1}{2}\right) + s \ge 0.$$

Given the hypothesis, $P(i, i) - P(i+1, i) \ge (P(i, i+1) - P(i, i))$ implies that $P(i, i) \ge P(i+1, i)$. By Lemma 3.3, this means that rows $i+1, i+2, \ldots, d-1$ are dominated. As a consequence of Lemma 3.4, column $i+1$ is then dominated by column i. The claim thus holds if $i \ge (d-1)/2$.

If $i < (d-1)/2$ then $P(i+1, i) \le P(i, i)$, because

$$\begin{aligned}
&8(P(i+1, i) - P(i, i)) \\
&\quad = s(d-7)(-d+1) - 8 - s(-2i+d-1)^2 + (2sd-8)(2i-d+1) \\
&\quad \le s(d-7)(-d+1) \le 0 \quad \text{if } d \ge 7.
\end{aligned}$$

Thus, $P(i, i+1) \le P(i, i)$ and the claim again holds by the above reasoning.

Lemma 3.6. *After reducing the matrix by row and column domination, for the remaining rows i, $P(i, i+1) \le P(i, j)$ if $j < i+1$.*

Proof. From Lemma 3.5 we have $P(i, i+1) \le P(i, i)$, and, for $j < i$, $P(i, i) \le P(i, j)$ since

$$P_1(i, i) - P_1(i, j) = (j-i)((d-1)-i)\left((d-i)\frac{s}{2} + 1\right) \le 0.$$

The result follows.

Lemma 3.7. *When considering optimal strategies for both players, it is only necessary to look at entries with $i \le j$ (or where $P_2(i, j)$ is used).*

Proof. As a result of Lemma 3.2, we know that player 1's optimal strategy is restricted to at most two consecutive rows, row i and row $i+1$ (where i is between 0 and $d-2$). If row i and row $i+1$ are not eliminated, column $i+1$ dominates columns $1, 2, \ldots, i$. Therefore, entries with $i > j$ will never be considered.

Suppose that $F(i, j)$ is a concave function of i on $[1, d-1] \times [1, d-1]$. According to Theorem 1 of [4], for a continuous game in which player 1's strategies correspond to real numbers between 1 and $d-1$, player 2's strategies correspond to real numbers between 1 and $d-1$, and F is the payoff function, player 1 has an optimal pure strategy x^*. Furthermore, according to [1], if only integers can be used then player 1's optimal mixed strategy involves at most strategies $\lfloor x^* \rfloor$ and $\lceil x^* \rceil$ (respectively the greatest integer less than or equal to x^* and the least integer greater than or equal to x^*). By Lemma 3.7, it suffices to show that $P_2(i, j)$ is a concave function of i. This follows since $\mathrm{d}^2 P_2/\mathrm{d}i^2 = -s(i+j+2) + 3 \le 0$. Since player 1 has an optimal pure strategy x^*, he wants to find the value of i corresponding to the largest member of the following set:

$$\{P_2(i, j^*)\colon j^* \text{ serves as a minimizer of } P_2(i, j) \text{ for the given } i \text{ value}\}.$$

In order to find the largest member, we need a method of finding the minimizer j^* for each i value. In Lemma 3.10, we give the formula for the minimizer; Lemmas 3.8 and 3.9 are needed first.

Lemma 3.8. *For any d, row i will be dominated by row $i-1$ if $i \geq \sqrt{3}d/3$.*

Proof. By Lemma 3.3, we need to show that $D := P(i-1, i-1) - P(i, i-1) \geq 0$. This holds because

$$D = \left(i - \frac{\sqrt{3}d}{3}\right)\left(\frac{3s}{2}\left(i + \frac{\sqrt{3}d}{3}\right) - \frac{s}{2} - 2\right) + \frac{d[s(3-\sqrt{3}) + 6 - 4\sqrt{3}]}{6} \geq 0 \quad \text{if } i \geq \frac{\sqrt{3}d}{3}.$$

Lemma 3.9. *Let $n = \lfloor \sqrt{3}d/3 \rfloor$. For $i \leq n$,*

$$\left.\frac{\mathrm{d}P_2}{\mathrm{d}j}(i, j)\right|_{j=i} < 0 < \left.\frac{\mathrm{d}P_2}{\mathrm{d}j}(i, j)\right|_{j=d-1}.$$

Proof. Let $T_1 = 6(\mathrm{d}P_2(i, j)/\mathrm{d}j)|_{j=i}$ and $T_2 = 6(\mathrm{d}P_2(i, j)/\mathrm{d}j)|_{j=d-1}$. Then

$$\begin{aligned}
T_1 &= 6i(sd - s - 2) + (-3sd^2 + 6sd - 3s + 6d - 6) - 3si^2 + 3si \\
&\leq 2\sqrt{3}d(sd - s - 2) - 3sd^2 + 6sd - 3s + 6d - 6 - sd^2 + \sqrt{3}sd \\
&= sd(d(2\sqrt{3} - 4) + (-\sqrt{3} + 6)) - 4\sqrt{3}d + 6d - 3s - 6 \\
&\leq sd(d(2\sqrt{3} - 4) + (-\sqrt{3} + 6)) \\
&< 0 \quad \text{if } d > \frac{\sqrt{3} - 6}{2\sqrt{3} - 4} \approx 7.99, \\
T_1 &\leq 24(6s - 2) + (-108s + 36) - 36s \\
&= -12 \\
&< 0 \quad \text{if } d = 7.
\end{aligned}$$

and

$$\begin{aligned}
T_2 &= 3sd^2 - 6sd - 6d + 3s + 6 - 3si^2 + 3si \\
&\geq 3sd^2 - 6sd - 6d + 3s + 6 - sd^2 + \sqrt{3}sd \\
&= d(2sd + (\sqrt{3} - 6)s - 6) + 3s + 6 \\
&> 0 \quad \text{if } d \geq 7.
\end{aligned}$$

Thus, $(\mathrm{d}P_2(i, j)/\mathrm{d}j)|_{j=i} < 0 < (\mathrm{d}P_2(i, j)/\mathrm{d}j)|_{j=d-1}$.

Lemma 3.10. *Suppose that $d \geq 7$ and that strategy i has not been dominated. The j^* that minimizes $P_2(i, j)$ is given by*

$$j^* = \frac{d-1}{2} + \frac{si^2 - si}{2(sd - s - 2)}. \tag{3.1}$$

Proof. Simple differentiation yields

$$\frac{\mathrm{d}P_2}{\mathrm{d}j}(i, j) = 0 \Rightarrow j = \frac{d-1}{2} + \frac{si^2 - si}{2(sd - s - 2)}.$$

Now that a formula for j^* has been found, it is possible to find x^*.

Lemma 3.11. *Strategy x^* is the value of i that solves*

$$\begin{aligned} &i^3(-6s^2) + i^2(-6ds^2 + 15s^2 + 12s) + i(-6d^2s^2 - 12ds^2 + 15s^2 + 48ds - 72) \\ &\quad + 6d^3s^2 - 9d^2s^2 + 2ds^2 + s^2 - 24d^2s + 24ds - 4s + 24d - 12 = 0. \end{aligned} \tag{3.2}$$

Proof. To find x^*, the first step is to replace any js in the expression for $P_2(i, j)$ with the formula for j^*, namely (3.1). Next, differentiate with respect to i. Multiplying this result by $12sd - 12s - 24$ gives the left-hand side of (3.2). Define $\phi(s, d, i)$ to equal the left-hand side of (3.2). The discriminant of $\mathrm{d}\phi/\mathrm{d}i$ is negative if $d \geq 7$, so there is one real root. Also, $\phi(s, d, 1) > 0 > \phi(s, d, d-1)$. (We omit the proofs of these two statements due to the complexity of the expressions involved.) The result follows.

Using (3.2), the critical denomination x can be found.

Theorem 3.2. *For $d \geq 7$, the critical denomination x for player 1 is $\lceil x^* \rceil$, where x^* is given by*

$$x^*(s, d) = \frac{a}{6s^2} + \frac{b}{c_1 s^2 f^{1/3}} - \frac{1}{c_2 s^2} f^{1/3}, \tag{3.3}$$

with

$$f = e + (4b^3 + e^2)^{1/2}, \qquad a = 4s + 5s^2 - 2ds^2, \qquad c_1 = 9 \cdot 2^{2/3}, \qquad c_2 = 18 \cdot 2^{1/3},$$

$$b = 1152s^2 - 360s^3 - 720ds^3 - 495s^4 + 396ds^4 + 72d^2s^4,$$

$$\begin{aligned} e = {} &43200s^3 + 57024s^4 - 72576ds^4 - 22032s^5 - 41472ds^5 \\ &+ 41076d^2s^5 - 19872s^6 + 20736ds^6 + 6480d^2s^6 - 7344d^3s^6. \end{aligned}$$

Proof. Since x^* is the root of (3.2), a cubic equation, Cardano's cubic formula can be used to find the root. By [1], the optimal strategies for player 1 are given by $\lfloor x^* \rfloor$ and $\lceil x^* \rceil$. Therefore, the critical denomination is $x = \lceil x^* \rceil$.

Now that the critical denomination for player 1 has been found, we proceed to find the critical denomination for player 2.

Lemma 3.12. *For each j, $1 \leq j \leq d-1$, $a_{i+1,j} - a_{ij}$ is decreasing in $i \leq j$.*

Proof. The claim holds because

$$(P_2(i+2, j) - P_2(i+1, j)) - (P_2(i+1, j) - P_2(i, j)) = -s(i + j + 3) + 3 < 0.$$

Theorem 3.3. *There exists a denomination y that will serve as a critical denomination for player 2. In optimal play, player 2 will always exchange his card if it is ranked lower than y, will always stand if he has a card ranked higher than y, and will exchange his card with a certain probability if he has a card whose denomination is y.*

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Proof. According to [5], if the columns of the matrix are discrete concave, then player 2's optimal mixed strategy involves either one column or two consecutive columns. In Lemma 3.12 we have shown that the columns satisfy this condition. The theorem follows from an argument similar to that used in the proof of Theorem 3.1.

Recall the continuous version of the game discussed earlier. If $P_2(i, j)$ is a concave function of i, then there is an optimal strategy x^* for player 1. From [4], if $P_2(i, j)$ is a convex function of j, then there is an optimal strategy y^* for player 2. From [1], the optimal columns in the discrete version are $\lfloor y^* \rfloor$ and $\lceil y^* \rceil$. Since $\mathrm{d}^2 P_2/\mathrm{d}j^2 = sd - s - 2 \geq 0$, there exists an optimal y^*. A method of finding this y^*, and the value of the critical denomination for player 2, is given in the next theorem.

Theorem 3.4. *The critical denomination for player 2 is $\lceil y^* \rceil$, where*

$$y^* = \frac{d-1}{2} + \frac{s(x^*)^2 - sx^*}{2(sd - s - 2)} \quad \textit{with } x^* \textit{ as given in (3.3).} \tag{3.4}$$

Proof. We already know player 1's optimal strategy, so player 2 wants to find the value of j that minimizes $P_2(x^*, j)$. While finding x^*, the formula for this j, (3.1), was discovered. Therefore, the optimal y^* is the value that results when x^* is substituted into (3.1). As a result, player 2's optimal mixed strategy involves at most strategies $\lfloor y^* \rfloor$ and $\lceil y^* \rceil$.

Now that we know how to determine the critical denominations, the next thing to work out is how often the critical denominations should be exchanged.

Theorem 3.5. *Let x be player 1's critical denomination and let y be player 2's critical denomination. Each player has an optimal pure strategy if and only if the following condition is satisfied, where $a = P(x-1, y-1)$, $b = P(x-1, y)$, $c = P(x, y-1)$, and $d = P(x, y)$:*

$$\operatorname{sgn}[(a-c)(b-d)] + \operatorname{sgn}[(a-b)(c-d)] > -2. \tag{3.5}$$

Proof. If condition (3.5) holds, then either one or two of the following claims are true: (i) $a > c$ and $b > d$, (ii) $a < c$ and $b < d$, (iii) $a = c$ and/or $b = d$, (iv) $a > b$ and $c > d$, (v) $a < b$ and $c < d$, (vi) $a = b$ and/or $c = d$.

Consider (i). If $P(x-1, y-1) > P(x, y-1)$ and $P(x-1, y) > P(x, y)$, then row $x-1$ dominates row x. After domination, a 1×2 matrix remains and whichever entry is smaller dominates the other for player 2. If the entries are the same, player 2 has two different pure optimal strategies. In the remainder of the situations in which condition (3.5) is satisfied (i.e. (ii)–(vi)), we can similarly show that the 2×2 matrix can be reduced to a 1×1 matrix. There are four different scenarios when this happens.

- Row $x-1$ and column $y-1$ remain: players 1 and 2 never exchange their critical denominations.
- Row $x-1$ and column y remain: player 1 never exchanges his critical denomination and player 2 always exchanges his critical denomination.
- Row x and column $y-1$ remain: player 1 always exchanges his critical denomination and player 2 never exchanges his critical denomination.
- Row x and column y remain: players 1 and 2 always exchange their critical denominations.

If condition (3.5) does not hold, then there is no possibility of row or column domination. Thus, player 1 must mix the remaining rows and player 2 must mix the remaining columns in their optimal strategies. The critical denominations are thus exchanged with some nonzero probability that is less than 1.

Theorem 3.6. *If condition (3.5) is not satisfied then player 1 should exchange his critical denomination with probability p_1, where*

$$p_1(x, y) = \frac{P_2(x-1, y-1) - P_2(x-1, y)}{P_2(x-1, y-1) - P_2(x-1, y) - P_2(x, y-1) + P_2(x, y)}, \tag{3.6}$$

and player 2 should exchange his critical denomination with probability p_2, where

$$p_2(x, y) = \frac{P_2(x-1, y-1) - P_2(x, y-1)}{P_2(x-1, y-1) - P_2(x-1, y) - P_2(x, y-1) + P_2(x, y)}. \tag{3.7}$$

Proof. The payoff matrix has been simplified to the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a = 2P(x-1, y-1) - 1, b = 2P(x-1, y) - 1, c = 2P(x, y-1) - 1$, and $d = 2P(x, y) - 1$.

For some p and q such that $0 < p, q < 1$, the optimal mixed strategies for players 1 and 2 are $(1-p, p)$ and $(1-q, q)$, respectively. From [7, p. 15], we have

$$p = \frac{a-b}{a-b-c+d} \quad \text{and} \quad q = \frac{a-c}{a-b-c+d}.$$

Substituting the values of a, b, c, and d into p and q gives (3.6) and (3.7).

4. The value of Le Her

Theorem 4.1. *The value, V, of Le Her with s suits and d denominations is the remaining entry in the payoff matrix once it has been reduced to a 1×1 matrix. If such a reduction is not possible, the value is given by*

$$V = 2\frac{P(x-1, y-1)P(x, y) - P(x-1, y)P(x, y-1)}{P(x-1, y-1) - P(x-1, y) - P(x, y-1) + P(x, y)} - 1. \tag{4.1}$$

Proof. The value in the second case comes from the formula for the value of a 2×2 matrix game.

The following question is an interesting one to consider: for what values of s and d is the value of the game positive? Since the entries in the matrix represent payoffs to player 1, a positive value means the game is advantageous to player 1; if the value is negative, the game is advantageous to player 2. It has already been shown that for one standard deck [6], the value of the game is positive. Is it true that for any number of standard decks of cards, the game is advantageous to player 1? The answer is no. To prove this we will need two lemmas.

Lemma 4.1. *For n standard decks of cards, the critical denomination for player 1 is 7 and the critical denomination for player 2 is 8.*

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Proof. Recall the definition of $\phi(s, d, i)$ in the proof of Lemma 3.11. In order to prove that the critical denomination for player 1 is 7, it suffices to show that, for any positive integer n, $\phi(4n, 13, 7) < 0$ and $\phi(4n, 13, 6) > 0$. This holds because $\phi(s, 13, 7) = -1542s^2 + 1208s - 204 < 0$ and $\phi(s, 13, 6) = 1194s^2 + 428s - 132 > 0$. If only rows 6 and 7 remain, then column 7 dominates columns 1–6 and column 8 dominates columns 9–13. (We are omitting the proof of this because of the lengthy algebra involved.) Therefore, the critical denomination for player 2 is 8.

Lemma 4.2. *For n standard decks of cards, player 1 will exchange a card of denomination* 7 *and player 2 a card of denomination* 8 *with some nonzero probability that is less than* 1.

Proof. If $d = 13$ then, after adding 1 to each of the entries and dividing by $2s^2/(sd)_3$, the payoff matrix reduces to

$$\begin{array}{cc} & \begin{array}{cc} 7 & \qquad\qquad 8 \end{array} \\ \begin{array}{c} 6 \\ 7 \end{array} & \begin{pmatrix} 1096s - 142 & 1099s - 145 \\ 1098s - 135 & 1095s - 138 \end{pmatrix}. \end{array}$$

It is easy to see that condition (3.5) is not satisfied, and that the lemma thus follows.

We can now present the interesting result.

Theorem 4.2. *If eighteen or fewer standard decks are used for playing Le Her, the game is advantageous to player 1. If nineteen or more standard decks are used, the game is advantageous to player 2.*

Proof. Using (4.1) with $d = 13$, $x = 7$, and $y = 8$, we obtain

$$V = \frac{-3(s^2 - 76s + 11)}{13(13s - 1)(13s - 2)}. \tag{4.2}$$

Thus, $V < 0$ if and only if $s > 38 + \sqrt{1433} \approx 75.9$. Since $\frac{76}{4} = 19$, the theorem is proved.

A consequence of Theorem 4.2 is that player 2 would have the advantage in standard Le Her if the cards were dealt with replacement. The next two propositions (stated without proof) give an idea of how the value of the game changes as s and d vary.

Proposition 4.1. *For a fixed d, the value of the game decreases as s increases.*

Proposition 4.2. *For a fixed s, the value of the game increases as d increases.*

5. More applications

Example 5.1. By Lemma 4.1, for any number of decks the critical denomination for player 1 is always 7 and the critical denomination for player 2 is always 8. However, the probability with which the critical denominations should be exchanged differs when two standard decks of cards are used, rather than one. Formulae (3.6) and (3.7) can be used to find the probabilities: if $s = 4$ and $d = 13$, we obtain $p_1 = 0.3750$ and $p_2 = 0.6250$; if $s = 8$ and $d = 13$, we obtain $p_1 = 0.4375$ and $p_2 = 0.4792$.

Example 5.2. Formulae (3.3) and (3.4) can be applied to the following variant of standard Le Her: instead of cards of different suits with the same denomination being regarded as the same in terms of ranking, we introduce tiebreakers with respect to the suits (e.g. spades > hearts > diamonds > clubs). Therefore, if player 1 had the 6 of spades and player 2 had

TABLE 1: Critical denominations (CDs) for different values of s and d.

s	d	$x^*(s,d)$	Player 1's CD	$y^*(s,d)$	Player 2's CD
2	25	13.1262	14	15.4603	16
2	50	26.7198	27	31.6586	32
2	53	28.3509	29	33.6022	34
4	53	28.2731	29	33.4863	34
7	23	11.9298	12	14.0024	15
8	23	11.9244	12	13.9947	14
1000	37	19.4974	20	23.0093	24

TABLE 2: Probabilities for exchanging critical denominations.

s	d	p_1	p_2	Reasoning behind probabilities
2	25	0	0	Row 13 dominates row 14; column 15 dominates column 16
2	50	1	1	Row 27 dominates row 26; column 32 dominates column 31
2	53	0.16	0.27	(3.5) not satisfied; use of (3.6) and (3.7)
4	53	0	0	Row 28 dominates row 29; column 33 dominates column 34
7	23	1	0	Row 12 dominates row 11; column 14 dominates column 15
8	23	1	1	Row 12 dominates row 11; column 14 dominates column 13
1000	37	1	0	Column 23 dominates column 24; row 20 dominates row 19

the 6 of hearts, player 1 would win. Suppose that six standard decks of cards are used in this version of the game. Then, because $x^*(6,52) = 27.7035$ and $y^*(6,52) = 32.8005$, the critical denominations for player 1 and player 2 are 28 and 33, respectively. The 28th and 33rd lowest-ranking cards are the 7 of spades and the 9 of clubs, respectively. Since $P_2(28,32) > P_2(27,32)$ and $P_2(28,33) > P_2(27,33)$, it is optimal for player 1 to always exchange the 7 of spades. Since $P_2(28,32) > P_2(28,33)$, player 2 should always exchange the 9 of clubs. By Propositions 4.1 and 4.2, this variant is more advantageous to player 1 than the original version played with six standard decks of cards.

The standard deck can be changed by adding and subtracting suits and/or denominations. In Table 1, we give the critical denominations for different s and d values.

From Theorems 3.5 and 3.6, it is possible to find out how often the critical denominations should be exchanged. This information is displayed in Table 2.

Table 3 shows how the game changes for large values of d. From it, we might guess that, for $s = 4$, the following occurs as $d \to \infty$: $x/d \to 0.544$, $y/d \to 0.648$, and $V \to 0.0705$. Indeed, it is possible to prove these three facts for any s.

Proposition 5.1. *Let x be player 1's critical denomination. As $d \to \infty$ we find that $x/d \to x_1$, where x_1 is the real root of the equation $z^3 + z^2 + z - 1 = 0$.*

Proof. Substitute ad for i in (3.2). As $d \to \infty$, (3.2) simplifies to

$$-6a^3d^3s^2 - 6a^2d^3s^2 - 6ad^3s^2 + 6d^3s^2 = 0$$
$$\Rightarrow a^3 + a^2 + a - 1 = 0.$$

The a value that satisfies the equation is 0.543 689, to 6 decimal places.

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TABLE 3: Game statistics for various values of d, with $s = 4$.

d	Player 1's CD (x)	Player 2's CD (y)	V
13	7	8	0.0251
100	54	64	0.0650
500	272	324	0.0694
1 000	544	647	0.0699
5 000	2 718	3 239	0.0704
1000 000	54 369	64 780	0.0705

TABLE 4: Game statistics for various values of s, with $d = 13$.

s	p_1	p_2	V
4	0.375	0.625	2.51×10^{-2}
12	0.458	0.431	7.32×10^{-3}
40	0.488	0.363	1.22×10^{-3}
100	0.495	0.345	-3.30×10^{-4}
400	0.499	0.336	-1.11×10^{-3}
100 000	0.500	0.333	-1.36×10^{-3}
400 000	0.500	0.333	-1.37×10^{-3}

Proposition 5.2. *Let y be player 2's critical denomination. As $d \to \infty$ we find that $y/d \to (1 + x_1^2)/2$.*

Proof. Substitute $x_1 d$ for x^* in (3.4). Then, as $d \to \infty$, we find that $y^*/d \to (1 + x_1^2)/2 \approx 0.647799$.

Proposition 5.3. *As $d \to \infty$ we find that $V \to 2P(x_1 d, ((1 + x_1^2)/2)d) - 1 \approx 0.070475$.*

In Table 4, we present the probability with which the critical cards should be exchanged and the value of the game for different numbers of standard decks.

Based on the statistics in Table 4, it seems that the following conclusions hold if the cards are dealt with replacement:

(i) Player 1 should exchange a card of denomination 7 with probability $\frac{1}{2}$.

(ii) Player 2 should exchange a card of denomination 8 with probability $\frac{1}{3}$.

(iii) The value of the game is -1.37×10^{-3}.

We conclude the paper by proving these three statements.

Proposition 5.4. *If cards are dealt from the deck with replacement, player 1 should exchange a card of denomination 7 with probability $\frac{1}{2}$.*

Proof. Using (3.6) with $d = 13$, $x = 7$, and $y = 8$, we find that $p_1 = (s - 1)/2s \to \frac{1}{2}$ as $s \to \infty$.

Proposition 5.5. *If cards are dealt from the deck with replacement, player 2 should exchange a card of denomination 8 with probability $\frac{1}{3}$.*

Proof. Using (3.7) with $d = 13$, $x = 7$, and $y = 8$, we find that $p_2 = (2s + 7)/6s \to \frac{1}{3}$ as $s \to \infty$.

Proposition 5.6. *If cards are dealt with replacement, the value is* $-3/13^3$.

Proof. By (4.2),

$$V = \frac{-3(s^2 - 76s + 11)}{13(13s - 1)(13s - 2)}.$$

Thus, as $s \to \infty$, $V \to -3/13^3 \approx -1.37 \times 10^{-3}$.

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References

[1] Benjamin, A. T. and Goldman, A. J. (1994). Localization of optimal strategies in certain games. *Naval Res. Logistics* **41,** 669–676.
[2] Benjamin, A. T. and Goldman, A. J. (2004). Analysis of the N-card version of the game Le Her. *J. Optimization Theory Appl.* **114,** 695–704.
[3] Ethier, S. (2006). Probabilistic aspects of gambling. Unpublished manuscript.
[4] Fan, K. (1953). Minimax theorems. *Proc. Nat. Acad. Sci. USA* **39,** 42–47.
[5] Howard, J. V. (1994). A geometric method of serving certain games. *Naval Res. Logistics* **41,** 133–136.
[6] Kuhn, H. (1968). James Waldegrave: excerpt from a letter. In *Precursors in Mathematical Economics: An Anthology*, eds William J. Baumol and Stephen M. Goldfeld. London School of Economics and Political Science, London, pp. 3–9.
[7] Owen, G. (1995). *Game Theory*, 3rd edn. Academic Press, San Diego, CA.