ANALYSIS OF THE EXPECTED SHORTFALL OF AGGREGATE DEPENDENT RISKS

BY

STAN ALINK, MATTHIAS LÖWE AND MARIO V. WÜTHRICH

ABSTRACT

We consider d identically and continuously distributed dependent risks $X_1, \ldots, X_d$. Our main result is a theorem on the asymptotic behaviour of expected shortfall for the aggregate risks: there is a constant c_d such that for large u we have $E\left[\sum_{i=1}^{d} X_i \middle| \sum_{i=1}^{d} X_i \le -u\right] \sim -uc_d$. Moreover we study diversification effects in two dimensions, similar to our Value-at-Risk studies in [2].

KEYWORDS

Archimedean copula, dependent random variables, diversification effect, extreme value theory, expected shortfall, Value-at-Risk.

1. INTRODUCTION

One of the central topics in modern insurance mathematics and finance is the search for new methods to calculate risk-adjusted solvency requirements for companies. Such methods should in particular be able to cope with different sorts of risks. Treating a particular kind of risk is still feasible using analytical tools. The main issue is to model and compute the aggregation effects of different, usually dependent risks.

In [2] and [13] a first step in this direction was undertaken. There d identically distributed dependent risks $X_1, \ldots, X_d$ were considered and results of the following type were obtained. (Note here that by $A \sim B$ we mean that $A/B \to 1$).

$$P\left[\sum_{i=1}^{d} X_i \le -u\right] \sim q_d \cdot P[X_1 \le -u], \quad \text{as } u \to \infty, \tag{1.1}$$

where the constant q_d quantifies the diversification effect between the dependent risks (for a precise statement see Theorem 3.2 in [2]).

From such analysis of the asymptotic behaviour of quantiles of the aggregate risks we were able to deduce as a main result an asymptotic Value-at-Risk estimate.

However, even though being very popular, Value-at-Risk has some disadvantageous properties, e.g. it is not a coherent risk measure (Value-at-Risk generally lacks the subadditivity property, cf. Artzner-Delbaen-Eber-Heath [3] or Alink-Löwe-Wüthrich [2], Theorem 3.5 for $\beta < 1$). Therefore various efforts are undertaken to look for more suitable, coherent risk measures. In many countries the regulators tend to use expected shortfall or worst conditional expectation, which in the case of continuous random variables are equivalent (see Acerbi-Tasche [1]). We do not want to enter the discussion here, about "good" and "bad" risk measures, we simply choose expected shortfall as our risk measure, which is coherent under the assumption that our random variables have continuous marginals (cf. Acerbi-Tasche [1]). We consider (for small p's) $E[X|X \le u_p]$, where u_p is the p-quantile of X. (To facilitate the analysis, we always assume losses to be negative, i.e. we study lower tails.)

Moreover it was pointed out to us by an anonymous referee that without much work the proof could be extended (and actually shortened) towards moment estimates of the sum we are considering. Therefore, even though our main interest lies in the analysis of expected shortfall (for which we need to take $\kappa = 1$ in Theorem 3.1 below), the proof also covers, for instance, the conditional variance (for which we would consider the case $\kappa = 2$).

This paper is organized as follows. In Section 2, we briefly describe our model. Section 3 contains the formulation of our main results, while Section 4 is devoted to examples. Finally in Section 5 we give the proofs, which are inspired by our previous results in [2]. We conclude this introduction with a quick review on the concept of copulas.

1.1. Copulas

With expected shortfall as our risk measure, we concentrate on the case of aggregating dependent risks. The dependency of the risks is modelled by copulas. Copulas are simply a convenient description for families of dependent random variables. The concept of copulas was introduced by Sklar [12]. The idea is that the dependence structure of a finite family of random variables is completely determined by their joint distribution function. For any $d \ge 2$, a d-dimensional copula is thus defined as a d-dimensional distribution function on $[0,1]^d$, with marginals that are uniformly distributed on $[0,1]$.

With the concept of copulas we separate a multivariate distribution function into two parts, one describing the dependence structure and the other one describing the behaviour of the marginals. Moreover, all distribution functions with continuous marginals have a copula associated with them and vice versa. This is the content of Sklar's theorem [12] (see Joe [7], Nelsen [11] or Section 2 in [2]).

In this article we focus on a special family of copulas, the Archimedean ones:

Definition 1.1. *Let $d \ge 2$. Let $\varphi : [0,1] \to [0,\infty]$ be strictly decreasing, convex and such that $\varphi(0) = \infty$ and $\varphi(1) = 0$. Define for $x_i \in [0,1]$, $i = 1,\ldots,d$:*

$$C^{\varphi}(x_1,\ldots,x_d) \stackrel{def.}{=} \varphi^{-1}\left(\sum_{i=1}^{d} \varphi(x_i)\right). \tag{1.2}$$

The function φ is called generator *of C^{φ}.*

https://doi.org/10.2143/AST.35.1.583164 Published online by Cambridge University Press

In the case $d = 2$ this definition automatically implies that C^{φ} is a copula. In the case $d \geq 3$, a further assumption is required for C^{φ} to be a copula: If for all k and $x > 0$ the k-th derivative of the inverse of φ, $\frac{d^k}{dx^k}\varphi^{-1}(x)$, exists and satisfies

$$(-1)^k \frac{d^k}{dx^k}\varphi^{-1}(x) \geq 0, \tag{1.3}$$

then C^{φ} is a distribution function, and hence a copula (cf. [10] and [2]). Copulas of this type will be called (strict) Archimedean copulas.

The importance of Archimedean copulas in practice lies in the fact that they are easy to construct, but still we obtain a rich family of dependence structures. Usually, Archimedean copulas depend on one parameter, only. This makes it easier — though still very difficult — to estimate copulas from data. One of the best studied Archimedean copulas is the *Clayton copula* with parameter $\alpha > 0$. It is generated by $\varphi(t) = t^{-\alpha} - 1$ and takes the form

$$C^{Cl,\alpha}(x_1, \dots, x_d) \overset{def.}{=} (x_1^{-\alpha} + \dots + x_d^{-\alpha} - d + 1)^{-1/\alpha}. \tag{1.4}$$

The limit $\alpha \to 0$ leads to independence, while $\alpha \to \infty$ leads to comonotonicity, i.e. complete positive dependence. For more examples we refer to Joe [7] and Nelsen [11].

With the notion of a copula in our hands the main results in this article can be described as follows. Assume the risks $X_1, \dots, X_d$ have the same continuous marginal distribution function F and $(X_1, \dots, X_d)$ has an Archimedean copula. Then we are able to compute the asymptotic behaviour of the expected shortfall, i.e. we are able to compute the decay of

$$E\left[\sum_{i=1}^{d} X_i \,\middle|\, \sum_{i=1}^{d} X_i \leq -u\right]$$

as u tends to infinity, or — more generally — moments of the form

$$E\left[\left(\sum_{i=1}^{d} X_i\right)^{\kappa} \,\middle|\, \sum_{i=1}^{d} X_i \leq -u\right]$$

for sufficiently small $\kappa \in \mathbb{N}$.

In this article we define expected shortfall as

$$-E(X \mid X \leq u).$$

As in the case of extreme value theorems which were proved in [2] it is possible to distinguish three different cases with respect to marginal distribution functions: the Fréchet case, the Gumbel case, and the Weibull case, of which only the two (most) interesting ones, the Fréchet and the Gumbel case will be considered here.

2. The model

As already mentioned in the introduction we study a multivariate model describing the diversification effect when aggregating d dependent risks. The dependence structure will be given by an Archimedean copula, and losses are assumed to be negative. More precisely our assumptions read as follows:

Assumption 2.1. *We assume that the random vector* $(X_1, \ldots, X_d)$ *satisfies:*

1) All coordinates X_i *are negative and have the same continuous marginal*

$$F(x) = P[X_1 \leq x].$$

2) $(X_1, \ldots, X_d)$ *has an Archimedean copula with generator* φ.

3) This generator φ *is regularly varying at* 0^+ *with index* $-\alpha$*, where* $\alpha > 0$.

The first condition is nescessary for our proof and seems rather restrictive, but when one has different marginals, one could take the heavier tail and see our same-marginal result as an upper bound for the various-marginal case.

For the last assumption let us recall the following definition (a standard reference on regular variation is Bingham-Goldie-Teugels [4]):

Definition 2.2. *A function f is called* regularly varying *at some point* x^+ *(or* x^-*, respectively)* with index $\alpha \in \mathbb{R}$ *if for all* $t > 0$

$$\lim_{s \downarrow x} \frac{f(st)}{f(s)} = t^{\alpha}, \tag{2.1}$$

(or $\lim_{s \uparrow x} \frac{f(st)}{f(s)} = t^{\alpha}$, *respectively).*

3. Results

In this section we formulate our central results. Depending on the extreme value behaviour of the underlying risks, we distinguish two cases: the Fréchet case and the Gumbel case.

3.1. Fréchet case

In the Fréchet case we look at (dependent) random variables that have a Fréchet-type distribution: their marginal distributions are regularly varying at $-\infty$ with paramater $-\beta$, for some $\beta > 0$. In our case we additionally assume that $\beta > 1$. The latter assumption is needed in order for the random variables to have a (finite) mean.

Theorem 3.1. (Fréchet case) *Let* $\kappa \in [0, \infty)$*, assume Assumption 2.1 and assume that F is regularly varying at* $-\infty$ *with parameter* $-\beta$*,* $\beta > \kappa$*. We have*

$$\lim_{u\to\infty}\frac{-1}{u^{\kappa}}E\left[\left(\sum_{i=1}^{d}X_i\right)^{\kappa}\middle|\sum_{i=1}^{d}X_i\le -u\right]=\frac{\beta}{\beta-\kappa}. \tag{3.1}$$

Remark 3.2. *Note that the limit is constant in α and d.*

Hence we find the following asymptotic behaviour for the conditional expectation ($\kappa=1$): As $u\to\infty$ we have

$$E\left[\sum_{i=1}^{d}X_i\middle|\sum_{i=1}^{d}X_i\le -u\right]\sim -\frac{\beta}{\beta-1}u, \tag{3.2}$$

which is essentially the asymptotic behaviour of the conditional expectation of the Pareto distribution (see Karamata's Theorem, [5] Theorem A3.6). The dependence strength comes now in via the following observation: For the expected shortfall, conditioned on an event with probability p we obtain the following result: Denote by $-u_p$ the p-quantile of $\sum_{i=1}^{d}X_i$. From the above theorem and our results in [2], Theorem 3.2, as $p\to 0$ we get

$$E\left[\sum_{i=1}^{d}X_i\middle|\sum_{i=1}^{d}X_i\le -u_p\right]\sim -\frac{\beta}{\beta-1}u_p\sim\frac{\beta}{\beta-1}F^{-1}\left(\frac{p}{q_d^F(\alpha,\beta)}\right), \tag{3.3}$$

where

$$q_d^F(\alpha,\beta)=\int_{\substack{x_i\ge 0\ \forall i\\ \sum_{i=1}^{d}1/x_i\ge 1}}\left[\frac{d^d}{dx_1\dots dx_d}\left(\sum_{i=1}^{d}x_i^{-\alpha\beta}\right)^{-1/\alpha}\right]dx_1\dots dx_d. \tag{3.4}$$

(Note here that the F in $q_d^F(\alpha,\beta)$ stands for Fréchet, and not for the marginal distribution function.) For $d=2$, $q_d^F(\alpha,\beta)$ can be calculated explicitly, (see Theorem 3.5 in [2]):

Let Y_α have density $f_\alpha=(1+x^\alpha)^{-1/\alpha-1}$, $\alpha>0$ and $x>0$, then

$$q_2^F(\alpha,\beta)=1+E\left[\left(1+Y_\alpha^{-1/\beta}\right)^{\beta-1}\right]. \tag{3.5}$$

For $\beta>1$, $q_2^F(\alpha,\beta)$ is increasing in α (see Theorem 3.5 in [2]). Hence we have found:

Corollary 3.3. *Choose $d=2$ and assume that (X_1,X_2) satisfies the assumptions of Theorem 3.1. For $p\to 0$ we have*

$$E[X_1+X_2\,|\,X_1+X_2\le -u_p]\sim\frac{\beta}{\beta-1}F^{-1}\left(\frac{p}{q_2^F(\alpha,\beta)}\right), \tag{3.6}$$

where the right-hand side of (3.6) is strictly decreasing in α.

This shows that the right-hand side of (3.6) is decreasing in α, i.e. the bigger α, the smaller the diversification effect. This is not surprising since α measures the dependence strength in the tails (see Juri-Wüthrich [8]). In the bivariate situation a coefficient for the dependence strength in the tails is the so-called tail dependence coefficient λ (see Embrechts-McNeil-Straumann [6]). For Archimedean copulas we have $\lambda = 2^{-1/\alpha}$ (see [8], Theorem 3.9), which is increasing in α.

3.2. Gumbel case

In the Gumbel case we look at (dependent) random variables that lie in the domain of attraction of the exponential limit law for exceedances: there is a $c \geq -\infty$ and a positive measurable function $s \mapsto a(s)$ such that for $t \in \mathbb{R}$ one has for marginals F that $\lim_{u \downarrow c} F(u + ta(u))/F(u) = e^t$.

Theorem 3.4. (Gumbel case) *Under Assumption 2.1 and F of Gumbel type we have that*

$$\lim_{u \to c} \frac{1}{a(u)} E\left[\sum_{i=1}^{d} X_i \,\middle|\, \sum_{i=1}^{d} X_i \leq du + a(u)\right] - \frac{du}{a(u)} = c_d^G(\alpha), \tag{3.7}$$

where

$$c_d^G(\alpha) = \frac{1}{q_d^G(\alpha)} \int_{\sum_{i=1}^{d} x_i \leq 1} \left(\sum_{i=1}^{d} x_i\right) \frac{d^d}{dx_1 \ldots dx_d} \left(\sum_{i=1}^{d} e^{-x_i \alpha}\right)^{-1/\alpha} dx_1 \ldots dx_d, \tag{3.8}$$

with q_d^G given by

$$q_d^G(\alpha) = \int_{\sum_{i=1}^{d} x_i \leq 1} \frac{d^d}{dx_1 \ldots dx_d} \left(\sum_{i=1}^{d} e^{-x_i \alpha}\right)^{-1/\alpha} dx_1 \ldots dx_d. \tag{3.9}$$

In particular we get

$$c_2^G(\alpha) = 1 + \frac{E\left[Y_\alpha^{-1/2} \log Y_\alpha\right]}{E\left[Y_\alpha^{-1/2}\right]} = -1, \tag{3.10}$$

where Y_α has probability density $f_\alpha = (1 + x^\alpha)^{-1/\alpha - 1}$ on $x > 0$.

Remark 3.5. *Note that $c_2^G(\alpha)$ is constant in α.*

We can now do similar considerations as in the Fréchet case, assume that F is strictly increasing, then as $u \downarrow c$:

https://doi.org/10.2143/AST.35.1.583164 Published online by Cambridge University Press

$$\begin{aligned} E\left[\sum_{i=1}^{d} X_i \,\middle|\, \sum_{i=1}^{d} X_i \le du + a(u)\right] &\approx du + c_d^G(\alpha)\, a(u) \\ &= dF^{-1}\Big(F\big(u + c_d^G(\alpha)\, a(u)/d\big)\Big) \approx dF^{-1}\Big(e^{c_d^G(\alpha)/d} F(u)\Big) \\ &\approx dF^{-1}\left(\frac{e^{c_d^G(\alpha)/d}}{q_d^G(\alpha)} P\left[\sum_{i=1}^{d} X_i \le du + a(u)\right]\right), \end{aligned} \tag{3.11}$$

where in the last step we have used formula (5.22) of [2]. (Note that here we use $\approx$ instead of $\sim$. $A \approx B$ can be read as $(A - du)/(B - du) \to 1$.)

Denote by u_p the p-quantile of $\sum_{i=1}^{d} X_i$. Then as $p \to 0$ we get

$$E\left[\sum_{i=1}^{d} X_i \,\middle|\, \sum_{i=1}^{d} X_i \le u_p\right] \approx dF^{-1}\left(\frac{p \cdot \exp\left\{c_d^G(\alpha)/d\right\}}{q_d^G(\alpha)}\right), \tag{3.12}$$

hence expected shortfall can be approximated asymptotically.

Using Theorem 3.9 of [2] we find:

Corollary 3.6. *Choose $d = 2$ and assume that (X_1, X_2) satisfies the assumptions of Theorem 3.4. For $p \to 0$ we have*

$$\begin{aligned} E\left[X_1 + X_2 \,\middle|\, X_1 + X_2 \le u_p\right] &\approx 2F^{-1}\left(\frac{p \cdot \exp\{-1/2\}}{q_2^G(\alpha)}\right) \\ &= 2F^{-1}\left(\frac{p \cdot \Gamma(1 + 1/\alpha)}{e \cdot \Gamma^2(1 + 1/(2\alpha))}\right), \end{aligned} \tag{3.13}$$

where we use [2], Theorem 2.9 for the equality and see that the right-hand side of (3.13) is strictly decreasing in α.

3.3. Conclusions

In Corollaries 3.3 and 3.6 we are able to study the asymptotic behaviour of expected shortfall, which gives upper and lower bounds for small p. The remarkable thing is that the estimate only depends on the marginals F and on the dependence strength α. I.e. in the Archimedean situation we can avoid the difficulty of choosing an explicit model (copula) for the dependence structure. All we need to estimate are the marginals and the (tail) dependence strength α (or the tail dependence coefficient $\lambda = 2^{-1/\alpha}$, resp.). As expected, the bounds are decreasing for increasing dependence strength α, i.e. the larger the dependence strength, the smaller the diversification effect.

4. Example

The results from the previous section can be used to estimate the expected shortfall in cases where the assumptions of that section are met. In this section we shall do the calculations for one such case. We shall also show the accuracy of our estimate for another case.

4.1. How can we use this result?

First we revisit the example given in [2]. There we took two dependent motor liability portfolios X_1 and X_2. As risk measure we considered Value-at-Risk at a certain probability level. Using Value-at-Risk we studied then the diversification effect when merging these two dependent portfolios to one big portfolio $X_1 + X_2$. Here we examine the same example, but this time we choose expected shortfall as our risk measure (which in our continuous setup is a coherent risk measure). Assume X_1 and X_2 have Archimedean copula generated by a regularly varying function with index $-\alpha$ at 0^+ ($\alpha > 0$). Moreover assume that $-X_1$ and $-X_2$ have translated Pareto marginals with translation $V_1 = 880$ and $V_2 = 820$, i.e. $Y_i := -(X_i + V_i)$ is Pareto distributed with $\theta = 80$ and $\beta = 3$: for $i = 1,2$.

$$P[X_i \le x] = P[X_i + V_i \le x + V_i] = \left(\frac{\theta}{-(x + V_i)}\right)^{\beta} \quad \text{for } x \le -(\theta + V_i). \tag{4.1}$$

We define expected shortfall for $p \in (0,1)$:

$$\mathrm{ES}_{X_i}(p) = -E[X_i | X_i < u_p(X_i)], \tag{4.2}$$

where $u_p(X_i)$ is the p-quantile of X_i.

Hence we have for $p = 0.5\%$

	portfolio 1	**portfolio 2**
translation V_i	880	820
mean $E[-X_i]$	1000	940
$u_p(X_i)$	-1347.8	-1287.8
$\mathrm{ES}_{X_i}(p)$	1581.8	1521.8

Now we merge these two dependent portfolios to one big portfolio and we study expected shortfall as a function of the dependence strength α:

$$\mathrm{ES}_{X_1+X_2}(p;\alpha) \stackrel{def.}{=} -E[X_1 + X_2 | X_1 + X_2 < u_p^{\alpha}(X_1 + X_2)], \tag{4.3}$$

where $u_p^{\alpha}(X_1 + X_2)$ is the p-quantile of $X_1 + X_2$. Using Corollary 3.3 on $(-Y_1, -Y_2)$ (note that this random vector has the same copula as (X_1, X_2), and furthermore identical marginals, which is nescessary in order to apply 3.3.), we see that we have the following approximation as $p \to 0$

$$\mathrm{ES}_{X_1+X_2}(p;\alpha) \sim \frac{\beta}{\beta-1}\theta\left(\frac{q_2^F(\alpha,\beta)}{p}\right)^{1/\beta} + V_1 + V_2 \overset{def.}{=} E_{X_1+X_2}(\alpha). \tag{4.4}$$

In order to quantify the benefits gained by merging the portfolios we introduce the diversification effect on expected shortfall.

Definition 4.1. *The diversification effect on expected shortfall, as a function of α is given by*

$$Div.eff.ES(\alpha) \overset{def.}{=} \frac{E_{X_1+X_2}(\infty) - E_{X_1+X_2}(\alpha)}{E_{X_1+X_2}(\infty) + E[X_1+X_2]}. \tag{4.5}$$

If we evaluate $E_{X_1+X_2}(\alpha)$ for different α's ($p = 0.5\%$) we obtain the following table (note that in the independent case we calculated the exact values, rather than the approximated values):

α	***indep.***	**0.5**	**1.0**	**1.5**	**2.0**	**3.0**	**4.0**	∞
$-E[X_1+X_2]$	1940	1940	1940	1940	1940	1940	1940	1940
$E_{X_1+X_2}(\alpha)$	2711	2918	3032	3066	3080	3092	3097	3104
Div.eff.ES(α)	33.7%	16.0%	6.2%	3.2%	2.0%	1.0%	0.6%	0%
Div.eff.VaR(α)	31.6%	17.8%	6.9%	3.6%	2.2%	1.1%	0.6%	0%

$\alpha = \infty$ belongs to the comonotonic case (complete positive dependence), and Div.eff.VaR(α) gives the comparison to the results obtained in [2] for Value-at-Risk.

Not surprisingly, we see that the diversification effect decreases for increasing dependence strength α. One also observes that the decrease is rather fast, i.e. already introducing slight dependencies in the tails reduces the diversification savings substantially.

For small α (i.e. close to the independent case), p should be even smaller than 0.5% in order for the approximation to be sharp. This is not a serious problem, however, since we can calculate the expected shortfall and the diversification effect directly in the independent case. A more detailed account of the accuracy of our approximation shall be given in the next subsection.

4.2. How accurate is the estimate?

Now we shall show the efficiency of our estimate for the case where X and Y are random variables with a dependence structure described by a Clayton copula and such that $-X$ and $-Y$ have Generalised Pareto distribution. The definitions of these can e.g. be found in [5], Definition 3.4.9 on page 162 and [11], (4.2.1), page 94. We shall recall them here:

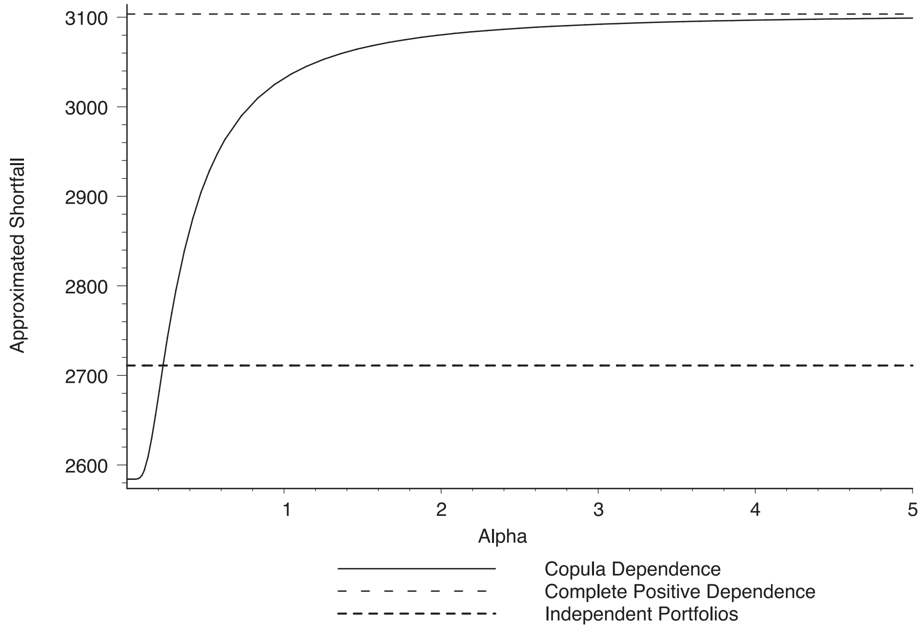

FIGURE 1: The approximated shortfall as a function of α.

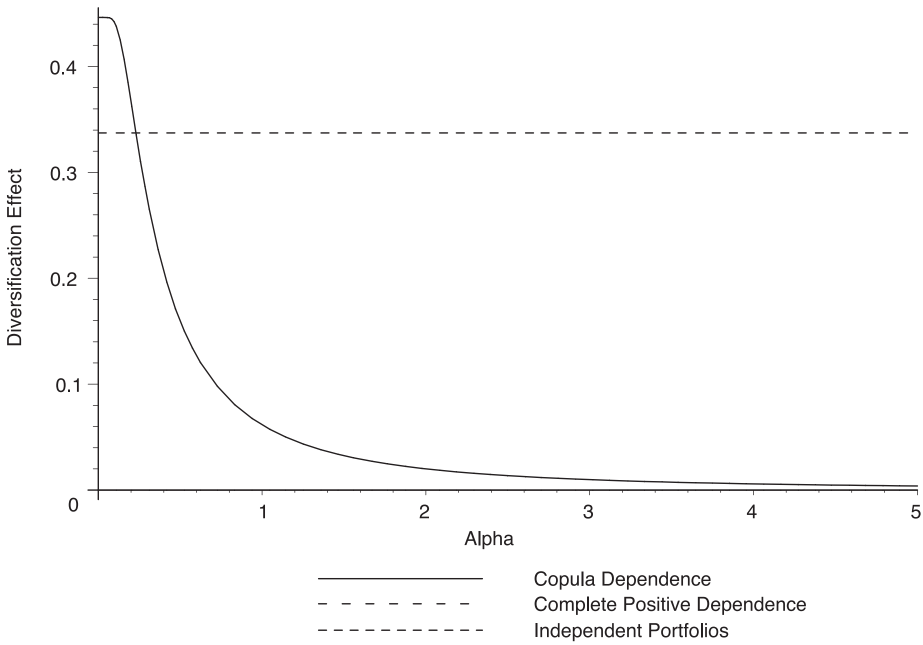

FIGURE 2: The diversification effect as a function of α. The complete positive dependence coincides with the x-axis.

Definition 4.2. *$-X$ and $-Y$ have Generalised Pareto marginals F_β^{GP}. I.e:*

$$F_\beta^{GP}(t) := P(X \leq t) = P(Y \leq t) = \left(1 - \frac{1}{\beta} t\right)^{-\beta}, \;\; \forall t \leq 0, \tag{4.6}$$

and X and Y have Clayton copula C_α^{Cl}, as given by (1.4)

This means that X and Y have joint distribution function F on $(-\infty,0]^2$ as follows:

$$\begin{aligned} F(x,y) &:= P(X \leq x, Y \leq y) = C_\alpha^{Cl}\left(F_\beta^{GP}(x), F_\beta^{GP}(y)\right) \\ &= \left(\left(1 - \frac{1}{\beta} x\right)^{\alpha\beta} + \left(1 - \frac{1}{\beta} y\right)^{\alpha\beta} - 1\right). \end{aligned} \tag{4.7}$$

According to Theorem 3.1 and especially (3.2) we have in the bivariate case:

$$E(X+Y \mid X+Y \leq -u) \sim -u \frac{\beta}{\beta - 1}, \tag{4.8}$$

for large u. In order to show the efficiency we calculate the value of $E(X+Y \mid X+Y \leq -u)$. With some straightforward calculations we see:

$$E(X+Y \mid X+Y \leq -u) = 2\,\frac{\int_{-\infty}^{-u} x\left(1 - \frac{1}{\beta} x\right)^{-\beta-1} dx + \int_{-u}^{0} x \cdot J_{\alpha,\beta}(x,u)\,dx}{\int_{-\infty}^{-u} \left(1 - \frac{1}{\beta} x\right)^{-\beta-1} dx + \int_{-u}^{0} J_{\alpha,\beta}(x,u)\,dx}, \tag{4.9}$$

where $J_{\alpha,\beta}$ is given by:

$$\begin{aligned} J_{\alpha,\beta}(x,u) &:= \left[\frac{d}{dx} F(x,y)\right]_{y=-\infty}^{-u-x} \\ &= \left(1 - \frac{x}{\beta}\right)^{\alpha\beta - 1}\left(\left(1 - \frac{x}{\beta}\right)^{\alpha\beta} + \left(1 + \frac{u+x}{\beta}\right)^{\alpha\beta} - 1\right)^{(-1/\alpha)-1}. \end{aligned} \tag{4.10}$$

We fed this formula into the computer-algebra-package Maple to draw the following result (Figure 3) for the case where $\alpha = 1$ and $\beta = 2$. The figure shows the exact value of $E(X+Y \mid X+Y \leq -u_p)$, divided by our estimate $-u_p\beta/(\beta-1)$, as a function of p $(= P(X+Y \leq -u_p))$.

Remark 4.3. Taking $u = 0$, one immediately sees that for negative random variables $X_1, \ldots, X_d$ typically

$$E(X+Y \mid X+Y \leq -u) = E(X+Y) \neq 0 = -u\frac{\beta}{\beta - 1}. \tag{4.11}$$

So our estimate is not exact for $u = 0$ (or $p = 1$, which is the same in this case), no matter what copula and marginal distribution one takes.

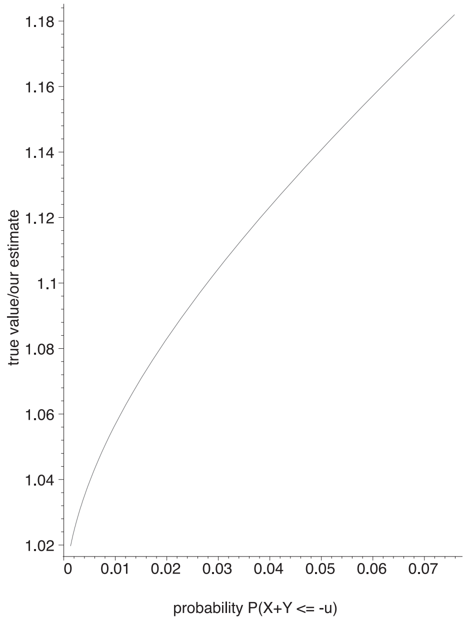

FIGURE 3: $\frac{E\left(X+Y \mid X+Y \le -u\right)}{-u\beta/\left(\beta-1\right)}$ as a function of $P(X+Y \le -u)$, for $\alpha = 1$ and $\beta = 2$.

This is not in contradiction with the results of [8] and [9]. Their results state that the Clayton copula $C^{Cl}(x, y)$ is the 'limiting copula' when one looks at the quotient $C(x\varepsilon, y\varepsilon)/C(\varepsilon, \varepsilon)$, and that the behaviour of the Clayton copula itself is invariant. But here we condition on $X + Y \le -u$ rather than $X \le -u \wedge Y \le -u$. So our estimate is slightly smaller than the real value, since we not only condition on (and thus divide by) the probability that both X and Y are very small, but also the probability that only one of them is very small. But as we take $\alpha > 0$ and thus positive dependency, this last probability is very small, but large enough to show up in Figure 3.

5. PROOFS

Proof of Theorem 3.1

The main idea here comes from [2]. The main theorem of that article states that

$$\lim_{u\to\infty} \frac{1}{F(-u)} P\left[\sum_{i=1}^{d} X_i \le -u\right] = q_d^F(\alpha, \beta), \tag{5.1}$$

for a certain constant $q_d^F(\alpha, \beta)$.

For simplicity, let us write S for $\sum_{i=1}^{d} X_i$. Let F_S be its distribution function. By the simple substitution $s = -t$ we obtain

$$\begin{aligned}
&\lim_{u\to\infty}\left(\frac{-1}{u}\right)^{\kappa} E\left[\left(\sum_{i=1}^{d} X_i\right)^{\kappa}\middle|\sum_{i=1}^{d} X_i \le -u\right] \\
&= \lim_{u\to\infty}\frac{\int_{-\infty}^{-u}(-t)^{\kappa}\,dF_S(t)}{u^{\kappa}F_S(-u)} \stackrel{t\to -s}{=\!=\!=} \lim_{u\to\infty}\frac{\int_{u}^{\infty}(s)^{\kappa}\,dF_S(-s)}{u^{\kappa}F_S(-u)}.
\end{aligned} \tag{5.2}$$

With (5.1) we can now see that the distribution function F_S is regular varying at $-\infty$ with parameter $-\beta$. We therefore may apply [4], Theorem 1.6.5 to the right hand side of the above equation to obtain

$$\lim_{u\to\infty}\left(\frac{-1}{u}\right)^{\kappa} E\left[\left(\sum_{i=1}^{d} X_i\right)^{\kappa}\middle|\sum_{i=1}^{d} X_i \le -u\right] \stackrel{[4]}{=} \frac{\beta}{\beta-\kappa}. \tag{5.3}$$

■

Proof of Theorem 3.4. For the lower bound note that

$$\begin{aligned}
&1 - E\left[\sum_{i=1}^{d}\frac{X_i - u}{a(u)}\middle|\sum_{i=1}^{d} X_i \le du + a(u)\right] \\
&\quad = E\left[1-\sum_{i=1}^{d}\frac{X_i - u}{a(u)}\middle|\sum_{i=1}^{d}\frac{X_i - u}{a(u)} \le 1\right]
\end{aligned} \tag{5.4}$$

has a positive argument in the integral. We define $Y_i(u) = (X_i - u)/a(u)$. Hence for all $\varepsilon > 0$

$$\begin{aligned}
&1 - E\left[\sum_{i=1}^{d}\frac{X_i - u}{a(u)}\middle|\sum_{i=1}^{d} X_i \le du + a(u)\right] \\
&= \int_0^{\infty} P\left[1-\sum_{i=1}^{d} Y_i(u) > z\middle|\sum_{i=1}^{d} Y_i(u) \le 1\right]dz \\
&= \int_0^{\infty} P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\middle|\sum_{i=1}^{d} Y_i(u) \le 1\right]dz \\
&= \int_0^{\infty}\frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{P\left[\sum_{i=1}^{d} Y_i(u) \le 1\right]}\,dz \\
&= \frac{F(u + a(u)/\varepsilon)}{P\left[\sum_{i=1}^{d} Y_i(u) \le 1\right]}\int_0^{\infty}\frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)}\,dz.
\end{aligned} \tag{5.5}$$

From the Gumbel assumption on F and formula (5.22) in [2], we find that the first term on the right-hand side in (5.5) satisfies

$$\lim_{u \to c} \frac{F(u + a(u)/\varepsilon)}{P\left[\sum_{i=1}^{d} Y_i(u) \le 1\right]} = \frac{e^{1/\varepsilon}}{q_d^G(\alpha)}. \tag{5.6}$$

It remains to study the integral. Choose $M > 1$ and $\varepsilon < d$ and divide the integral into two parts:

$$\begin{aligned} &\int_0^\infty \frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)} dz \\ &= \int_0^M \frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)} dz + \int_M^\infty \frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)} dz. \end{aligned} \tag{5.7}$$

To the first term we apply the dominated convergence theorem, the second term becomes arbitrarily small for large M.

Term 1. For $z > 0$

$$P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right] \le d \cdot P\left[Y_1(u) < (1 - z)/d\right] \le d \cdot F(u + a(u)/d). \tag{5.8}$$

Hence for all large u we have that

$$\frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)} \le d \cdot \frac{F(u + a(u)/d)}{F(u + a(u)/\varepsilon)} \le (d + 1) \exp\{1/d - 1/\varepsilon\}. \tag{5.9}$$

Henceforth we have found an uniform upper bound, which implies that our function is L^1 on $[0, M]$. There remains to prove pointwise convergence in z so that we can apply the dominated convergence theorem to the first term on the right-hand side of (5.7).

We introduce the events $\{Y_1(u) < 1/\varepsilon\}$.

$$\begin{aligned} &\frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)} \\ &= P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z \,\middle|\, Y_1(u) < 1/\varepsilon\right] + \frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z,\, Y_1(u) \ge 1/\varepsilon\right]}{F(u + a(u)/\varepsilon)}. \end{aligned} \tag{5.10}$$

Lemma 5.3 of [2] states:

$$\lim_{u \to c} P(X_i \le u + x_i a(u), i = 1, \ldots, d \,|\, X_1 \le u + a(u)/\varepsilon) = e^{-1/\varepsilon} \left(\sum_{i=1}^{d} e^{-\alpha x_i}\right)^{-1/\alpha}. \tag{5.11}$$

When we apply this to the first term on the right-hand side of (5.10), we find

$$
\begin{aligned}
e^{-1/\varepsilon} f_{1,\varepsilon}(z) &\stackrel{def.}{=} \lim_{u\to c} P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z \,\middle|\, Y_1(u) < 1/\varepsilon\right] \\
&= e^{-1/\varepsilon} \int\limits_{\substack{\sum_i x_i < 1-z \\ x_1 < 1/\varepsilon}} \left[\frac{d^d}{dx_1 \dots dx_d}\left(\sum_{i=1}^{d} e^{-\alpha x_i}\right)^{-1/\alpha}\right] dx_1 \dots dx_d .
\end{aligned}
\tag{5.12}
$$

To the second term on the right-hand side of (5.10) we give an estimate which is similar to (5.12) in [2].

$$
\begin{aligned}
&\limsup_{u\to c} \frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z,\, Y_1(u) \geq 1/\varepsilon\right]}{F(u + a(u)/\varepsilon)} \\
&\leq \limsup_{u\to c} \frac{(d-1)\cdot P\left[Y_2(u) < (1-z)/d,\, Y_1(u) \geq 1/\varepsilon\right]}{F(u + a(u)/\varepsilon)} \\
&\leq \limsup_{u\to c} \frac{(d-1)\cdot F(u + a(u)(1-z)/d)}{F(u + a(u)/\varepsilon)} \\
&\quad \times \left(1 - \frac{\varphi^{-1}\left(\varphi\left(F(u + a(u)(1-z)/d)\right) + \varphi\left(F(u + a(u)/\varepsilon)\right)\right)}{F(u + a(u)(1-z)/d)}\right) \\
&= (d-1)\, e^{-1/\varepsilon}\left[e^{(1-z)/d} - \left(e^{-\alpha(1-z)/d} + e^{-\alpha/\varepsilon}\right)^{-1/\alpha}\right] \\
&\leq (d-1)\, e^{-1/\varepsilon} e^{(1-z)/d}\left[1 - \left(1 + e^{-\alpha/e + \alpha/d}\right)^{-1/\alpha}\right] \stackrel{def.}{=} e^{-1/\varepsilon} f_{2,\varepsilon}(z).
\end{aligned}
\tag{5.13}
$$

Now we come to the last term on the right-hand side of (5.7). For $M > 1$,

$$
\begin{aligned}
&\int_M^\infty \frac{P\left[\sum_{i=1}^{d} Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)}\, dz \\
&\leq d \int_M^\infty \frac{F(u + (1-z)\, a(u)/d)}{F(u + a(u)/\varepsilon)}\, dz \\
&= d\, \frac{F\left(u - \frac{M-1}{d}\, a(u)\right)}{F(u + a(u)/\varepsilon)} \int_{(M-1)/d}^\infty \frac{F(u - x a(u))}{F\left(u - \frac{M-1}{d}\, a(u)\right)}\, dx \\
&= d\, \frac{F\left(u - \frac{M-1}{d}\, a(u)\right)}{F(u + a(u)/\varepsilon)} \int_{(M-1)/d}^\infty P\left[Y_1(u) < -x \mid Y_1(u) < -(M-1)/d\right] dx \\
&= d\, \frac{F\left(u - \frac{M-1}{d}\, a(u)\right)}{F(u + a(u)/\varepsilon)}\, E\left[-Y_1(u) \mid -Y_1(u) > (M-1)/d\right].
\end{aligned}
\tag{5.14}
$$

Next we consider the expectation in the expression above:

$$\begin{aligned} E\left[-Y_1(u)\middle|-Y_1(u) > \frac{M-1}{d}\right] &= E\left[-\frac{X_1-u}{a(u)}\middle|-\frac{X_1-u}{a(u)} > \frac{M-1}{d}\right] \\ &= \frac{M-1}{d} + \frac{1}{a(u)} E\left[-X_1 - v_M(u) \middle| -X_1 > v_M(u)\right], \end{aligned} \tag{5.15}$$

where $v_M(u) = (M-1)a(u)/d - u$. Now we may use the that we are working with marginals which have Gumbel type, henceforth (see [5], formula (3.3.34))

$$\begin{aligned} &\limsup_{u\to c} \frac{1}{a(u)} E\left[-X_1 - v_M(u)\middle|-X_1 > v_M(u)\right] = \limsup_{u\to c} \frac{a(-v_M(u))}{a(u)} \\ &= \limsup_{u\to c} \frac{a\left(-\frac{M-1}{d}a(u)+u\right)}{a(u)} = 1, \end{aligned} \tag{5.16}$$

where in the last step we have used that $\lim_{u\to c} a'(u) = 0$ (see [5], Theorem 3.3.26 and formula (3.3.31)).

Hence we find for all $\varepsilon < d$ and all $M > 1$ (see (5.12), (5.13), (5.14), (5.16))

$$\begin{aligned} &\limsup_{u\to c} \int_0^\infty \frac{P\left[\sum_{i=1}^d Y_i(u) < 1 - z\right]}{F(u + a(u)/\varepsilon)} dz \\ &\quad \le e^{-1/\varepsilon}\left(\int_0^M f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x) dx + de^{-(M-1)/d}((M-1)/d + 1)\right). \end{aligned} \tag{5.17}$$

The function $f_{1,\varepsilon}$ is increasing in ε. Moreover

$$\int_0^M f_{2,\varepsilon}(x)\,dx = (d-1)\,d\left(e^{1/d} - e^{(1-M)/d}\right)\left[1 - \left(1 + e^{-\alpha/\varepsilon + \alpha/d}\right)^{-1/\alpha}\right], \tag{5.18}$$

which converges to 0 for $\varepsilon \to 0$. Hence we find (see (5.5), (5.6), (5.17))

$$\begin{aligned} &\limsup_{M\to\infty} \limsup_{\varepsilon\to 0} \left(1 - E\left[\sum_{i=1}^d Y_i(u)\middle|\sum_{i=1}^d Y_i(u) \le 1\right]\right) \\ &\quad \le \frac{1}{q_d^G(\alpha)} \int_0^\infty \int_{\sum_i x_i < 1-z} \left[\frac{d^d}{dx_1 \ldots dx_d}\left(\sum_{i=1}^d e^{-\alpha x_i}\right)^{-1/\alpha}\right] dx_1 \ldots dx_d\, dz \\ &\quad = \frac{1}{q_d^G(\alpha)} \int_{\sum_{i=1}^d x_i \le 1} \left(1 - \sum_{i=1}^d x_i\right)\left[\frac{d^d}{dx_1 \ldots dx_d}\left(\sum_{i=1}^d e^{-x_i \alpha}\right)^{-1/\alpha}\right] dx_1 \ldots dx_d \\ &\quad = 1 - \frac{1}{q_d^G(\alpha)} \int_{\sum_{i=1}^d x_i \le 1} \left(\sum_{i=1}^d x_i\right)\left[\frac{d^d}{dx_1 \ldots dx_d}\left(\sum_{i=1}^d e^{-x_i \alpha}\right)^{-1/\alpha}\right] dx_1 \ldots dx_d. \end{aligned} \tag{5.19}$$

Exchanging the two integrations finishes to proof of the lower bound. The same upper bound is found only considering the term coming from $f_{1,\varepsilon}$. This finishes the proof of (3.7).

Now, for the case $d = 2$ we find

$$\begin{aligned}
c_2^G(\alpha) &= \frac{2}{q_2^G(\alpha)} \int_{x_1+x_2\le 1} x_1 \left[\frac{d^2}{dx_1 dx_2} \left(\sum_{i=1}^{2} e^{-x_i \alpha} \right)^{-1/\alpha} \right] dx_1 dx_2 \\
&= \frac{2}{q_2^G(\alpha)} \int_{-\infty}^{\infty} x e^{-\alpha x} \left(e^{-\alpha x} + e^{-\alpha(1-x)} \right)^{-1/\alpha - 1} dx \\
&= \frac{2}{q_2^G(\alpha)} \int_{-\infty}^{\infty} x e^{x} \left(1 + e^{-\alpha(1-2x)} \right)^{-1/\alpha - 1} dx \\
&\overset{y = e^{-(1-2x)}}{=} \frac{2}{q_2^G(\alpha)} \left(\int_{-\infty}^{\infty} \frac{e^{1/2}}{4} (1 + \log(y))\, y^{-1/2} \left(1 + y^{\alpha}\right)^{-1/\alpha - 1} dy \right) \\
&= \frac{2}{q_2^G(\alpha)} \left(\frac{e^{1/2}}{4} E\left[Y_\alpha^{-1/2} (1 + \log Y_\alpha) \right] \right).
\end{aligned} \tag{5.20}$$

Recall (5.39) from [2]:

$$q_2^G(\alpha) = \frac{e^{1/2}}{2} E\left[Y_\alpha^{-1/2} \right], \tag{5.21}$$

and find:

$$c_2^G(\alpha) = 1 + \frac{E\left[Y_\alpha^{-1/2} \log Y_\alpha \right]}{E\left[Y_\alpha^{-1/2} \right]}. \tag{5.22}$$

This proves the left equality of (3.10); for a proof of the right equality we introduce $\gamma < 0$ and generalize:

$$\begin{aligned}
\frac{E\left(Y_\alpha^{\gamma} \log(Y)\right)}{E\left(Y_\alpha^{\gamma}\right)} &= \frac{\int_0^{\infty} y^{\gamma} \log(y) \left(1 + y^{\alpha}\right)^{-\frac{1}{\alpha} - 1} dy}{\int_0^{\infty} y^{\gamma} \left(1 + y^{\alpha}\right)^{-\frac{1}{\alpha} - 1} dy} = \frac{d}{d\gamma} \log \left[\int_0^{\infty} y^{\gamma} \left(1 + y^{\alpha}\right)^{-\frac{1}{\alpha} - 1} dy \right] \\
&\overset{z = y^{\alpha}}{=} \frac{d}{d\gamma} \log \left[\frac{1}{\alpha} \int_0^{\infty} z^{\frac{\gamma+1}{\alpha} - 1} (1 + z)^{-\frac{1}{\alpha} - 1} dz \right] \overset{s = \frac{z}{1+z}}{=} \frac{d}{d\gamma} \log \left[\frac{1}{\alpha} \int_0^{1} s^{\frac{\gamma+1}{\alpha} - 1} (1 + s)^{\frac{\alpha - \gamma}{\alpha} - 1} ds \right] \\
&= \frac{d}{d\gamma} \log \left[\frac{1}{\alpha} B\left(\frac{\gamma + 1}{\alpha}, \frac{\alpha - \gamma}{\alpha} \right) \right] = \frac{d}{d\gamma} \log \left[\frac{\Gamma\left(\frac{\gamma+1}{\alpha}\right) \Gamma\left(1 - \frac{\gamma}{\alpha}\right)}{\alpha \Gamma\left(\frac{\alpha+1}{\alpha}\right)} \right] \\
&= \frac{d}{d\gamma} \left[\log \Gamma\left(\frac{\gamma + 1}{\alpha} \right) + \log(-\gamma) + \log \Gamma\left(-\frac{\gamma}{\alpha} \right) \right] \\
&= \frac{1}{\alpha} (\log \Gamma)' \left(\frac{\gamma + 1}{\alpha} \right) + \frac{1}{\gamma} - \frac{1}{\alpha} (\log \Gamma)' \left(-\frac{\gamma}{\alpha} \right).
\end{aligned}$$

https://doi.org/10.2143/AST.35.1.583164 Published online by Cambridge University Press

Now we take $\gamma = -1/2$ and find:

$$\frac{E\left(Y_\alpha^{-1/2}\log(Y_\alpha)\right)}{E\left(Y_\alpha^{-1/2}\right)} = -2, \tag{5.23}$$

which, together with (5.22) finishes proof of Theorem 3.4. ■

Acknowledgement

We would like to thank Holger Knöpfel and Ronald Kortram for their help with analytical problems, and the anonymous referee for her or his suggestions how to improve and shorten the proof of Theorem 3.1.

BIBLIOGRAPHY

Acerbi, C. and Tasche, D. (2002) On the coherence of expected shortfall, *Journal of Banking and Finance* **26(7)**, 1487-1503.

Alink, S., Löwe, M. and Wüthrich, M.V. (2004) Diversification of aggregate dependent risks, *Insurance: Math. Econom*. **35**, 77-95.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999) Coherent measures of risk. *Math. Fin.* **9/3**, 203-228.

Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987) Regular variation. Cambridge University Press, Cambridge.

Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997) Modelling extremal events for insurance and finance. Springer, Berlin.

Embrechts, P., McNeil, A. and Straumann, D. (2002) Correlation and Dependency in Risk Management: Properties and Pitfalls. In *Risk Management: Value at Risk and Beyond*, M. Dempster (Ed.), Cambridge University Press, Cambridge, 176-223.

Joe, H. (1997) Multivariate Models and Dependence Concepts. Chapman and Hall, London.

Juri, A. and Wüthrich, M.V. (2002) Copula convergence theorems for tail events. *Insurance: Math. Econom*. **30**, 405-420.

Juri, A. and Wüthrich, M.V. (2003) Tail dependence from a distributional point of view. *Extremes* **6**, 213-246.

Kimberling, C. (1974) A probabilistic interpretation of complete monotonicity. *Aequationes Mathematica* **10**, 152-164.

Nelsen, R.B. (1999) An Introduction to Copulas. Springer, New York.

Sklar, A. (1959) Fonctions de répartition à *n* dimensions et leur marges. *Publications de l'Institut de Statistique de l'Université de Paris* **8**, 229-231.

Wüthrich, M.V (2003) Asymptotic Value-at-Risk estimates for sums of dependent random variables. *Astin Bulletin* **33/(1)**, 75-92.

Stan Alink
Katholieke Universiteit Nijmegen
Subfaculteit Wiskunde
Toernooiveld 1 – 6500 GL Nijmegen
The Netherlands

MATTHIAS LÖWE
Westfälischen Wilhelms Universität Münster
Fachbereich Mathematik
Institut für Mathematische Statistik
Einsteinstrasse 62 – 48149 Münster
Germany

MARIO V. WÜTHRICH
Winterthur Insurance
Römerstrasse 17, P.O.Box 357
CH-8401 Winterthur
Switzerland