LMS J. Comput. Math. 19 (Special issue A) (2016) 205–219 © 2016 Authors
doi:10.1112/S1461157016000346

Computing Jacobi forms

Nathan C. Ryan, Nicolás Sirolli, Nils-Peter Skoruppa and Gonzalo Tornaría

Abstract

We describe an implementation for computing holomorphic and skew-holomorphic Jacobi forms of integral weight and scalar index on the full modular group. This implementation is based on formulas derived by one of the authors which express Jacobi forms in terms of modular symbols of elliptic modular forms. Since this method allows a Jacobi eigenform to be generated directly from a given modular eigensymbol without reference to the whole ambient space of Jacobi forms, it makes it possible to compute Jacobi Hecke eigenforms of large index. We illustrate our method with several examples.

1. *Introduction*

Jacobi forms play a central role in the theory of automorphic forms (for example, via the Fourier–Jacobi expansion of orthogonal modular forms), in quantum field theory (where they appear as characters of infinite-dimensional Lie algebras) and in algebraic geometry (where they provide an indispensable tool for the construction of functions with prescribed behavior of their divisors). A philosophical reason for this might be that any given space of scalar-valued or vector-valued elliptic modular forms of integral or half-integral weight can be naturally embedded into a space of Jacobi forms of integral weight and lattice index on the full modular group [**14**].

In addition to their centrality and importance, they have the striking property that there are various methods to compute their Fourier expansions or even to describe them by explicit formulas. The main techniques to compute Jacobi forms are: theta blocks [**5**]; pullback of Jacobi forms of lattice index of singular or critical weight, where the latter are essentially invariants of Weil representations of $\mathrm{SL}(2,\mathbb{Z})$ [**1**]; the Taylor expansion of a Jacobi form around $z = 0$ (see, for example, [**13**]); and modular symbols [**9**, **11**, **12**]. In this paper we describe an implementation based on the latter.

Why do we focus on the modular symbol method? Theta blocks work nicely for small weights and produce appealing explicit formulas, but miss more and more Jacobi forms the larger the weight becomes. Similarly, it is not yet clear in what generality the most recent method of pulling back singular and critical weight Jacobi forms of lattice index works. The Taylor expansion method always works and is easy to implement but becomes computationally harder as the index and, accordingly, the dimension of the spaces of Jacobi forms increase. In contrast, the modular symbol method allows one to compute directly a desired Jacobi eigenform without having to generate first a whole space of Jacobi forms and then cut it down in a second step to the eigenspace one is looking for. More precisely, we start with a modular symbol representing

Received 22 February 2016.

2010 Mathematics Subject Classification 11F50, 11-04 (primary).

Contributed to the Twelfth Algorithmic Number Theory Symposium (ANTS-XII), Kaiserslautern, Germany, 29 August–2 September 2016.

The first author was supported by CNIC grant no. IIA-1341090 from the Office of International Integrative Activities at the NSF. The second author was fully supported by the Agencia Nacional de Investigación e Innovación, under grant code PD_NAC_2013_1_11010.

an elliptic eigen-newform we are interested in. The articles referenced above that are related to the modular symbol method propose closed formulas for the Jacobi newform associated to this elliptic modular form. This allows, for example, for an automatic generation of Tunnell-like formulas for (the squares of) the central values of the L-function attached to the twist of a given elliptic curve over the rationals.

Whereas a ready-to-compute description of Jacobi forms in terms of modular symbols is given in [**12**] for weight $k = 2$ (based on the results in [**11**]), the formulas in [**9**], which cover weights $k \geqslant 3$, need some reformulation along the lines of [**12**]. Since the necessary steps for this are not completely obvious we found it worthwhile to deduce and describe these steps in detail in this note. The resulting formulas are summarized in Theorem 4.15.

The paper is organized as follows. We begin by discussing a couple of examples. After showing the kind of data the algorithm can produce, we give some background to Jacobi forms and then proceed to state and prove the mentioned formula we implemented. We then highlight some details of our implementation and conclude with an Appendix of various tables as proof of concept of our formulas and implementations. The implementation of the formula we derive is available at [**8**].

2. *Examples*

We start with some examples that we computed via an implementation of (5.5). These two examples are two whose correctness can be checked independently as they have been computed elsewhere in the literature.

First, we construct the (unique up to normalization) holomorphic Jacobi cuspform $\phi_{2,37}$ of weight 2 and index 37. The coefficients in this case are indexed by pairs of integers n, r so that $r^2 < 4 \cdot 37 \cdot n$. The n are the exponents of $q = e^{2\pi i \tau}$ ($\tau \in \mathbb{H}$) and the r are the exponents of $\zeta = e^{2\pi i z}$ ($z \in \mathbb{C}$):

$$\begin{aligned}
\phi_{2,37}(\tau, z) = {} & q(-2\,(\zeta + \zeta^{-1}) + 4\,(\zeta^2 + \zeta^{-2}) + 3\,(\zeta^4 + \zeta^{-4}) \\
& \quad + 3\,(\zeta^5 + \zeta^{-5}) - 4\,(\zeta^6 + \zeta^{-6}) - 4\,(\zeta^7 + \zeta^{-7}) \\
& \quad - (\zeta^8 + \zeta^8) + 6\,(\zeta^9 + \zeta^{-9}) + 3\,(\zeta^{11} + \zeta^{-11}) + 3\,(\zeta^{12} + \zeta^{-12})) \\
& + q^2\,(\,4\,(\zeta + \zeta^{-1}) + (\zeta^2 + \zeta^{-2}) - (\zeta^3 + \zeta^{-3}) - 6\,(\zeta^4 + \zeta^{-4}) \\
& \quad - (\zeta^5 + \zeta^{-5}) + 2\,(\zeta^6 + \zeta^{-6}) - 4\,(\zeta^7 + \zeta^{-7}) + 6\,(\zeta^8 + \zeta^{-8}) \\
& \quad + (\zeta^{11} + \zeta^{-11}) - 2\,(\zeta^{12} + \zeta^{-12}) + (\zeta^{13} + \zeta^{-13}) \\
& \quad - 3\,(\zeta^{14} + \zeta^{-14}) + (\zeta^{15} + \zeta^{-15}) + 2\,(\zeta^{16} + \zeta^{-16}) - (\zeta^{17} + \zeta^{-17})) \\
& + \ldots .
\end{aligned}$$

More coefficients can be found in Table A.1. These coefficients can be independently checked by comparing them to [**3**, Table 4].

Second, we construct the holomorphic cuspform $\phi_{10,1}$ of weight 10 and index 1. In this case the Fourier expansion is indexed by pairs of integers n, r so that $r^2 < 4 \cdot 1 \cdot n$, with the notation otherwise as above:

$$\begin{aligned}
\phi_{10,1}(\tau, z) = {} & q(-2 + (\zeta + \zeta^{-1})) \\
& + q^2\,(\,36 - 16\,(\zeta + \zeta^{-1}) - 2\,(\zeta^2 + \zeta^{-2})) \\
& + q^3\,(\,-272 + 99\,(\zeta + \zeta^{-1}) + 36\,(\zeta^2 + \zeta^{-2}) + (\zeta^3 + \zeta^{-3})) \\
& + \ldots .
\end{aligned}$$

More coefficients can be found in Table A.3. These coefficients can be independently checked by comparing them to the first coefficient in the Fourier–Jacobi expansion of the Siegel modular cuspform of degree 2, weight 10, and level 1 (see, for example, [**6**, Maass form of weight 10]).

3. *Jacobi forms*

The basic reference for holomorphic Jacobi forms is the book of Eichler and Zagier [**3**], whereas for skew-holomorphic Jacobi forms we refer the reader to [**10**]. Denote by $J^-_{k,m}$ and $J^+_{k,m}$ the spaces of holomorphic and skew-holomorphic Jacobi forms of weight k and index m, respectively. Thus $J^\varepsilon_{k,m}$, for $\varepsilon = \pm 1$ and integers $k \geqslant 2$ and $m \geqslant 1$, is the space of smooth and periodic functions $\phi(\tau, z)$ with $\tau \in \mathbb{H}$, $z \in \mathbb{C}$, having a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{\Delta,r\in\mathbb{Z},\ \varepsilon\Delta\geqslant 0\\ \Delta\equiv r^2 \bmod 4m}} c_\phi(\Delta,r)e^{2\pi i(((r^2-\Delta)/4m)u+((r^2+|\Delta|)/4m)iv+rz)} \quad (\tau = u+iv), \tag{3.1}$$

where the coefficients $c_\phi(\Delta, r)$ depend on r only modulo $2m$ (see [**3**, Theorem 2.2]), and such that

$$\phi\left(\frac{-1}{\tau}, \frac{z}{\tau}\right)e^{-2\pi i m(z^2/\tau)} = \phi(\tau,z)\cdot\begin{cases}\tau^k & \text{if } \varepsilon=-1,\\ \overline{\tau}^{k-1}|\tau| & \text{if } \varepsilon=+1.\end{cases}$$

This transformation formula, used twice, implies that $c_\phi(\Delta, -r) = (-1)^{k-1}\,\varepsilon\, c_\phi(\Delta, r)$. Also note that there is a skew-linear involution $\jmath : J^\varepsilon_{k,m} \to J^\varepsilon_{k,m}$, given by $\phi(\tau, z) \mapsto \overline{\phi(-\overline{\tau}, -\overline{z})}$, which satisfies $c_{\jmath\phi}(\Delta, r) = \overline{c_\phi(\Delta, r)}$.

As mentioned above, the Fourier coefficients are indexed by pairs of integers $(\Delta, r \bmod 2m)$ with $r^2 \equiv \Delta \pmod{4m}$ and $\Delta \leqslant 0$ if ϕ is holomorphic and $\Delta \geqslant 0$ if ϕ is skew-holomorphic. We remark that, for holomorphic Jacobi forms like $\phi_{2,37}$ and $\phi_{10,1}$ in § 2, we have $(r^2 - \Delta)/4m = (r^2 + |\Delta|)/4m$ and so an alternative is to use Fourier coefficients $a_\phi(n, r) = c_\phi(r^2 - 4mn, r)$ indexed by pairs of integers n, r with $r^2 - 4mn \leqslant 0$ and obtain a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{n=0}^{\infty} q^n\left(\sum_{r^2\leqslant 4mn} a_\phi(n,r)\,\zeta^r\right) \quad (q = e^{2\pi i\tau},\ \zeta = e^{2\pi i z}),$$

as illustrated by those two examples. However, in what follows it will be more convenient to index the coefficients as in (3.1), which works for both holomorphic as well as skew-holomorphic Jacobi forms.

A form in $J^\varepsilon_{k,m}$ is cuspidal if $c(0, r) = 0$ for every r. For $k = 2$ and $\varepsilon = +1$ there are certain *trivial cuspforms*. These are those Jacobi cuspforms for which $c(\Delta, r)$ is non-zero at most if Δ is a perfect square (see the definition of T_{r_0} in [**11**, p. 514]). Their Fourier coefficients are trivial to compute.

We denote by $S^\varepsilon_{k,m}$ the subspace of $J^\varepsilon_{k,m}$ consisting of the cuspidal forms *which are orthogonal to the trivial cuspforms*. Let $S^\varepsilon_{2k-2}(m)$ denote the space of classical holomorphic modular cuspforms of weight $2k - 2$ for $\Gamma_0(m)$ whose L-functions have functional equation with sign ε. The following result was proved in [**15**, Theorem 5] when $\varepsilon = -1$. For the case $\varepsilon = 1$ it was announced in [**10**, Main Theorem]; its proof will be given in [**2**].

THEOREM 3.2. *Assume $k \geqslant 2$. For any fixed fundamental discriminant Δ_0 and any fixed integer r_0 such that $\Delta_0 \equiv r_0^2 \pmod{4m}$ and $\operatorname{sgn}\Delta_0 = \varepsilon$, there is a Hecke equivariant map*

$$\mathscr{S}_{\Delta_0,r_0} : S^\varepsilon_{k,m} \to S^\varepsilon_{2k-2}(m)$$

given by

$$\mathscr{S}_{\Delta_0,r_0}(\phi) = \sum_{n\geqslant 1}\left(\sum_{d\mid n}\left(\frac{\Delta_0}{d}\right)c_\phi\left(\frac{n^2}{d^2}\Delta_0, \frac{n}{d}r_0\right)\right)q^n \quad (q = e^{2\pi i\tau}).$$

Some linear combination of these maps is injective, and its image comprises all newforms in $S^\varepsilon_{2k-2}(m)$. Furthermore, $\mathscr{S}_{\Delta_0,r_0}$ sends newforms to newforms.

We remark that the sum of the images of the maps $\mathscr{S}_{\Delta_0,r_0}$ can be explicitly described. It consists of the *certain space* introduced in [15]. Moreover, the theorem remains valid also for Eisenstein series ϕ (with a suitable definition of the constant term of $\mathscr{S}_{\Delta_0,r_0}(\phi)$).

4. *Formulas*

In [12], a formula is given that takes as input a cuspidal modular symbol σ of weight 2, a fundamental discriminant Δ_0 and a square root r_0 of Δ_0 mod $4m$, and produces the (Δ, r)th coefficient of a Jacobi form of weight 2 and index m associated to σ for every discriminant Δ such that Δ and $\Delta\Delta_0$ are not squares. The formula is given as a sum of terms involving the so-called intersection numbers of two geodesics, one connecting the roots of indefinite quadratic forms of discriminant $\Delta\Delta_0$ and one induced by σ.

In this section we state and prove an extension of the formula in [12] to weight $k \geqslant 3$. We also make it more suitable for computation. We start by fixing some notation and defining the intersection number.

4.1. *A pairing for polynomials*

Given a non-negative integer w, let $\mathrm{GL}(2,\mathbb{C})$ act on the space $\mathbb{C}[X,Y]_w$ of homogeneous polynomials of degree w by

$$(A\cdot P)(X,Y) = P(A^{-1}\left(\begin{smallmatrix} X \\ Y\end{smallmatrix}\right)).$$

We let $P(\alpha) = P(\alpha, 1)$ for $\alpha \in \mathbb{C}$ and $P(\infty) = P(1,0)$. Note that $(A\cdot P)(A\alpha) = P(\alpha)$.

Given $P_1 = \sum_{l=0}^{w} a_l X^l Y^{w-l}$ and $P_2 = \sum_{l=0}^{w} b_l X^l Y^{w-l}$ polynomials in $\mathbb{C}[X,Y]_w$, let

$$[P_1 \mid P_2] = \sum_{l=0}^{w} (-1)^l \binom{w}{l}^{-1} a_l\, b_{w-l}.$$

PROPOSITION 4.1. *The bilinear pairing $[\cdot \mid \cdot]$ satisfies the following properties.*

(i) $[(xY - X)^w \mid P] = P(x)$ *for every* $P \in \mathbb{C}[X,Y]_w$ *and* $x \in \mathbb{C}$.

(ii) $[P_1 \mid P_2] = (-1)^w\, [P_2 \mid P_1]$.

(iii) $[A\cdot P_1 \mid A\cdot P_2] = \det(A)^{-w}\, [P_1 \mid P_2]$ *for every* $P_1, P_2 \in \mathbb{C}[X,Y]_w$ *and* $A \in \mathrm{GL}(2,\mathbb{C})$.

Proof. The first and second assertions are clear. To prove the third assertion, it suffices to consider matrices A of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a'\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right)$. The first two cases are easy, so let $A = \left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right)$, with $b \in \mathbb{C}$, and assume $P_1 = X^s Y^{w-s}$ and $P_2 = X^t Y^{w-t}$. Then

$$\begin{aligned}
[A\cdot P_1 \mid A\cdot P_2] &= [(X-bY)^s\, Y^{w-s} \mid (X-bY)^t\, Y^{w-t}] \\
&= \sum_{l=w-t}^{s} (-1)^l \binom{w}{l}^{-1} \binom{s}{l} (-b)^{s-l} \binom{t}{w-l} (-b)^{t-w+l} \\
&= (-1)^s \frac{s!\,t!}{w!} \frac{b^{s+t-w}}{(s+t-w)!} \sum_{l=w-t}^{s} (-1)^{t-w+l} \binom{s+t-w}{t-w+l} \\
&= (-1)^s \frac{s!\,t!}{w!} \frac{b^{s+t-w}}{(s+t-w)!} \sum_{i=0}^{s+t-w} (-1)^{i} \binom{s+t-w}{i} \\
&= \begin{cases} (-1)^s \binom{w}{s}^{-1} & \text{if } s+t-w = 0, \\ 0 & \text{otherwise} \end{cases} \\
&= [P_1 \mid P_2].
\end{aligned}$$

□

4.2. *Modular symbols and the intersection number*

Let $k \geqslant 2$ be an integer. Following [16], we denote by $\mathbb{M}_2$ the free abelian group generated by symbols $\{\alpha, \beta\}$ with $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$, modulo the relations

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0,$$

and modulo any torsion. We let $\mathbb{M}_{2k-2} = \mathbb{M}_2 \otimes \mathbb{Z}[X, Y]_{2k-4}$. Then $\mathbb{M}_{2k-2}$ is a $\mathrm{GL}(2, \mathbb{Z})$-module, via

$$A \cdot (\{\alpha, \beta\} \otimes P) = \{A\alpha, A\beta\} \otimes A \cdot P.$$

Let m be a non-negative integer. We let $\mathbb{M}_{2k-2}(m)$ denote the space of modular symbols of weight $2k-2$ and level m, that is, the quotient of $\Gamma_0(m)$-coinvariants of $\mathbb{M}_{2k-2}$. It has finite rank. Furthermore, $\mathbb{M}_{2k-2}(m)$ comes equipped with the action of Hecke operators. We denote the subspace of *new modular symbols* by $\mathbb{M}^{\mathrm{new}}_{2k-2}(m)$.

Given $P \in \mathbb{Z}[X, Y]_{2k-4}$ and $A \in \mathrm{GL}(2, \mathbb{Z})$, let $[P, A]$ denote the *Manin symbol*

$$[P, A] = A \cdot (\{0, \infty\} \otimes P) \in \mathbb{M}_{2k-2}(m).$$

By [16, Proposition 8.3] these symbols span $\mathbb{M}_{2k-2}(m)$. This implies that every modular cuspform $f \in S_{2k-2}(m)$ induces a Hecke equivariant map

$$\begin{aligned} I_f : \mathbb{M}_{2k-2}(m) &\longrightarrow \mathbb{C} \\ [P, A] &\mapsto \int_0^{i\infty} (f|_{2k-2}[A])(t)\, P(t, 1)\, dt. \end{aligned}$$

Let $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]$ denote the free abelian group generated by symbols (α) with $\alpha \in \mathbb{P}^1(\mathbb{Q})$, and let $\mathbb{B}_{2k-2} = \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \otimes \mathbb{Z}[X, Y]_{2k-4}$. Then $\mathbb{B}_{2k-2}$ is a $\mathrm{GL}(2, \mathbb{Z})$-module via

$$A \cdot ((\alpha) \otimes P) = (A\alpha) \otimes A \cdot P.$$

We define $\mathbb{B}_{2k-2}(m)$ to be the quotient of $\Gamma_0(m)$-coinvariants of $\mathbb{B}_{2k-2}$. We have a map of $\mathrm{GL}(2, \mathbb{Z})$-modules $\partial : \mathbb{M}_{2k-2}(m) \to \mathbb{B}_{2k-2}(m)$ induced by

$$\{\alpha, \beta\} \otimes P \mapsto ((\beta) - (\alpha)) \otimes P.$$

We let $\mathbb{S}_{2k-2}(m) = \ker \partial$, and we let $\mathbb{S}^{\mathrm{new}}_{2k-2}(m) = \mathbb{S}_{2k-2}(m) \cap \mathbb{M}^{\mathrm{new}}_{2k-2}(m)$.

For $\varepsilon = \pm 1$ we denote by $\mathbb{M}^{\varepsilon}_{2k-2}(m)$ the subspace where $g = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts as multiplication by $(-1)^{k-1}\varepsilon$. Given $\sigma \in \mathbb{M}_{2k-2}(m)$, we denote

$$\sigma^{\varepsilon} = \sigma + (-1)^{k-1}\, \varepsilon\, (g \cdot \sigma) \in \mathbb{M}^{\varepsilon}_{2k-2}(m).$$

Finally, we define $\mathbb{S}^{\varepsilon}_{2k-2}(m) = \mathbb{S}_{2k-2}(m) \cap \mathbb{M}^{\varepsilon}_{2k-2}(m)$.

Let $Q \in \mathbb{Z}[X, Y]_2$ be a binary quadratic form with integral coefficients. We define an *intersection number* map $C_Q : \mathbb{M}_{2k-2} \to \mathbb{Q}$ by

$$C_Q \cdot \{\alpha, \beta\} \otimes P = \tfrac{1}{2}(\operatorname{sgn} Q(\alpha) - \operatorname{sgn} Q(\beta))\, [P \mid Q^{k-2}]. \tag{4.2}$$

By Proposition 4.1, we have that $C_{A \cdot Q} \cdot (A \cdot \sigma) = C_Q \cdot \sigma$ for every $A \in \mathrm{GL}(2, \mathbb{Z})$ and every $\sigma \in \mathbb{M}_{2k-2}$.

REMARK 4.3. Assume that Q has positive discriminant, and write $Q = (uX + vY)(wX + tY)$ with $u, v, w, t \in \mathbb{R}$. We associate to Q the *Heegner cycle*

$$C_Q = \operatorname{sgn}(ut - vw)\{-v/u, -t/w\} \otimes Q^{k-2}.$$

Then $C_Q \cdot \sigma$ can be interpreted as an intersection number on $\mathbb{C}[\mathfrak{L}] \otimes \mathbb{Z}[X,Y]_{2k-4}$, where $\mathfrak{L}$ is the set of oriented hyperbolic lines in the Poincaré upper half plane, and where a symbol $\{\alpha,\beta\}$ is identified with the hyperbolic line with α as 'starting point' and β as 'end point'. Furthermore, when $k=2$ the intersection number $C_Q \cdot C_R$ agrees with the one introduced in [12].

4.3. *Formulas for Fourier coefficients of Jacobi forms*

For the rest of this section we assume that (Δ_0, r_0) is a fixed *m-admissible pair*, that is, Δ_0 is a fundamental discriminant and r_0 is an integer such that $\Delta_0 \equiv r_0^2 \pmod{4m}$. Furthermore, we assume that $\mathrm{sgn}(\Delta_0) = \varepsilon$.

The key idea of [11] and [9] for obtaining explicit formulas for Jacobi forms in terms of modular symbols is to consider the Hecke equivariant map

$$\Sigma_{\Delta_0,r_0} : S^{\varepsilon}_{k,m} \xrightarrow{\mathscr{S}_{\Delta_0,r_0}} S^{\varepsilon}_{2k-2}(m) \longrightarrow \mathrm{Hom}(\mathbb{M}_{2k-2}(m)\otimes\mathbb{C},\mathbb{C})$$
$$f \mapsto I_f$$

and to dualize it. Identifying $S_{k,m}$ with its dual space via the map $\phi \mapsto \langle \cdot, \jmath\phi\rangle$, we get a Hecke equivariant map $\Sigma^*_{\Delta_0,r_0} : \mathbb{M}_{2k-2}(m) \otimes \mathbb{C} \to S^{\varepsilon}_{k,m}$ that satisfies

$$\langle \phi, \jmath\Sigma^*_{\Delta_0,r_0}(\sigma)\rangle = \Sigma_{\Delta_0,r_0}(\phi)(\sigma^{\varepsilon}) \tag{4.4}$$

for every $\sigma \in \mathbb{M}_{2k-2}(m)$ and every $\phi \in S^{\varepsilon}_{k,m}$. By Theorem 3.2, since the map which associates its periods to a modular form is injective, some linear combination of the maps $\Sigma^*_{\Delta_0,r_0}$ is surjective. Furthermore, every newform in $S^{\varepsilon}_{k,m}$ can be obtained from some $\sigma \in \mathbb{S}^{\varepsilon,\mathrm{new}}_{2k-2}$.

From here on assume that $k \geqslant 3$, and let

$$b_{k,m} = \frac{2}{\sqrt{\varepsilon}}\left(\frac{2\varepsilon}{mi}\right)^{k-2},$$

where $i = \sqrt{-1}$.

We denote $\mathscr{Q}_m = \{[ma,b,c] : a,b,c \in \mathbb{Z}\}$, where $[a,b,c]$ represents the binary quadratic form $aX^2+bXY+cY^2 \in \mathbb{Z}[X,Y]_2$. Denote by $\chi_{m,\Delta_0} : \mathscr{Q}_m \to \{0,1,-1\}$ the genus character introduced in [4, Proposition 1]. Given integers Δ, r, we let

$$\mathscr{Q}_m(\Delta,r) = \{[ma,b,c] \in \mathscr{Q}_m : b^2 - 4mac = \Delta, b \equiv r \bmod 2m\}.$$

Note that if $\Delta \neq \square$ and $[ma,b,c] \in \mathscr{Q}_m(\Delta,r)$ then $ac \neq 0$.

With this notation in mind, given $A \in \mathrm{SL}(2,\mathbb{Z})$, we let $\mathscr{L}^A_{\Delta_0,r_0} : \mathbb{H}\times\mathbb{C} \to \mathbb{C}[X,Y]_{2k-4}$ be the kernel map defined in [9]. For fixed $x \in \mathbb{C}$ the function $\mathscr{L}^A_{\Delta_0,r_0}(\cdot)(x)$ belongs to $S^{\varepsilon}_{k,m}$, and its (Δ,r)th Fourier coefficient is given by $b_{k,m}\,\mathscr{C}^A_{\Delta_0,r_0}(\Delta,r)(x)$, where $\mathscr{C}^A_{\Delta_0,r_0}(\Delta,r) \in \mathbb{R}[X,Y]_{2k-4}$ is given by

$$\begin{aligned}\mathscr{C}^A_{\Delta_0,r_0}(\Delta,r)(x) = &\sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0,rr_0)\\ A^{-1}\cdot Q=[a,b,c],\, ac<0}} \chi_{m,\Delta_0}(Q)\,\mathrm{sgn}(a)\,(A^{-1}\cdot Q)(x)^{k-2}\\ &+ \sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0,rr_0)\\ A^{-1}\cdot Q=[0,b,c],\, 0\leqslant c<N}} F_Q(x) - \sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0,rr_0)\\ A^{-1}\cdot Q=[a,b,0],\, 0\leqslant a<N}} G_Q(x)\\ &+ \mathscr{Z}^A_{\Delta_0,r_0}(\Delta,r)(x).\end{aligned} \tag{4.5}$$

Here N is a certain non-negative integer; $F_Q(X)$, $G_Q(X)$ are certain polynomials over $\mathbb{Q}$ of degree at most $k-1$ (they have explicit descriptions which we do not need). Furthermore, $\mathscr{Z}^A_{\Delta_0,r_0}(\Delta,r)(x)$ is a correction term which we describe below.

The following result is proved in [9, Proposition 4].

PROPOSITION 4.6. *The map $\mathscr{L}^A_{\Delta_0,r_0}$ is a kernel map, in the sense that it satisfies*

$$\langle \phi, \mathscr{L}^A_{\Delta_0,r_0}(\cdot)(\overline{x})\rangle = \Sigma_{\Delta_0,r_0}(\phi)([P_x, A]^\varepsilon) \tag{4.7}$$

for every $\phi \in S^\varepsilon_{k,m}$ and $x \in \mathbb{C}$, where $P_x = (xY - X)^{2k-4} \in \mathbb{C}[X, Y]_{2k-4}$.

REMARK 4.8. In the statement of [**9**, Proposition 4] there is a tiny mistake: in the left-hand side of (4.7) the kernel map $\mathscr{L}^A_{\Delta_0,r_0}$ appears evaluated in $-\overline{x}$, but it should be $\overline{x}$.

The following lemma relates $\Sigma^*_{\Delta_0,r_0}$ to the kernel map introduced above.

LEMMA 4.9. *For every $P \in \mathbb{Z}[X, Y]_{2k-4}$ and $A \in \mathrm{SL}(2, \mathbb{Z})$, we have that*

$$\Sigma^*_{\Delta_0,r_0}([P, A])(\tau, z) = \overline{b_{k,m}}/_{b_{k,m}}\, [P \mid \mathscr{L}^A_{\Delta_0,r_0}(\tau, z)]. \tag{4.10}$$

Proof. By (4.4) and (4.7) we have

$$\langle \phi, \mathscr{L}^A_{\Delta_0,r_0}(\cdot)(\overline{x})\rangle = \Sigma_{\Delta_0,r_0}(\phi)([P_x, A]^\varepsilon) = \langle \phi, \jmath\Sigma^*_{\Delta_0,r_0}([P_x, A])\rangle$$

for every $\phi \in S^\varepsilon_{k,m}$. Hence, $\Sigma^*_{\Delta_0,r_0}([P_x, A]) = \jmath(\mathscr{L}^A_{\Delta_0,r_0}(\cdot)(\overline{x}))$. By the definition given in [**9**, Proposition 4], there exist $\phi_l \in S^\varepsilon_{k,m}$ with real Fourier coefficients such that $\mathscr{L}^A_{\Delta_0,r_0}(\cdot)(x) = b_{k,m}\sum_l \phi_l\, x^l$ for every $x \in \mathbb{C}$. This implies that

$$\jmath(\mathscr{L}^A_{\Delta_0,r_0}(\cdot)(\overline{x})) = \overline{b_{k,m}}/_{b_{k,m}}\, \mathscr{L}^A_{\Delta_0,r_0}(\cdot)(x) \tag{4.11}$$

for every $x \in \mathbb{C}$. Since $P(x) = [P_x \mid P]$ for any polynomial $P \in \mathbb{C}[X, Y]_{2k-4}$, we conclude that

$$\Sigma^*_{\Delta_0,r_0}([P_x, A])(\tau, z) = \overline{b_{k,m}}/b_{k,m}\, \mathscr{L}^A_{\Delta_0,r_0}(\tau, z)(x) = \overline{b_{k,m}}/b_{k,m}\, [P_x \mid \mathscr{L}^A_{\Delta_0,r_0}(\tau, z)].$$

Now (4.10) follows by linearity, since the polynomials P_x generate $\mathbb{C}[X, Y]_{2k-4}$. □

For a pair of integers Δ, r, we let $\xi_{\Delta,r} : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{R}$ be the map given by

$$\xi_{\Delta,r}(\alpha) = \gamma(\zeta_{m,\Delta\Delta_0,rr_0,\alpha,\Delta_0}(k-1) + (-1)^{k-1}\,\varepsilon\,\zeta_{m,\Delta\Delta_0,rr_0,-\alpha,\Delta_0}(k-1)),$$

where $\zeta_{m,\Delta,r,\alpha,\Delta_0}(s)$ denotes the Dirichlet series defined in [**9**, p. 67], and where $\gamma \in \mathbb{R}$ is as in [**9**, Proposition 4] (with k replaced by $k-1$). By properties of these Dirichlet series, the function $\xi_{\Delta,r}$ satisfies $\xi_{\Delta,r}(A\alpha) = \xi_{\Delta,r}(\alpha)$ for every $A \in \Gamma_0(m)$.

For $A \in \mathrm{SL}(2, \mathbb{Z})$, the correction term $\mathscr{Z}^A_{\Delta_0,r_0}(\Delta, r) \in \mathbb{R}[X, Y]$ appearing in (4.5) is given by

$$\mathscr{Z}^A_{\Delta_0,r_0}(\Delta, r) = \xi_{\Delta,r}(A0)\, X^{2k-4} - \xi_{\Delta,r}(A\infty)\, Y^{2k-4}.$$

LEMMA 4.12. *For each Δ, r there is a map $\Xi_{\Delta,r} : \mathbb{B}_{2k-2}(m) \to \mathbb{R}$ satisfying*

$$\Xi_{\Delta,r}(\partial[P, A]) = [P \mid \mathscr{Z}^A_{\Delta_0,r_0}(\Delta, r)]$$

for every $P \in \mathbb{Z}[X, Y]_{2k-4}$ and $A \in \mathrm{GL}(2, \mathbb{Z})$.

Proof. The correspondence $(\alpha) \otimes P \mapsto -\xi_{\Delta,r}(\alpha)\, P(\alpha)$ induces a map $\Xi_{\Delta,r} : \mathbb{B}_{2k-2}(m) \to \mathbb{R}$, since $\xi_{\Delta,r}(A\alpha) = \xi_{\Delta,r}(\alpha)$ for every $A \in \Gamma_0(m)$.

Given $P \in \mathbb{Z}[X, Y]_{2k-4}$ and $A \in \mathrm{GL}(2, \mathbb{Z})$, we have

$$\begin{aligned}\Xi_{\Delta,r}(\partial[P, A]) &= \Xi_{\Delta,r}(((A\infty) - (A0)) \otimes A \cdot P) = \xi_{\Delta,r}(A0)\, P(0) - \xi_{\Delta,r}(A\infty)\, P(\infty)\\ &= \xi_{\Delta,r}(A0)\, [P \mid X^{2k-4}] - \xi_{\Delta,r}(A\infty)\, [P \mid Y^{2k-4}]\\ &= [P \mid \mathscr{Z}^A_{\Delta_0,r_0}(\Delta, r)],\end{aligned}$$

where we used that $[P \mid X^{2k-4}] = P(0)$ and $[P \mid Y^{2k-4}] = P(\infty)$. □

 https://doi.org/10.1112/S1461157016000346

PROPOSITION 4.13. *Let $P \in \mathbb{Z}[X,Y]_{2k-4}$ and let $A \in \mathrm{SL}(2,\mathbb{Z})$. Let $\phi \in S^{\varepsilon}_{k,m}$ be given by $\phi = 1/\overline{b_{k,m}}\,\Sigma^{*}_{\Delta_0,r_0}([P,A])$. Then for every Δ such that $\Delta\Delta_0 \neq \square$, we have*

$$c_\phi(\Delta,r) = \sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0,rr_0)\\ A^{-1}\cdot Q=[a,b,c],\,ac<0}} \chi_{m,\Delta_0}(Q)\,\mathrm{sgn}(a)\,[A\cdot P \mid Q^{k-2}] + \Xi_{\Delta,r}(\partial[P,A]). \tag{4.14}$$

Proof. Since $\Delta\Delta_0$ is not a square, the second and third summands of $\mathscr{C}^{A}_{\Delta_0,r_0}(\Delta,r)(x)$ in (4.5) are empty. Hence using Lemmas 4.9 and 4.12, and (4.5), we get that

$$\begin{aligned} c_\phi(\Delta,r) &= [P \mid \mathscr{C}^{A}_{\Delta_0,r_0}(\Delta,r)] \\ &= \sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0,rr_0)\\ A^{-1}\cdot Q=[a,b,c],\,ac<0}} \chi_{m,\Delta_0}(Q)\,\mathrm{sgn}(a)[P \mid (A^{-1}\cdot Q)^{k-2}] + \Xi_{\Delta,r}(\partial[P,A]). \end{aligned}$$

Since by Proposition 4.1 we have that $[P \mid (A^{-1}\cdot Q)^{k-2}] = [A\cdot P \mid Q^{k-2}]$, this completes the proof. $\square$

With these preliminary results and notation in hand, we now prove the main formula, which extends [**12**, Theorem 3] to weights $k \geqslant 3$.

THEOREM 4.15. *Given $\sigma \in \mathbb{M}_{2k-2}(m)$, let $\phi = -1/\overline{b_{k,m}}\,\Sigma^{*}_{\Delta_0,r_0}(\sigma)$. Then for every Δ such that $\Delta\Delta_0 \neq \square$, we have that*

$$c_\phi(\Delta,r) = \sum_{Q\in\mathscr{Q}_m(\Delta\Delta_0,rr_0)} \chi_{m,\Delta_0}(Q)\,C_Q\cdot\sigma + \Xi_{\Delta,r}(\partial\sigma). \tag{4.16}$$

REMARK 4.17. When σ is cuspidal we have that $\Xi_{\Delta,r}(\partial\sigma) = \Xi_{\Delta,r}(0) = 0$, and hence the right-hand side of (4.16) becomes simpler. In particular, we do not need to compute the Dirichlet series appearing in the definition of $\xi_{\Delta,r}(\alpha)$. For completeness, though, we observe that, for level 1, these Dirichlet series are partial zeta functions of quadratic number fields, and for higher levels, when certain congruences in the summation have to be observed, they become partial ray class zeta functions. And so, in principle, the required special values could be computed.

REMARK 4.18. Each summand in the right-hand side of (4.16) is well defined for $\sigma \in \mathbb{M}_{2k-2}$, but not for $\sigma \in \mathbb{M}_{2k-2}(m)$. However, since $\Gamma_0(m)$ acts on $\mathscr{Q}_m(\Delta\Delta_0, rr_0)$, by the $\Gamma_0(m)$-invariance of both the intersection number and the genus character, the right-hand side of (4.16) is well defined for $\sigma \in \mathbb{M}_{2k-2}(m)$.

REMARK 4.19. The proof given is based on results that require the hypothesis $k \geqslant 3$ (for example, [**9**, Proposition 4]). When $k = 2$, Theorem 4.15 is valid if we assume furthermore that $\sigma \in \mathbb{S}_{2k-2}(m)$ and that $\Delta \neq \square$; this is proved in [**12**, Theorem 3].

Proof. We assume without loss of generality that $\sigma = [P,A]$. Let $Q \in \mathscr{Q}_m(\Delta\Delta_0, rr_0)$, and denote $A^{-1}\cdot Q$ by $[a,b,c]$. Note that $ac \neq 0$, since $\mathrm{disc}(A\cdot Q) = \mathrm{disc}\,Q \neq \square$. Since $Q(A0) = c$ and $Q(A\infty) = a$, we have

$$C_Q\cdot[P,A] = C_Q\cdot(\{A0, A\infty\}\otimes A\cdot P) = \frac{\mathrm{sgn}(c)-\mathrm{sgn}(a)}{2}\,[A\cdot P \mid Q^{k-2}].$$

We have that $\operatorname{sgn}(c) - \operatorname{sgn}(a)$ is non-zero if and only if $ac < 0$, and in that case it equals $-2\operatorname{sgn}(a)$. Hence

$$C_Q \cdot [P, A] = \begin{cases} -\operatorname{sgn}(a)\,[A \cdot P \mid Q^{k-2}], & ac < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Summing over all $Q \in \mathscr{Q}_m(\Delta\Delta_0, rr_0)$, we conclude that

$$\sum_{Q \in \mathscr{Q}_m(\Delta\Delta_0, rr_0)} \chi_{m,\Delta_0}(Q)\, C_Q \cdot [P, A] = -\sum_{\substack{Q \in \mathscr{Q}_m(\Delta\Delta_0, rr_0) \\ A^{-1}\cdot Q = [a,b,c],\, ac<0}} \chi_{m,\Delta_0}(Q) \operatorname{sgn}(a)\,[A \cdot P \mid Q^{k-2}].$$

The result now follows from Proposition 4.13. □

5. *Details on the implementation*

In Theorem 4.15 we stated a formula for computing Fourier coefficients of Jacobi forms. In this section we describe the practical issues related to carrying out computations using that formula. In particular, we rewrite this formula without making explicit mention of intersection numbers, we describe why the support for the infinite sums in the formula is finite and we identify which quadratic forms we need to consider when computing with the formula. We also describe a few auxiliary things we implemented. The code is available at [**8**].

Again for this section, we assume that (Δ_0, r_0) is a fixed *m-admissible pair* and that $\operatorname{sgn}(\Delta_0) = \varepsilon$.

5.1. *Ready-to-compute formulas*

The formulas above are appealing but not so useful for computation. In the following lemma we give the formula that we implement for computing the intersection numbers appearing in (4.16).

Given $\sigma \in \mathbb{M}_{2k-2}$, using that $\{\alpha, \beta\} = \{\infty, \beta\} - \{\infty, \alpha\}$, we can write

$$\sigma = \sum_i n_i \{\infty, s_i\} \otimes P_i. \tag{5.1}$$

LEMMA 5.2. *Let $\sigma \in \mathbb{M}_{2k-2}$ be as in (5.1), and let Q be a binary quadratic form with integral coefficients such that* $\operatorname{disc} Q \neq \square$. *Then*

$$C_Q \cdot \sigma = \operatorname{sgn} Q(\infty) \sum_{Q(\infty)\, Q(s_i) < 0} n_i\,[P_i \mid Q^{k-2}]. \tag{5.3}$$

Proof. By definition of the intersection number map C_Q we have that

$$C_Q \cdot \sigma = \sum_i n_i\, C_Q \cdot (\{\infty, s_i\} \otimes P_i) = \sum_i n_i \frac{\operatorname{sgn} Q(\infty) - \operatorname{sgn} Q(s_i)}{2}\,[P_i \mid Q^{k-2}].$$

Since $\operatorname{disc} Q \neq \square$, we have $Q(\infty)\, Q(s_i) \neq 0$, hence $(\operatorname{sgn} Q(\infty) - \operatorname{sgn} Q(s_i))/2$ is non-zero if and only if $Q(\infty)\, Q(s_i) < 0$, and in that case it equals $\operatorname{sgn} Q(\infty)$. □

PROPOSITION 5.4. *Given $\sigma \in \mathbb{S}_{2k-2}(m)$ as in (5.1), let $\phi = -1/\overline{b_{k,m}}\, \Sigma^*_{\Delta_0,r_0}(\sigma)$. Then for every Δ such that $\Delta\Delta_0 \neq \square$ we have that $c_\phi(\Delta, r) = \widetilde{c}(\Delta, r) + (-1)^{k-1}\, \varepsilon\, \widetilde{c}(\Delta, -r)$, where*

$$\widetilde{c}(\Delta, r) = \sum_i n_i \sum_{\substack{Q \in \mathscr{Q}_m(\Delta\Delta_0, rr_0) \\ Q(\infty) > 0,\, Q(s_i) < 0}} \chi_{m,\Delta_0}(Q)\,[P_i \mid Q^{k-2}]. \tag{5.5}$$

Proof. Combining Lemma 5.2 and Theorem 4.15, we get that

$$c_\phi(\Delta, r) = \sum_i n_i \sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0, rr_0)\\ Q(\infty)\,Q(s_i)<0}} \operatorname{sgn} Q(\infty)\chi_{m,\Delta_0}(Q)\,[P_i \mid Q^{k-2}].$$

Splitting the inner sum above, it is enough to show that

$$\sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0, rr_0)\\ Q(\infty)<0,\, Q(s_i)>0}} -\chi_{m,\Delta_0}(Q)\,[P_i \mid Q^{k-2}] = (-1)^{k-1}\,\varepsilon \sum_{\substack{Q\in\mathscr{Q}_m(\Delta\Delta_0, -rr_0)\\ Q(\infty)>0,\, Q(s_i)<0}} \chi_{m,\Delta_0}(Q)\,[P_i \mid Q^{k-2}].$$

This follows by considering the bijection between the supports given by $Q \mapsto -Q$. □

5.2. *Quadratic forms in the support*

The sums in (5.5) are indexed by indefinite quadratic forms in $\mathscr{Q}_m(\Delta\Delta_0, rr_0)$ with $Q(\infty) > 0$ and $Q(s) < 0$. The following lemma shows that these sets are finite, and also gives explicit bounds for the coefficients of the quadratic forms that we need to consider.

LEMMA 5.6. *Let $D_{\max} > 0$. Let $Q = [a, b, c]$ be a quadratic form with $a > 0$. Assume that $0 < \operatorname{disc} Q \leqslant D_{\max}$. Further, let $s = p/q \in \mathbb{Q}$ and $Q(s) < 0$. Then*

$$a \leqslant \frac{D_{\max}\, q^2}{4},$$
$$\lfloor -2as - \sqrt{D_{\max}} \rfloor < b < \lceil -2as + \sqrt{D_{\max}} \rceil,$$
$$\left\lceil \frac{b^2 - D_{\max}}{4a} \right\rceil \leqslant c < \left\lceil \frac{b^2 - D_0}{4a} \right\rceil,$$

where $D_0 = (b + 2as)^2$.

Proof. By hypothesis, $q^2 Q(s)$ is a negative integer. Furthermore, $q^2Q(s) = a(p + bq/2a)^2 - Dq^2/4a \geqslant -Dq^2/4a$, where $D = \operatorname{disc} Q$. In particular, $-1 \geqslant -Dq^2/4a$, which proves the first inequality. The inequalities involving b follow from the fact that $(-b - \sqrt{D})/2a < s < (-b + \sqrt{D})/2a$. The lower bound on c follows easily from the bound on D. The upper bound on c is equivalent to the inequality $D_0 < D$. The latter follows from the fact that

$$D_0 = D + 4a(as^2 + bs + c) = D + 4aQ(s).$$ □

5.3. *Every coefficient can be computed*

As mentioned before, in order to compute the coefficient $c(\Delta, r)$, the formulas given by (5.5) require that $\Delta\Delta_0 \neq \square$. A reasonable question to ask, then, is whether or not it is possible to compute every coefficient of a given Jacobi form. The following lemma answers the question in the affirmative.

LEMMA 5.7. *Given an m-admissible pair (Δ, r) there exists an m-admissible pair (Δ_1, r_1) with Δ_1 a negative fundamental discriminant such that $\Delta\Delta_1 \neq \square$.*

Proof. Let (Δ_0, r_0) be any m-admissible pair with Δ_0 a negative, odd fundamental discriminant. If $\Delta\Delta_0 \neq \square$, we are done. Otherwise, let p be a prime such that $p \equiv 1 \mod \gcd(4m, \Delta_0, \Delta)$. Let $\Delta_1 = p\Delta_0$ and $r_1 = r_0$. Then Δ_1 is square-free and $\Delta_1 \equiv 1 \mod 4$, whence it is a negative fundamental discriminant. Furthermore, $\Delta_1 \equiv \Delta_0 \equiv r_1^2 \mod 4m$, and $\Delta_1\Delta = p\Delta_0\Delta \neq \square$. Hence (Δ_1, r_1) satisfies the required conditions. □

5.4. *Choice of m-admissible pair*

The starting point for the formula in Proposition 5.4 is a choice of m-admissible pair (Δ_0, r_0). We find Δ_0 among the negative fundamental discriminants that are squares modulo $4m$ if the form we are computing is holomorphic, and among the positive fundamental discriminants that are squares modulo $4m$ if the form is skew-holomorphic. Once a Δ_0 is chosen, we compute all the square roots of Δ_0 modulo $4m$ that are less than $2m$. In practice we choose the smallest Δ_0 we can in order to keep the support of the sums in (5.5) as small as possible.

5.5. *Genus character*

The last part of formula (5.5) we have not described is the genus character. We remark that the implementation of the formula for the genus character $\chi_{m,\Delta_0}(Q)$ as described in [**4**, Proposition 1] is straightforward.

5.6. *Effectiveness of the implementation*

In this section we make some brief comments about range of Jacobi forms we have computed using our implementation of the formulas proved above and an idea of the time involved to carry out some of the computations. Timings are summarized in Table 1. The computations are done using Sage 6.7 [**17**] and the Cython code posted at [**8**]. The code was run on an Intel 2.7 GHz processor running RHEL 7.0.0.

TABLE 1. *Timings for the computation of particular examples of Jacobi forms, not including the time to compute the modular symbol. We do not identify which particular form in each space we compute, but instead aim to illustrate how the timing depends on the weight and the index. We do point out that if a form in the Appendix is an element of a space in this table, the timing reported is for that form.*

Space	$\Delta_{\max}$	Sample time to compute $c(\Delta, r)$ for $\lvert\Delta\rvert < \Delta_{\max}$ for an element of the space (s)
$S^-_{2,37}$	10 000	38.5
$S^+_{2,11}$	10 000	32.7
$S^+_{2,15}$	1 000	9.51
$S^+_{2,389}$	10 000	37.2
$S^+_{2,5077}$	100	745
$S^-_{10,1}$	1 000	87.4
$S^-_{40,1}$	100	7.51
$S^-_{50,1}$	100	9.1
$S^-_{100,1}$	100	20.8

In order to illustrate the effectiveness of our method, we fixed the weight at 2 and computed expansions of forms in various indices. We also fixed the index at 1 and computed expansions of forms of various weights. As Table 1 shows, were able to compute fairly quickly in weight 2 and fairly high index and in index 1 and fairly high weight.

Appendix. *Tables of coefficients*

In this article we have given formulas for the Fourier expansion of Jacobi forms. Consider a Jacobi form ϕ of weight k and index m. Then ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{\Delta, r\in\mathbb{Z},\ \varepsilon\Delta\geqslant 0\\ \Delta\equiv r^2 \bmod 4m}} c_\phi(\Delta, r)\, e^{2\pi i(((r^2-\Delta)/4m)u+((r^2+|\Delta|)/4m)iv+rz)} \quad (\tau = u + iv),$$

where $\varepsilon = -1$ if ϕ is holomorphic and $\varepsilon = 1$ if ϕ is skew-holomorphic. The way our formula (5.5) works is that it produces the coefficients $c_\phi(\Delta, r)$. We point out that the coefficients $c_\phi(\Delta, r)$ depend only on Δ and r (mod $2m$); if k is even and $m = 1$ or prime, they only depend on Δ. Furthermore, they satisfy that $c_\phi(\Delta, -r) = (-1)^{k-1}\,\varepsilon\, c_\phi(\Delta, r)$.

TABLE A.1. *Coefficients $c(\Delta, r)$ of the holomorphic Jacobi cuspform in $S^-_{2,37}$ corresponding to the modular symbol $\{\infty, -1/23\} - \{\infty, -1/32\} + \{\infty, -1/34\} - \{\infty, 0\} \in \mathbb{S}^-_2(37)$. The data in the third column are the result of using (5.5) with $\Delta_0 = -4$ and $r = 12$. The data in the fourth column are the result of using (5.5) with $\Delta_0 = -3$ and $r = 21$. NA means that $\Delta\Delta_0 = \square$ and our formula does not apply. The values of the coefficients agree with those in [3].*

Δ	r	$c_{-4,12}(\Delta, r)$	$c_{-3,21}(\Delta, r)$
-3	21	1	NA
-4	12	NA	1
-7	17	-1	-1
-11	27	1	1
-12	32	-1	NA
-16	24	NA	-2
-27	11	-3	NA
-28	34	3	3
-36	36	NA	-2
-40	16	2	2
-44	20	-1	-1
-47	29	-1	-1
-48	10	0	NA

TABLE A.2. *Coefficients $c(\Delta, r)$ of the skew-holomorphic Jacobi cuspform in $S^+_{2,11}$ corresponding to the modular symbol $\{\infty, -1/9\} - 2\{\infty, -1/8\} + \{\infty, 0\} \in \mathbb{S}^+_2(11)$.*

Δ	r	$c(\Delta, r)$
1	1	1
4	2	-3
5	7	5
9	3	-2
12	10	5
16	4	4
20	8	5
25	5	0
33	11	0
36	6	6
37	9	5
44	0	0
45	1	0
48	2	10
49	7	-3

TABLE A.3. *Coefficients $c(\Delta, r)$ of the holomorphic Jacobi cuspform in $S^-_{10,1}$ corresponding to the modular symbol $\{\infty, 0\} \otimes X^{14}Y^2 \in \mathbb{S}^-_{18}(1)$. These coefficients agree with the first coefficient of the Fourier–Jacobi expansion of the Siegel modular cuspform of degree 2, weight 10, and level 1.*

Δ	r	$c(\Delta, r)$
−3	1	−1
−4	0	2
−7	1	16
−8	0	−36
−11	1	−99
−12	0	272
−15	1	240
−16	0	−1056
−19	1	253
−20	0	1800
−23	1	−2736
−24	0	1464
−27	1	4284
−28	0	−12 544
−31	1	6816
−32	0	19008
−35	1	−27 270
−36	0	4554
−39	1	6864
−40	0	−39 880
−43	1	66 013
−44	0	26 928
−47	1	−44 064
−48	0	−12 544

TABLE A.4. *Coefficients $c(\Delta, r)$ of the skew-holomorphic Jacobi cuspform in $S^+_{2,15}$ corresponding to the modular symbol $\{\infty, 1/5\} + \{\infty, -1/2\} - \{\infty, -2/5\} - \{\infty, 0\} \in \mathbb{S}^+_2(15)$. The values of the coefficients are consistent with those in [7].*

Δ	r	$c(\Delta, r)$
1	1	1
1	11	1
4	2	−2
4	8	2
9	3	−2
16	4	0
16	14	0
21	9	8
24	12	8
25	5	0
36	6	4
40	10	0
45	15	0
49	7	−1
49	13	1

TABLE A.5. *Coefficients $c(\Delta, r)$ of the skew-holomorphic Jacobi cuspform in $S^+_{2,389}$ corresponding to the unique rational newform in $S^+_2(389)$, which in turn corresponds to the elliptic curve E with Cremona label* `389a1`. *Note that for a fundamental discriminant Δ, $c(\Delta, r)$ vanishes if and only if the twist E_Δ has positive rank, in accordance with the Birch–Swinnerton-Dyer conjecture.*

Δ	r	$c(\Delta, r)$
1	1	0
4	2	0
5	303	1
9	3	0
13	205	-1
16	4	0
17	79	-1
20	172	1
24	56	1
25	5	0
28	240	-1
36	6	0
41	279	-1
44	40	-1
45	131	-1
49	7	0
52	368	-1
64	8	0
65	125	0
68	158	3
69	153	-1
73	97	-1
76	290	1
77	323	1
80	344	0
81	9	0
85	181	-1
93	69	0
96	112	-2
97	137	1

References

1. H. BOYLAN, *Jacobi forms, finite quadratic modules and Weil representations over number fields*, Lecture Notes in Mathematics 2130 (Springer, Cham, 2015), with a foreword by N.-P. Skoruppa.
2. H. BOYLAN, N.-P. SKORUPPA and H. ZHOU, 'The arithmetic theory of skew-holomorphic Jacobi forms', Preprint, 2016.
3. M. EICHLER and D. ZAGIER, *The theory of Jacobi forms*, Progress in Mathematics 5 (Birkhäuser, Boston, 1985).
4. B. GROSS, W. KOHNEN and D. ZAGIER, 'Heegner points and derivatives of L-series. II', *Math. Ann.* 278 (1987) no. 1–4, 497–562.
5. V. GRITSENKO, N. SKORUPPA and D. ZAGIER, Theta blocks, in preparation.
6. The LMFDB Collaboration, 'The L-functions and modular forms database', 2015, http://beta.lmfdb.org (accessed 20 December 2015).
7. A. PACETTI and G. TORNARÍA, 'Computing central values of twisted L-series: the case of composite levels', *Exp. Math.* 17 (2008) no. 4, 459–471.
8. N. RYAN, N. SIROLLI, N.-P. SKORUPPA and G. TORNARÍA, 'Jacobi modular forms data', 2016, http://www.unix.bucknell.edu/~ncr006/jmf-data/index.html (accessed 10 February 2016).
9. N.-P. SKORUPPA, 'Binary quadratic forms and the Fourier coefficients of elliptic and Jacobi modular forms', *J. reine angew. Math.* 411 (1990) 66–95.

10. N.-P. Skoruppa, 'Developments in the theory of Jacobi forms', *Automorphic functions and their applications (Khabarovsk, 1988)* (Institute for Applied Mathematics of the USSR Academy of Sciences, Khabarovsk, 1990) 167–185.

11. N.-P. Skoruppa, 'Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms', *Invent. Math.* 102 (1990) no. 3, 501–520.

12. N.-P. Skoruppa, 'Heegner cycles, modular forms and Jacobi forms', *Sém. Théor. Nombres Bordeaux* (2) 3 (1991) no. 1, 93–116.

13. N.-P. Skoruppa, 'Computations of Siegel modular forms of genus two', *Math. Comp.* 58 (1992) no. 197, 381–398.

14. N.-P. Skoruppa, 'A remark on Jacobi forms of lattice index', *Waseda Spring Number Theory Conference 2011* (2011).

15. N.-P. Skoruppa and D. Zagier, 'Jacobi forms and a certain space of modular forms', *Invent. Math.* 94 (1988) no. 1, 113–146.

16. W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics 79 (American Mathematical Society, Providence, RI, 2007), with an appendix by Paul E. Gunnells.

17. W. A. Stein *et al.*, Sage Mathematics Software (Version 6.7), The Sage Development Team, 2015, http://www.sagemath.org.

Nathan C. Ryan
Department of Mathematics
Bucknell University
Lewisburg
Pennsylvania 17387
USA

nathan.ryan@bucknell.edu

Nicolás Sirolli
Universidad de la República
Montevideo
Uruguay

nsirolli@dm.uba.ar

Current address:
Departamento de Matemática
Ciudad Universitaria, Pabellón I
(C1428EGA) Buenos Aires
Argentina

Nils-Peter Skoruppa
Universität Siegen
Department Mathematik
57068 Siegen
Germany

nils.skoruppa@uni-siegen.de

Gonzalo Tornaría
Centro de Matemática
Facultad de Ciencias
Iguá 4225
(11400) Montevideo
Uruguay

tornaria@cmat.edu.uy