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Atomic Decomposition and Boundedness of Operators on Weighted Hardy Spaces

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Abstract. In this article, we establish a new atomic decomposition for $f \in L^2_w \cap H^p_w$, where the decomposition converges in L^2_w-norm rather than in the distribution sense. As applications of this decomposition, assuming that T is a linear operator bounded on L^2_w and $0 < p \leq 1$, we obtain (i) if T is uniformly bounded in L^p_w-norm for all w-p-atoms, then T can be extended to be bounded from H^p_w to L^p_w; (ii) if T is uniformly bounded in H^p_w-norm for all w-p-atoms, then T can be extended to be bounded on H^p_w; (iii) if T is bounded on H^p_w, then T can be extended to be bounded from H^p_w to L^p_w.

1 Introduction

The study of H^p spaces has been going on for a long time. The classical H^p spaces on the unit circle or upper half-plane are defined by the aid of complex function theory. Stein and Weiss [13] extended the definitions of these spaces to higher dimensional cases by a system of conjugate harmonic functions. Fefferman and Stein [2] gave real characterizations of H^p spaces by several maximal functions, the Littlewood–Paley function, and the Lusin function. Coifman [1] and Latter [9] gave explicit representation theorems for elements in H^p, that is, atomic decomposition theorems. Using Muckenhoupt's weights w, Garcia-Cuerva [4] characterized weighted Hardy spaces H^p_w by several maximal functions; moreover, he used the auxiliary maximal function S^*_M to get the atomic decomposition of H^p_w. Gundy and Wheeden [7] gave a characterization of H^p_w in terms of the Lusin area integral. Recently Garcia-Cuerva and Martell [5] gave another equivalent expression of elements in H^p_w via a wavelet characterization. It is important to emphasize that to prove the boundedness of many classes of operators defined on H^p spaces, it suffices to verify the boundedness of operators acting on all atoms. The best known class of operators with this property is the class of Calderón–Zygmund operators. A complete argument for verifying Calderón–Zygmund operators bounded from H^p to L^p and bounded on H^p can be found in [6, Chapter III, §7] or [11, §7.3].

Garcia-Cuerva and Rubio de Francia [6, pp. 322–325] used smoothly truncated kernels to deal with the boundedness of convolution operators on $H^p(\mathbb{R}^n)$. Here we are trying to generalize their results, not only to more universal linear operators, but also to weighted cases.

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The main purpose of this article is to give a criterion of the boundedness of operators on H^p_w. We first establish a new atomic decomposition for $L^2_w(\mathbb{R}^n) \cap H^p_w(\mathbb{R}^n)$, where the decomposition converges in L^2_w-norm instead of in the distribution sense.

Theorem 1.1 *Let $0 < p \leq 1$ and $w \in A_2$. Set $N = [n(2/p - 1)]$ the integer part of $n(2/p-1)$. For $f \in L^2_w(\mathbb{R}^n) \cap H^p_w(\mathbb{R}^n)$, there exist a sequence $\{a_i\}$ of w-$(p,2,N)$-atoms and a sequence $\{\lambda_i\}$ of real numbers satisfying $\sum |\lambda_i|^p \leq C\|f\|^p_{[b]H^p_w}$ such that $f = \sum \lambda_i a_i$, where the series converges in $L^2_w(\mathbb{R}^n)$ and hence a subsequence converges almost everywhere.*

As a consequence of Theorem 1.1, we obtain the following.

Corollary 1.2 *Let $0 < p \leq 1$ and $w \in A_2$. For a linear operator T bounded on $L^2_w(\mathbb{R}^n)$, if $Tf \in H^p_w(\mathbb{R}^n)$ and $\|Tf\|_{H^p_w} \leq C\|f\|_{H^p_w}$ for $f \in L^2_w \cap H^p_w$, then T can be extended to a bounded operator from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$.*

Corollary 1.3 *Let $0 < p \leq 1$ and $w \in A_2$. For a linear operator T bounded on $L^2_w(\mathbb{R}^n)$, T can be extended to a bounded operator from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ if and only if there exists an absolute constant C such that $\|Ta\|_{L^p_w} \leq C$ for any w-$(p,2,N)$-atom a.*

Corollary 1.4 *Let $0 < p \leq 1$ and $w \in A_2$. For a linear operator T bounded on $L^2_w(\mathbb{R}^n)$, T can be extended to a bounded operator on $H^p_w(\mathbb{R}^n)$ if and only if there exists an absolute constant C such that $\|Ta\|_{H^p_w} \leq C$ for any w-$(p,2,N)$-atom a.*

Remark It follows from Corollary 2 that, for $0 < p \leq 1$ and $w \in A_2$, the identity operator on $H^p_w(\mathbb{R}^n)$ extends to a bounded operator from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$. One could be curious to know if such an extension concludes a fallacious result $H^p_w = L^p_w$. The answer is negative. We start with the identity operator $\mathbf{1}$ on $L^2_w(\mathbb{R}^n) \cap H^p_w(\mathbb{R}^n)$. By Corollary 1.2 it has an extension $\widetilde{\mathbf{1}}$; however, $\widetilde{\mathbf{1}}$ is different from $\mathbf{1}$ outside the $L^2_w(\mathbb{R}^n) \cap H^p_w(\mathbb{R}^n)$.

Throughout this paper the letter C will denote a positive constant that may vary from line to line but will remain independent of the main variables.

2 Preliminaries

By a weight we always mean the Muckenhoupt A_p weight. Let us recall the definition and properties of A_p weight. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\int_I w(x)\,dx\right)\left(\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq C|I|^p \quad \text{for every cube } I \subset \mathbb{R}^n,$$

where C is a positive constant independent of I. By the definition of A_2, we know $w \in A_2$ if and only if $w^{-1} \in A_2$. For $p = 1$, we say that $w \in A_1$ if

$$\frac{1}{|I|}\int_I w(x)\,dx \leq C \cdot \operatorname*{ess\,inf}_{x \in I} w(x) \quad \text{for every cube } I \subset \mathbb{R}^n.$$

A function w satisfies the condition A_∞ if $w \in A_p$ for some $p \geq 1$. It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$ and $w \in A_q$ for some $1 < q < p$. We thus use $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the *critical index* of w and set weighted measure $w(E) = \int_E w(x)\,dx$. For any cube I and $\lambda > 0$, we denote by λI the cube concentric with I whose each edge is λ times as long. It is known that for $w \in A_p$, $p \geq 1$, w satisfies the doubling condition.

Given a weight function w on $\mathbb{R}^n$, as usual we use $L_w^q(\mathbb{R}^n)$, $0 < q < \infty$, to express the space of all functions satisfying

$$\|f\|_{L_w^q}^q \equiv \int_{\mathbb{R}^n} |f(x)|^q w(x)\,dx < \infty,$$

when $q = \infty, L_w^\infty$ will be taken to mean L^∞ and $\|f\|_{L_w^\infty} = \|f\|_{L^\infty}$. Similarly to the classical Hardy spaces, the weighted Hardy space $H_w^p(\mathbb{R}^n)$, $0 < p \leq 1$ can be defined by the area function.

For $0 < p \leq 1$, let $\psi(x)$ be a radial Schwartz function supported on the unit ball and satisfying

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\},$$

$$\int_{\mathbb{R}^n} \psi(x) x^\alpha dx = 0 \quad \text{for given multi-index } \alpha \text{ with } |\alpha| \leq N.$$

Set $\psi_t(x) = t^{-n}\psi(x/t)$. For $f \in \mathscr{S}'(\mathbb{R}^n)$, the space of tempered distributions, the *Lusin area function* is defined by

$$S(f)(x) = \Big(\int_0^\infty \int_{|x-y|<t} |\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

and the *Littlewood–Paley g function* is defined by

$$g(f)(x) = \Big(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\Big)^{1/2}.$$

It follows from [12, p. 89] that $g(f)(x) \leq CS(f)(x)$, and it is well known that $\|S(f)\|_{L_w^2} \leq C\|f\|_{L_w^2}$ for $w \in A_2$. The *weighted Hardy space* $H_w^p(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ for which $S(f) \in L_w^p(\mathbb{R}^n)$ with quasi-norm $\|f\|_{H_w^p}^p = \|S(f)\|_{L_w^p}^p$. The space can also be defined in terms of non-tangential maximal function, radial maximal function, and wavelet characterization [4, 5, 7].

We can characterize the element in H_w^p in terms of atoms as well.

Definition On $\mathbb{R}^n$, let $0 < p \leq 1 \leq q \leq \infty$, $p < q$, and $w \in A_q$. For $s \in \mathbb{Z}$ satisfying $s \geq [n(q_w/p - 1)]$, a real-valued function $a \in L_w^q$ is called a w-(p, q, s)-*atom* if the following hold:

(i) a is supported on a cube I,
(ii) $\|a\|_{L_w^q} \leq w(I)^{1/q-1/p}$,

(iii) $\int_{\mathbb{R}}^{n} a(x)x^{\alpha}\, dx = 0$ for every multi-index α with $|\alpha| \le s$.

It is known that the atomic decomposition of H_w^p can be expressed as follows.

Theorem A ([4, 10]) *Let $0 < p \le 1 \le q \le \infty$, $p < q$, and $w \in A_q$. For each $f \in H_w^p(\mathbb{R}^n)$, there exist a sequence $\{a_i\}$ of w-(p,q,s)-atoms, $s \ge [n(q_w/p-1)]$, and a sequence $\{\lambda_i\}$ of real numbers with $\sum |\lambda_i|^p \le C\|f\|_{H_w^p}^p$ such that $f = \sum \lambda_i a_i$ both in the sense of distributions and in H_w^p norm. Moreover,*

$$\|f\|_{H_w^p} \approx \inf\Big\{ \Big(\sum_i |\lambda_i|^p\Big)^{1/p} : \sum_i \lambda_i a_i \text{ is a decomposition of } f \text{ into } w\text{-}(p,q,s)\text{-atoms}\Big\}.$$

3 Proofs of Main Results

Let ψ be the function given in Section 2 and

$$\mathscr{S}_\infty(\mathbb{R}^n) = \Big\{ f \in \mathscr{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)x^{\alpha}\, dx = 0 \text{ for any multi-index } \alpha \Big\}$$

with the same topology as $\mathscr{S}(\mathbb{R}^n)$. It is known that $\mathscr{S}_\infty(\mathbb{R}^n)$ is dense in L_w^2 (see [14, Chapter 7, Theorem 1]). To prove Theorem 1.1, we need the *Calderón reproducing formula* for weighted L^2.

Lemma 3.1 *Let $w \in A_2$. If $f \in L_w^2$. Then*

$$f(x) = \int_0^\infty \psi_t * \psi_t * f(x)\, \frac{dt}{t},$$

where the integral converges in L_w^2.

Proof First we would like to point out that the Fourier transform was the main tool to get the classical Calderón reproducing formula on L^2. Obviously, this method cannot be applied to get this lemma. One may imagine L_w^2 as a space of homogeneous type and hence, Lemma 3.1 would follow directly from the Calderón reproducing formula on spaces of homogeneous type as given in [8]. This, however, does not work because convolutions given in Lemma 3.1 are taken in the Lebesgue measure without weight w. The proof of Lemma 3.1 is based on the classical Calderón reproducing formula in which, for $f \in \mathscr{S}_\infty(\mathbb{R}^n)$, the integral converges in $\mathscr{S}(\mathbb{R}^n)$ (see [3, p. 122, Theorem 3]). That shows Lemma 3.1 for $f \in \mathscr{S}_\infty(\mathbb{R}^n)$ because the L_w^2 norm is dominated by a certain seminorm of $\mathscr{S}(\mathbb{R}^n)$.

For general $f \in L_w^2$ and given $\eta > 0$, since $\mathscr{S}_\infty(\mathbb{R}^n)$ is dense in L_w^2, there exists

$g \in \mathscr{S}_\infty(\mathbb{R}^n)$ such that $f = g + b$ with $\|b\|_{L_w^2} \le \eta$. Then

$$\Big\| f - \int_\varepsilon^K \psi_t * \psi_t * f(\,\cdot\,)\frac{dt}{t}\Big\|_{L_w^2} \le$$
$$\Big\| g - \int_\varepsilon^K \psi_t * \psi_t * g(\,\cdot\,)\frac{dt}{t}\Big\|_{L_w^2} + \|b\|_{L_w^2} + \Big\| \int_\varepsilon^K \psi_t * \psi_t * b(\,\cdot\,)\frac{dt}{t}\Big\|_{L_w^2}.$$

Since $w^{-1} \in A_2$, by a duality argument and the Littlewood–Paley theory on L_w^2, there exists a constant C independent of ε and K such that

$$\begin{aligned}
&\Big\| \int_\varepsilon^K \psi_t * \psi_t * b(\,\cdot\,)\frac{dt}{t}\Big\|_{L_w^2} \\
&\quad\le \sup_{\|h\|_{L^2_{w^{-1}}}\le 1} \Big(\int_{\mathbb{R}^n}\int_\varepsilon^K |\psi_t * b(y)|^2 \frac{dt}{t} w(y)\,dy\Big)^{1/2} \\
&\qquad \times \Big(\int_{\mathbb{R}^n}\int_\varepsilon^K |\psi_t * h(y)|^2 \frac{dt}{t} w^{-1}(y)\,dy\Big)^{1/2} \\
&\quad\le \sup_{\|h\|_{L^2_{w^{-1}}}\le 1} \Big(\int_{\mathbb{R}^n}\int_\varepsilon^K |\psi_t * b(y)|^2 \frac{dt}{t} w(y)\,dy\Big)^{1/2} \|g(h)\|_{L^2_{w^{-1}}} \\
&\quad\le C\Big(\int_{\mathbb{R}^n}\int_\varepsilon^K |\psi_t * b(y)|^2 \frac{dt}{t} w(y)\,dy\Big)^{1/2} \\
&\quad\le C\|g(b)\|_{L_w^2} \le C\|b\|_{L_w^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big\| f - \int_\varepsilon^K \psi_t * \psi_t * f(\,\cdot\,)\frac{dt}{t}\Big\|_{L_w^2} \\
&\qquad\le \Big\| g - \int_\varepsilon^K \psi_t * \psi_t * g(\,\cdot\,)\frac{dt}{t}\Big\|_{L_w^2} + (1+C)\eta \\
&\qquad\le C\eta \quad \text{as } \varepsilon \to 0 \text{ and } K \to \infty.
\end{aligned}$$

Since η is arbitrary, the proof of Lemma 3.1 is complete. ■

Proof of Theorem 1.1 For $k \in \mathbb{Z}$, let

$$\Omega_k = \{x \in \mathbb{R}^n : S(f)(x) > 2^k\},$$
$$B_k = \Big\{ \text{dyadic cube } Q : w(Q \cap \Omega_k) > \frac{1}{2}w(Q) \text{ and } w(Q \cap \Omega_{k+1}) \le \frac{1}{2}w(Q)\Big\}.$$

It is clear that if a cube $Q \in B_k$, then $Q \notin B_j$ for $j \neq k$. For each dyadic cube Q, we denote its tent by

$$\widehat{Q} = \{(x,t) : x \in Q \text{ and } \sqrt{n}|Q|^{1/n} < t \leq 2\sqrt{n}|Q|^{1/n}\}.$$

For $f \in L^2_w$, by Lemma 3.1 we claim

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in B_k} \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y)\, \frac{dydt}{t},$$

where $\widetilde{Q} \in B_k$ are maximal dyadic cubes in B_k and the series converges in L^2_w, and hence a subsequence converges almost every $x \in \mathbb{R}^n$.

Assume the claim for the moment. Let $a_{\widetilde{Q}}(x)$ and $\lambda_{\widetilde{Q}}$ be defined by

$$a_{\widetilde{Q}}(x) = C^{-1} w(5\sqrt{n}\widetilde{Q})^{(\frac{1}{2}-\frac{1}{p})} \Bigg\{ \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}} \Bigg\}^{-1/2}$$

$$\times \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y) \frac{dydt}{t}$$

$$\lambda_{\widetilde{Q}} = C w(5\sqrt{n}\widetilde{Q})^{(\frac{1}{p}-\frac{1}{2})} \Bigg\{ \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}} \Bigg\}^{1/2},$$

where the constant C is the same as the one in (3.1).

We first verify that $a_{\widetilde{Q}}(x)$ is a w-$(p,2,N)$-atom. It is easy to see that $a_{\widetilde{Q}}(x)$ is supported on $5\sqrt{n}\widetilde{Q}$ and the vanishing moment conditions follow from the assumption of ψ. To verify the size condition of atom, by the duality between L^2_w and $L^2_{w^{-1}}$,

$$\Bigg\| \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} \psi_t(\,\cdot\, - y)\psi_t * f(y) \frac{dydt}{t} \Bigg\|_{L^2_w}$$

$$= \sup_{\|h\|_{L^2_{w^{-1}}} \leq 1} \Bigg\langle \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} \psi_t(\,\cdot\, - y)\psi_t * f(y) \frac{dydt}{t}, h \Bigg\rangle$$

$$\leq C \sup_{\|h\|_{L^2_{w^{-1}}} \leq 1} \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} |Q||\psi_t * h(y)||\psi_t * f(y)| \frac{dydt}{t^{n+1}}.$$

The last inequality is due to the definition of $\widehat{Q}$ and hence, if $(y,t)\in\widehat{Q}$, $|Q|\approx t^n$. It is clear that

$$|Q| = \int_Q w(x)^{1/2} w(x)^{-1/2}\,dx \le w(Q)^{1/2}[w^{-1}(Q)]^{1/2},$$

so

$$\begin{aligned}
&\Big\| \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} \psi_t(\cdot - y)\psi_t * f(y)\frac{dydt}{t}\Big\|_{L^2_w} \\
&\qquad \le C \sup_{\|h\|_{L^2_{w^{-1}}}\le 1} \Big(\sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} \\
&\qquad\quad \times \Big(\sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} w^{-1}(Q)|\psi_t * h(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}.
\end{aligned}$$

For any $Q\in B_k$ and $(y,t)\in\widehat{Q}$, we have $Q\subset\{x\in\mathbb{R}^n : |x-y|<t\}$, and hence

$$\begin{aligned}
&\sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} w^{-1}(Q)|\psi_t * h(y)|^2 \frac{dydt}{t^{n+1}} \\
&\qquad \le \int_0^\infty \int_{\mathbb{R}^n} w^{-1}(\{x\in\mathbb{R}^n : |x-y|<t\})|\psi_t * h(y)|^2 \frac{dydt}{t^{n+1}} \\
&\qquad = \int_0^\infty \int_{\mathbb{R}^n} \int_{|x-y|<t} |\psi_t * h(y)|^2 w^{-1}(x)\,dx \frac{dydt}{t^{n+1}} \\
&\qquad = \int_{\mathbb{R}^n} S(h)^2(x) w^{-1}(x)\,dx \le C\|h\|^2_{L^2_{w^{-1}}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big\| \sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} \psi_t(\cdot - y)\psi_t * f(y)\frac{dydt}{t}\Big\|_{L^2_w} \\
&\qquad \le C\Big(\sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},
\end{aligned} \tag{3.1}$$

which proves the size condition.

To show $\{\lambda_{\widetilde{Q}}\} \in \ell^p$, doubling condition of w and Hölder's inequality yield

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in B_k} |\lambda_{\widetilde{Q}}|^p &\le C \sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in B_k} w(\widetilde{Q})^{(1-\frac{p}{2})} \\
&\quad \times \Big(\sum_{\substack{Q\subset\widetilde{Q}\\ Q\in B_k}} \int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}} \Big)^{p/2} \\
&\le C \sum_{k\in\mathbb{Z}} \Big(\sum_{\widetilde{Q}\in B_k} w(\widetilde{Q}) \Big)^{(1-\frac{p}{2})} \\
&\quad \times \Big(\sum_{Q\in B_k} \int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}} \Big)^{p/2}.
\end{aligned} \tag{3.2}$$

To estimate the last term in (3.2), we define the weighted Hardy–Littlewood maximal function by

$$M_w f(x) = \sup_{x\in Q} \frac{1}{w(Q)} \int_Q |f(x)|w(x)\,dx.$$

Let $\widetilde{\Omega}_k = \{x \in \mathbb{R}^n : M_w(\chi_{\Omega_k})(x) > \frac{1}{2}\}$. Then $\Omega_k \subset \widetilde{\Omega}_k$. Since M_w is of weak type $(1,1)$ with respect to $w(x)dx$, $w(\widetilde{\Omega}_k) \le Cw(\Omega_k)$ which yields

$$\begin{aligned}
C2^{2k} w(\Omega_k) \ge 2^{2k+2} w(\widetilde{\Omega}_k) &\ge \int_{\widetilde{\Omega}_k \setminus \Omega_{k+1}} [S(f)(x)]^2 w(x)dx \\
&= \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\psi_t * f(y)|^2 \chi_{\{x\in\widetilde{\Omega}_k\setminus\Omega_{k+1}:|x-y|<t\}} w(x)\,dx \frac{dydt}{t^{n+1}} \\
&\ge \sum_{Q\in B_k} \int_{\widehat{Q}} \int_{\mathbb{R}^n} |\psi_t * f(y)|^2 \chi_{\{x\in\widetilde{\Omega}_k\setminus\Omega_{k+1}:|x-y|<t\}} w(x)\,dx \frac{dydt}{t^{n+1}}.
\end{aligned}$$

For any $Q \in B_k$ and $(y,t) \in \widehat{Q}$, we have $Q \subset \widetilde{\Omega}_k$ and $Q \subset \{x \in \mathbb{R}^n : |x-y| < t\}$. That yields

$$\begin{aligned}
\int_{\mathbb{R}^n} \chi_{\{x\in\widetilde{\Omega}_k\setminus\Omega_{k+1}:|x-y|<t\}} w(x)\,dx &\ge w(Q \cap (\widetilde{\Omega}_k \setminus \Omega_{k+1})) \\
&= w(Q) - w(Q \cap \Omega_{k+1}) \\
&\ge w(Q)/2,
\end{aligned}$$

and hence

$$\sum_{Q\in B_k} \int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}} \le C2^{2k} w(\Omega_k). \tag{3.3}$$

Note that $\sum_{\widetilde{Q}\in B_k} w(\widetilde{Q}) \le w(\widetilde{\Omega}_k) \le Cw(\Omega_k)$, since $\widetilde{Q}$'s are disjoint and contained in $\widetilde{\Omega}_k$. Plugging (3.3) into (3.2), we get

$$\sum_{k\in\mathbb{Z}}\sum_{\widetilde{Q}\in B_k}|\lambda_{\widetilde{Q}}|^p \le C\sum_{k\in\mathbb{Z}} w(\Omega_k)^{(1-\frac{p}{2})}2^{kp}w(\Omega_k)^{\frac{p}{2}} \le C\|S(f)\|^p_{L^p_w} = C\|f\|^p_{H^p_w}.$$

We return to the proof of the claim. This is equivalent to showing

$$\Big\|\sum_{|k|>M}\sum_{Q\in B_k}\int_{\widehat{Q}}\psi_t(\cdot - y)\psi_t * f(y)\frac{dydt}{t}\Big\|_{L^2_w} \longrightarrow 0 \quad \text{as } M\to\infty.$$

By the same proof as in (3.1) and (3.3), we obtain

$$\begin{aligned}
&\Big\|\sum_{|k|>M}\sum_{Q\in B_k}\int_{\widehat{Q}}\psi_t(\cdot - y)\psi_t * f(y)\frac{dydt}{t}\Big\|_{L^2_w} \\
&\qquad \le C\Big(\sum_{|k|>M}\sum_{Q\in B_k}\int_{\widehat{Q}} w(Q)|\psi_t * f(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2} \\
&\qquad \le C\Big(\sum_{|k|>M}2^{2k}w(\Omega_k)\Big)^{1/2}.
\end{aligned}$$

The last term tends to zero as M goes to infinity because

$$\sum_{k\in\mathbb{Z}}2^{2k}w(\Omega_k) \le C\|f\|^2_{L^2_w} < \infty. \qquad \blacksquare$$

Proof of Corollary 1.2 For each w-(p,q,N)-atom a supported on I, by Hölder's inequality,

$$\|a\|^p_{L^p_w} \le \|a^p\|_{L^{q/p}_w}w(I)^{1-p/q} = \|a\|^p_{L^q_w}w(I)^{1-p/q} \le 1.$$

Applying Theorem 1.1, for $f\in L^2_w\cap H^p_w$ we have $f=\sum\lambda_i a_i$ almost everywhere, where the a_i's are w-$(p,2,N)$-atoms and $\sum|\lambda_i|^p \le C\|f\|^p_{H^p_w}$. Thus,

$$\|f\|^p_{L^p_w} \le \sum|\lambda_i|^p\|a\|^p_{L^p_w} \le \sum|\lambda_i|^p \le C\|f\|^p_{H^p_w}.$$

Given $f\in L^2_w\cap H^p_w$, the L^2_w boundedness and H^p_w boundedness of T give $Tf\in L^2_w\cap H^p_w$ and, by the above estimate, $\|Tf\|_{L^p_w} \le C\|Tf\|_{H^p_w} \le C\|f\|_{H^p_w}$. Since $L^2_w\cap H^p_w$ is dense in H^p_w, T can be extended to a bounded operator from H^p_w to L^p_w. $\blacksquare$

Proof of Corollary 1.3 Suppose that T is bounded from H^p_w to L^p_w. For a w-$(p,2,N)$-atom a, then $a\in H^p_w$. It follows from Theorem A that $\|Ta\|_{L^p_w} \le C\|a\|_{H^p_w} \le C$.

Conversely, Theorem 1.1 shows that for $f \in H^p_w \cap L^2_w$ we have $f = \sum_{i=1}^{\infty} \lambda_i a_i$ in L^2_w, where a_i's are w-$(p,2,N)$-atoms and $\sum |\lambda_i|^p \leq C\|f\|^p_{H^p_w}$. Since T is linear and bounded on L^2_w,

$$\Big\| Tf - \sum_{i=1}^{M} \lambda_i Ta_i \Big\|_{L^2_w} = \Big\| T\Big(f - \sum_{i=1}^{M} \lambda_i a_i \Big) \Big\|_{L^2_w}$$

$$\leq C \Big\| f - \sum_{i=1}^{M} \lambda_i a_i \Big\|_{L^2_w} \longrightarrow 0 \quad \text{as } M \longrightarrow \infty.$$

Hence, there exists a subsequence (we also write the same indices) such that $Tf = \sum_{i=1}^{\infty} \lambda_i Ta_i$ almost everywhere. Fatou's lemma yields

$$\int_{\mathbb{R}^n} |Tf|^p w(x)\, dx \leq \liminf_{M\to\infty} \int_{\mathbb{R}^n} \Big| \sum_{i=1}^{M} \lambda_i Ta_i \Big|^p w(x)\, dx$$

$$\leq \sum_{i=1}^{\infty} |\lambda_i|^p \int_{\mathbb{R}^n} |Ta_i|^p w(x)\, dx$$

$$\leq C\|f\|^p_{H^p_w}.$$

Since $H^p_w \cap L^2_w$ is dense in H^p_w, T can be extended to a bounded operator from H^p_w to L^p_w. ∎

Proof of Corollary 1.4 If T is bounded on H^p_w, then by Theorem A,

$$\|Ta\|_{H^p_w} \leq C\|a\|_{H^p_w} \leq C.$$

For $f \in H^p_w \cap L^2_w$, we have the atomic decomposition $f = \sum_{i=1}^{\infty} \lambda_i a_i$ in L^2_w. Let ψ be the function given in Section 2. Then

$$\psi_t * Tf = \sum_{i=1}^{\infty} \lambda_i \psi_t * Ta_i \quad \text{in } L^2_w.$$

Hence, there is a subsequence (we also write the same indices) such that

$$\psi_t * Tf = \sum_{i=1}^{\infty} \lambda_i \psi_t * Ta_i \quad \text{almost everywhere.}$$

Fatou's lemma and Minkowski's inequality imply that

$$\begin{aligned} S(Tf)(x) &= \Big(\int_0^\infty \int_{|x-y|<t} |\psi_t * Tf(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} \\ &= \Big(\int_0^\infty \int_{|x-y|<t} \liminf_{M\to\infty} \Big| \sum_{i=1}^{M} \lambda_i \psi_t * Ta_i(y)\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} \\ &\le \liminf_{M\to\infty} \Big(\int_0^\infty \int_{|x-y|<t} \Big| \sum_{i=1}^{M} \lambda_i \psi_t * Ta_i(y)\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} \\ &\le \sum_{i=1}^{\infty} |\lambda_i| \Big(\int_0^\infty \int_{|x-y|<t} |\psi_t * Ta_i(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} \\ &= \sum_{i=1}^{\infty} |\lambda_i| S(Ta_i)(x). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^n} [S(Tf)(x)]^p w(x)\, dx &= \int_{\mathbb{R}^n} \liminf_{M\to\infty} \Big(\sum_{i=1}^{M} |\lambda_i| S(Ta_i)(x)\Big)^p w(x) dx \\ &\le \liminf_{M\to\infty} \int_{\mathbb{R}^n} \Big(\sum_{i=1}^{M} |\lambda_i| S(Ta_i)(x)\Big)^p w(x)\, dx \\ &\le \sum_{i=1}^{\infty} |\lambda_i|^p \int_{\mathbb{R}^n} [S(Ta_i)(x)]^p w(x)\, dx \\ &= \sum_{i=1}^{\infty} |\lambda_i|^p \|Ta_i\|_{H_w^p}^p \le C\|f\|_{H_w^p}^p. \end{aligned}$$

Since $H_w^p \cap L_w^2$ is dense in H_w^p, T can be extended to a bounded operator on H_w^p. ■

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