AN APPROXIMATIVE PROPERTY OF SPACES OF CONTINUOUS FUNCTIONS

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1. Introduction. A Banach space X is said to have *property* (*PROXBID*) if the canonical image of X in its bidual X^{**} is proximal. In other words, if $J : X \to X^{**}$ is the canonical embedding, then it is required that every element of X^{**} have at least one best approximation (i.e., nearest point) from the closed subspace $J(X)$. We show below that, if X is the space of (real or complex) continuous functions on a compact set, or the space of (real or complex) continuous functions that vanish at infinity on a locally compact set, then X has property (PROXBID). At this point we should mention the existence of a variety of examples **[2, 8]** of Banach spaces which lack property (PROXBID).

Let us recall that the classical Gelfand–Naimark Theorem (e.g. **[10**, p. 4]) allows us to identify any commutative C^*-algebra A with either the space $C(\Lambda)$ (if A has an identity) or else the space $C_0(\Lambda)$ (if A has no identity). Here Λ is the pure state space of A. Thus we may conclude that every commutative C^*-algebra has property (PROXBID), and likewise, so does the (real) space of all self-adjoint elements of any such algebra. Let us further recall that, in **[8]**, Holmes and Kripke have shown (constructively) that the noncommutative C^*-algebra of all compact operators on a Hilbert space is proximinal in the algebra of all bounded operators, and consequently has property (PROXBID). More recently, and more generally, Fakhoury **[4]** has shown (non-constructively) that any " M-ideal " in a Banach space X is proximal in X, and, in fact, that a continuous choice of best approximation can be defined on X. (An extensive presentation of the theory of M-ideals has been given by Alfsen and Effros **[1]**; here it is pertinent only to note that every closed two-sided ideal in a C^*-algebra is an M-ideal.) In any event, we are led to raise the general question: Which C^*-algebras have property (PROXBID)? Making use of the universal representation of a given C^*-algebra A, we see that, in order to answer this question affirmatively for A, it would suffice to show that A is proximinal in the W^*-algebra generated by A. Indeed, this is in effect what was done in **[8]** and those authors' earlier paper **[6]** (for the special cases under consideration there).

The fact that the spaces $C_0(\Lambda, R)$ have property (PROXBID) has already been established by J. Blatter **[2]**, as a special case of his very extensive theory of best approximation from certain subspaces of $C_0(\Lambda, X)$, where X is a (real) Banach space whose dual is an abstract L-space. His proof requires in particular the use of theorems of Kakutani and Seever to identify the bidual $C_0(\Lambda, R)^{**}$ with another space $C(\Lambda_1, R)$ and $C_0(\Lambda, R)$ with a subalgebra of this latter space. By contrast, our proof deals directly with the elements of the bidual *qua* functions on the first dual, and is therefore relatively self-contained. Although we necessarily employ certain special properties of spaces of continuous functions, we feel that this kind of proof may be more useful as a model for proofs of property (PROXBID) in other spaces for which a Kakutani-type representation theorem is not available. We further remark that the present proof was originally developed for the case of real-valued functions, as another application of the " interposition " method of approximation as propounded in [6, 7].

2. The existence of best approximations. Let Λ be a compact Hausdorff space and let $C(\Lambda)$ (resp. $C(\Lambda, R)$) be the linear space of all continuous complex-valued (resp. real-valued) functions on Λ. When endowed with the usual supremum norm, these spaces are Banach spaces. A *state* on such a Banach space is a positive linear functional of norm one. If e is the identity in such a space (that is, the function everywhere equal to unity), then a state can also be described as a norm-one linear functional whose value at e is 1. Thus for the Banach spaces under discussion we can say that a state is simply a Borel probability measure on Λ.

THEOREM 1. *Let X be one of the spaces $C(\Lambda)$ or $C(\Lambda, R)$, and let Φ be an arbitrary element of the bidual X^{**}. Then the distance from Φ to the canonical image $J(X)$ is attained. Hence X has property* (PROXBID).

Proof. We shall deal only with the more general (and more difficult) case where $X = C(\Lambda)$. The alternative case is handled analogously, and certain simplifications are possible; these will be indicated at the end of the proof.

We introduce the following notation:

$\mathbb{C}$ = complex plane;
d_H = Hausdorff metric on the compact nonempty subsets of $\mathbb{C}$;
$\mathcal{N}(t)$ = the directed family of open t-neighbourhoods, for $t \in \Lambda$;
$\Omega(t, N)$ = the set of all states in $C(\Lambda)^*$ with support in N ($N \in \mathcal{N}(t)$);
$A_N(t)$ = closure of $\Phi(\Omega(t, N))$.

Now, for each $t \in \Lambda$, consider the net $N \mapsto A_N(t)$ defined on $\mathcal{N}(t)$. The range of this net lies in the compact metric space $M(\Phi; d_H)$ whose elements are the compact convex and nonempty subsets of the disk in $\mathbb{C}$ with centre 0 and radius $\|\Phi\|$. We put

$$A(t) = \lim_{N \in \mathcal{N}(t)} A_N(t); \tag{1}$$

this limit exists in $M(\Phi; d_H)$ by virtue of the compactness of this space and monotonicity of the net $N \mapsto A_N(t)$. It may be verified that

$$A(t) = \bigcap_{N \in \mathcal{N}(t)} A_N(t). \tag{2}$$

We show that the map $A: \Lambda \to 2^{\mathbb{C}}$ just defined is upper semicontinuous. This requires us to choose any nonempty open set $G \subset \mathbb{C}$ and then show that $\{t \in \Lambda : A(t) \subset G\}$ is open. Let t_0 belong to this set. Then, by (1), there is an $N \in \mathcal{N}(t_0)$ for which $A_N(t_0) \subset G$. Hence, if $t \in N$, we have by (2) that

$$A(t) \subset A_N(t) = A_N(t_0) \subset G,$$

and so A is upper semicontinuous.

Now let $r(t)$ be the Chebyshev radius [5, § 33] of $A(t)$; that is,

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$$r(t) = \inf_{\beta \in \mathbb{C}} \sup_{\alpha \in A(t)} |\alpha - \beta|, \tag{3}$$

and put

$$R(A) = \sup \{r(t): t \in \Lambda\}.$$

Clearly $R(A) \leqq \|\Phi\|$. Following Olech [9, p. 288] we introduce the map $B: \Lambda \to 2^{\mathbb{C}}$ defined by

$$B(t) = \{\beta \in \mathbb{C}: A(t) \subset D(\beta, R(A))\},$$

where $D(\beta, R)$ is the closed disk in $\mathbb{C}$ with centre β and radius R. Olech proved that the values $B(t)$ are compact, convex and nonempty subsets of $\mathbb{C}$ and that B is lower semicontinuous. Thus, by the Michael selection principle, there is an $f_0 \in C(\Lambda)$ for which $f_0(t) \in B(t)$, for all $t \in \Lambda$. We shall complete the proof by showing that any such f_0 is a best approximation to Φ from $J(C(\Lambda))$, and that $\text{dist}(\Phi, J(C(\Lambda))) = R(A)$.

First we establish that $R(A) \leqq \text{dist}(\Phi, J(C(\Lambda)))$, by showing that

$$R(A) \leqq \|\Phi - J(f)\| \quad \text{for all} \quad f \in C(\Lambda). \tag{4}$$

Given $\varepsilon > 0$, choose $t \in \Lambda$ so that $r(t) > r(A) - \varepsilon$. If $f \in C(\Lambda)$, we can choose $N \in \mathcal{N}(t)$ for which $\text{osc}(f; N) < \varepsilon$, and then

$$|\langle f, \mu \rangle - f(t)| = |\langle f - f(t)e, \mu \rangle| \leqq \|(f - f(t)e)|N\| < \varepsilon,$$

whenever $\mu \in \Omega(t, N)$. (Recall that e is the identity in $C(\Lambda)$.) Taking into account (1), (3), and the fact that $f(t) \in A(t)$, we may assume that N has also been chosen so small that there is an $\alpha \in A_N(t)$ for which $|\alpha - f(t)| > r(t) - \varepsilon$. By definition of $A_N(t)$, there exists a $\mu \in \Omega(t, N)$ for which $|\alpha - \Phi(\mu)| < \varepsilon$. Consequently

$$\|\Phi - J(f)\| \geqq |\Phi(\mu) - \langle f, \mu \rangle| \geqq |\alpha - f(t)| - |\alpha - \Phi(\mu)| - |f(t) - \langle f, \mu \rangle| \geqq r(t) - 3\varepsilon \geqq R(A) - 4\varepsilon,$$

and this establishes the inequality (4).

It remains to prove that

$$\|\Phi - J(f_0)\| \leqq R(A).$$

Let $\varepsilon > 0$ and $t \in \Lambda$. There is an $N_t \in \mathcal{N}(t)$ for which

$$\max \{\text{osc}(f_0; N_t), d_H(A_{N_t}(t), A(t))\} < \varepsilon.$$

Since f_0 is a selection of B, we know that $A(t) \subset D(f_0(t), R(A))$. Therefore, if $\mu \in \Omega(t, N_t)$,

$$|\Phi(\mu) - \langle f_0, \mu \rangle| \leqq |\Phi(\mu) - \alpha| + |\alpha - f_0(t)| + |f_0(t) - \langle f_0, \mu \rangle| \leqq R(A) + 2\varepsilon, \tag{5}$$

if $\alpha \in A(t)$ satisfies $|\alpha - \Phi(\mu)| < \varepsilon$. Next we cover Λ by a finite number of these neighbourhoods N_t, say by $N_1 \equiv N_{t_1}, \ldots, N_k \equiv N_{t_k}$, and then choose a partition of unity subordinate to this cover, say $p_1, \ldots, p_k$. Consider now any state μ in $C(\Lambda)^*$. The measures $\text{sgn}(p_i \mu)$ belong to $\Omega(t_i, N_i)$ and we have, by (5),

$$\begin{aligned}|\Phi(\mu)-\langle f_0,\mu\rangle| &= |\Phi(\Sigma p_i\mu)-\langle f_0,p_i\mu\rangle| \\ &= |\Sigma\|p_i\mu\|(\Phi(\operatorname{sgn}(p_i\mu))-\langle f_0,\operatorname{sgn}(p_i\mu)\rangle)| \\ &\leqq \Sigma\|p_i\mu\|(R(A)+2\varepsilon) \\ &= R(A)+2\varepsilon,\end{aligned}$$

since

$$\Sigma\|p_i\mu\| = \Sigma\langle e,p_i\mu\rangle = \Sigma\langle p_i,\mu\rangle = \langle e,\mu\rangle = 1.$$

In this way we obtain the inequality

$$|\Phi(\mu)-\langle f_0,\mu\rangle| \leqq R(A), \tag{6}$$

whenever μ is a state in $C(\Lambda)^*$.

Finally, if μ is any element of the unit sphere of $C(\Lambda)^*$, there are states μ_1 and μ_2, and complex numbers s and t with $|s|+|t| = 1$ such that $\mu = s\mu_1+t\mu_2$. It follows from (6) that

$$\begin{aligned}|\Phi(\mu)-\langle f_0,\mu\rangle| &\leqq |s||\Phi(\mu_1)-\langle f_0,\mu_1\rangle|+|t||\Phi(\mu_2)-\langle f_0,\mu_2\rangle| \\ &\leqq |s|R(A)+|t|R(A) \\ &= R(A),\end{aligned}$$

and the proof of Theorem 1 is complete.

Remarks. (a) As mentioned earlier, the proof of Theorem 1 applies equally well to the case where $X = C(\Lambda, R)$. Let us note, however, the following simplifications, which also serve to illustrate the constructions involved in the preceding proof. We can define a pair of bounded real-valued functions l and u on Λ by

$$\left.\begin{matrix}u(t)\\ l(t)\end{matrix}\right\} = \lim_{N\in\mathcal{N}(t)}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right.\{\Phi(\mu):\mu\in\Omega(t,N)\}.$$

These functions u and l are respectively upper and lower semicontinuous on Λ, and clearly $l(\cdot)\leqq u(\cdot)$. In our earlier notation we have

$$\begin{aligned}A(t) &= [l(t),u(t)], \\ R(T) &= \tfrac{1}{2}\|u-l\|_\infty, \\ B(t) &= [u(t)-R(A),l(t)+R(A)].\end{aligned}$$

The assertions about the map B and its values $B(t)$, for whose proofs we made references to [9], are clearly valid in the present case. Further, the problem of obtaining a continuous selection of B (to serve as a best approximation to Φ) does not now require the use of the Michael selection principle (although it of course applies), but instead yields to the original Dieudonné interposition theorem [3, p. 75]. Thus, as we mentioned in the introduction, the proof that $C(\Lambda, R)$ has property (PROXBID) can be viewed as a further application of the " interposition method " of approximation as developed in [6, 7].

(b) In the course of proving Theorem 1, we showed that any continuous selection of the map B was a best approximation to Φ. Now we observe that the converse holds, that is, any best approximation $f \in C(\Lambda)$ must satisfy $f(t) \in B(t)$ for all $t \in \Lambda$. Indeed, given $\alpha \in A(t)$ and $\varepsilon > 0$, there is an $N \in \mathcal{N}(t)$ and a $\mu \in \Omega(t, N)$ for which $|\alpha - \Phi(\mu)| < \varepsilon$. Therefore

$$\begin{aligned} |\alpha - f(t)| &< |\alpha - \Phi(\mu)| + |\Phi(\mu) - \langle f, \mu \rangle| + |\langle f, \mu \rangle - f(t)| \\ &\leqq 2\varepsilon + \|\Phi - J(f)\| \\ &= 2\varepsilon + R(A), \end{aligned}$$

showing that $A(t) \subset D(f(t), R(A))$, or, in other words, that $f(t) \in B(t)$.

Now we turn to the case where Λ is a (generally non-compact) locally compact Hausdorff space and X is either the Banach space $C_0(\Lambda)$ of continuous complex-valued functions on Λ which vanish at infinity or the analogous real space $C_0(\Lambda, R)$. If we replace Λ by its one-point compactification $\bar{\Lambda}$, then X can be identified with the subspace of $C(\bar{\Lambda})$ (resp. $C(\bar{\Lambda}, R)$) consisting of those functions in X which vanish at the point at infinity. With this viewpoint it is clear that the next theorem establishes that X has property (PROXBID).

THEOREM 2. *Let Λ be a compact Hausdorff space and $t_0 \in \Lambda$. Let X be the hyperplane in either $C(\Lambda)$ or $C(\Lambda, R)$ consisting of all functions f for which $f(t_0) = 0$. Then X has property* (PROXBID).

Proof. As before, we shall deal only with the more general case of complex-valued functions. Let δ_0 be the state " evaluation at t_0 ", so that $X = \ker(\delta_0)$. We can identify the bidual X^{**} with the hyperplane $\{\delta_0\}^{\perp}$ in $C(\Lambda)^{**}$. Choose $\Phi \in X^{**}$. Let $D(0, R_0)$ be the smallest disk in $\mathbb{C}$ which contains the set $A(t_0)$ (notation as in the proof of Theorem 1), and put $R(A, t_0) = \max\{R(A), R_0\}$. We claim that

$$R(A, t_0) = \operatorname{dist}(\Phi, J(X)), \tag{7}$$

and that this distance is attained at some point in $J(X)$.

Suppose that $R(A, t_0) = R_0$. We choose an $f \in X$ and show that

$$R(A, t_0) \leqq \|\Phi - J(f)\|. \tag{8}$$

Given $\varepsilon > 0$, we know that for all sufficiently small $N \in \mathcal{N}(t_0)$ we have

$$\max\{\operatorname{osc}(f; N), d_H(A_N(t_0), A(t_0))\} < \varepsilon.$$

Also, by definition of R_0, there exists $\alpha \in A(t_0)$ such that $|\alpha| > R_0 - \varepsilon$, and consequently there is a $\mu \in \Omega(t_0, N)$ for which $|\Phi(\mu)| > R_0 - 2\varepsilon$. Therefore

$$\|\Phi - J(f)\| \geqq |\Phi(\mu) - \langle f, \mu \rangle| \geqq |\Phi(\mu)| - |\langle f, \mu \rangle| \geqq R_0 - 3\varepsilon \equiv R(A, t_0) - 3\varepsilon,$$

since $f(t_0) = 0$. This proves (8).

Now introducing the map

$$t \mapsto B(t) = \{\beta \in \mathbb{C} : A(t) \subset D(\beta, R(A, t_0))\},$$

we have (by [9] again) that B is lower semicontinuous and that (by construction) $0 \in B(t_0)$. If we put

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$$\bar{B}(t) = \begin{cases} B(t) & (t \neq t_0), \\ 0 & (t = t_0), \end{cases}$$

we have again that $\bar{B}$ is lower semicontinuous and consequently there is a continuous selection f_0 for $\bar{B}$. Clearly $f_0 \in X$. We claim that f_0 satisfies $\|\Phi - J(f_0)\| \leqq R(A, t_0)$. The argument at this point is very nearly identical to the analogous part of Theorem 1, and we omit it. This inequality establishes (7), and completes the proof of Theorem 2 in the case where $R(A, t_0) = R_0$. In the alternative case, we have $R(A, t_0) = R(A)$. But now $0 \in B(t_0)$ automatically. Thus we may define $\bar{B}$ and proceed as before. The proof of Theorem 2 is now complete.

REMARK. 1 Again we observe that best approximations to Φ from $J(X)$ can be characterized as the continuous selections of a certain mapping, namely $\bar{B}$.

REMARK 2. As in Theorem 2, let X be the subspace of either $C(\Lambda)$ or $C(\Lambda, R)$ consisting of all functions vanishing at $t_0 \in \Lambda$. According as X is real or complex, let Y be either $C(\Lambda, R)$ or $C(\Lambda)$. Let $\Phi \in Y^{**}$. By using the techniques of Theorems 1 and 2, it can be shown that, if X is real,

$$\text{dist}(\Phi, J(X)) = \text{dist}(\Phi, J(Y)) + \text{dist}(P_c(\Phi), J(X)),$$

where $P_c(\Phi)$ is the set of all best approximations to Φ from $J(Y)$ and

$$d(U, V) = \inf\{\|u - v\| : u \in U, v \in V\}.$$

It may also be shown that this equation must be replaced by an inequality ($\leqq$) when X is complex.

REFERENCES

1. E. Alfsen and E. Effros, Structure in real Banach spaces, *Ann. Math.* **96** (1972), 98–173.

2. J. Blatter, *Grothendieck Spaces in Approximation Theory*, Amer. Math. Soc. Memoir 120 (Providence, R.I., 1972).

3. J. Dieudonné, Une généralisation des espaces compacts, *J. Math. Pures Appl.* **23** (1944), 65–76.

4. H. Fakhoury, Projections de meilleure approximation continues dans certains espaces de Banach, *C.R. Acad. Sci. Paris* **276** (1973), A45–A48.

5. R. Holmes, *A Course on Optimization and Best Approximation* (Springer-Verlag, Berlin-Heidelberg-New York, 1972).

6. R. Holmes and B. Kripke, Approximation of bounded functions by continuous functions, *Bull. Amer. Math. Soc.* **71** (1965), 896–897.

7. R. Holmes and B. Kripke, Interposition and approximation, *Pacific J. Math.* **24** (1968), 103–110.

8. R. Holmes and B. Kripke, Best approximation by compact operators, *Indiana Univ. Math. J.* **21** (1971), 255–263.

9. C. Olech, Approximation of set-valued functions by continuous functions, *Collect. Math.* **19** (1968), 285–293.

10. S. Sakai, *C*-Algebras and W*-Algebras* (Springer-Verlag, New York-Heidelberg-Berlin, 1971).

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