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A CLASS OF ONE-RELATOR GROUPS WITH CENTRE

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Let the group H have presentation $\langle a_1, \ldots, a_m;\ a_1^{p_1} = a_2^{p_1}, \ldots, a_{m-1}^{p_{m-1}} = a_m^{p_{m-1}} \rangle$ where $m \geqslant 3$, $p_i \geqslant 2$ and $(p_i, p_j) = 1$ if $i \neq j$. We show that H is a one-relator group precisely if H can be obtained from a suitable group $\langle a, b;\ a^p = b^p \rangle$ by repeated applications of a (two-stage) procedure consisting of applying central Nielsen transformations followed by adjoining a root of a generator. We conjecture that any one-relator group G with non-trivial centre and G/G' not free abelian of rank two can be obtained in the same way from a suitable group $\langle a, b;\ a^p = b^q \rangle$.

1. INTRODUCTION

Let G be a non-cyclic one-relator group with non-trivial centre and with commutator quotient group G/G' not free abelian of rank two. It was shown by Pietrowski [9] that G has a presentation of the form

$$\langle a_1, \ldots, a_m;\ a_1^{p_1} = a_2^{q_1}, \ldots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle, \tag{1}$$

where $m \geqslant 2$, $p_i \geqslant 2$, $q_i \geqslant 2$ and $(p_i, q_j) = 1$ for $i > j$. This leads to the problem as to which groups H satisfying these conditions actually have a one-relator presentation. In [7] Meskin, Pietrowski and Steinberg showed that this is always the case if $m = 3$, and in general that any such group H is generated by the pair (a_1, a_m) and has centre generated by $a_1^{p_1 \cdots p_{m-1}} = a_m^{q_1 \cdots q_{m-1}}$; moreover, if H is one-relator on (a_1, a_m) and $p_m \equiv \pm 1$ modulo $q_1 q_2 \cdots q_{m-1}$, then

$$\langle a_1, \cdots, a_{m+1};\ a_1^{p_1} = a_2^{q_1}, \cdots, a_m^{p_m} = a_{m+1}^{q_m} \rangle \tag{2}$$

is one-relator on (a_1, a_{m+1}) for any choice of $q_m \geqslant 2$. On the negative side, it was shown by Collins [3] that the group $\langle a_1, a_2, a_3, a_4;\ a_1^2 = a_2^2,\ a_2^5 = a_3^5, a_3^3 = a_4^3 \rangle$ is not a one-relator group. Collins also showed in [3] that any generating set (x, y) of a group of the form (1) is Nielsen equivalent to one of the form (a_1^r, a_m^s), for suitable integers r and s.

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In the present paper we give a simple extension, in terms of Nielsen transformations, of the recipe of [7] mentioned above for constructing new one-relator groups of form (1) from known ones. Our main result is that this procedure gives, in particular, all one-relator groups of the form (1) which satisfy the additional conditions $p_i = q_i$ $(1 \leqslant i \leqslant m-1)$. This yields a very easy algorithm for testing whether or not a given presentation (1) with $p_i = q_i$ $(1 \leqslant i \leqslant m-1)$ is a one relator group. Thus it turns out, for example, that the group $\langle a_1, a_2, a_3, a_4;\ a_1^2 = a_2^2,\ a_2^{p_2} = a_3^{p_2},\ a_3^3 = a_4^3\rangle$ is not a one-relator group for any choice of $p_2 \geqslant 2$. This extends the example of Collins mentioned above.

2. A Nielsen transformation procedure

Let (a_1^r, a_m^s) be a generating set of a group H of the form (1) above. Applying either of the Nielsen transformations

$$(a_1^r, a_m^s) \longrightarrow \left(a_1^r, a_m^{\varepsilon s} a_1^{rkp_1p_2\cdots p_{m-1}}\right)$$

or

$$(a_1^r, a_m^s) \longrightarrow \left(a_1^{\varepsilon r} a_m^{s\ell q_1 q_2 \cdots q_{m-1}}, a_m^s\right),$$

where $\varepsilon = \pm 1$ and k, ℓ are integers, will again yield such a generating set, since, for example, $a_m^{\varepsilon s} a_1^{rkp_1p_2\cdots p_{m-1}} = a_m^{\varepsilon s + rkq_1q_2\cdots q_{m-1}}$. We shall call these *central Nielsen transformations*.

It is clear that if (x, y) is a one-relator generating set of a group G, then so is any pair (x_1, y_1) which is Nielsen equivalent to (x, y). In particular, if (a_1^r, a_m^s) is a one-relator generating set of a group H of form (1), and (a_1^λ, a_m^μ) is Nielsen equivalent to this, then we have a presentation $\langle x, y;\ R(x,y) = 1\rangle$ (say) of H, with x corresponding to a_1^λ and y to a_m^μ . We now choose $q_m \geqslant 2$ and take H_1 to be the free product of H with the infinite cycle on a_{m+1}, amalgamating $a_{m+1}^{q_m} = y^\eta$, where $\eta = 1$ if $\mu > 0$ and $\eta = -1$ if $\mu < 0$. Now on the one hand H_1 is a one-relator group, since it has presentation $\langle x, a_{m+1};\ R(x, a_{m+1}^{q_m}) = 1\rangle$, and on the other hand it has presentation

$$\langle a_1, \ldots, a_{m+1};\ a_1^{p_1} = a_2^{q_1}, \ldots, a_m^{p_m} = a_{m+1}^{q_m}\rangle, \tag{3}$$

where $p_m = |\mu|$, so that H_1 is of the form (1) if $p_m \geqslant 2$, while if $p_m = 1$ this is again the case since then, by deleting the generator $a_m = a_{m+1}^{q_m}$, and renumbering a_{m+1}, (3) can be replaced by

$$\langle a_1, \ldots, a_m;\ a_1^{p_1} = a_2^{q_1}, \ldots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}q_m}\rangle. \tag{4}$$

In either case the uniqueness result of Pietrowski [9], namely that the isomorphism class of a group of form (1) is uniquely determined by the sequence $p_1, q_1, \cdots, p_{m-1}, q_{m-1}$

together with its mirror image $q_{m-1}, p_{m-1}, \dots, q_1, p_1$, shows that the group H_1 is not isomorphic to H. Furthermore, H_1 has a one-relator generating set of the same kind as the generating set (a_1^r, a_m^s) of H, namely (a_1^λ, a_{m+1}) if H_1 has presentation (3), and (a_1^λ, a_m) if (4) applies. Of course the above process applies equally well if we adjoin a root of a_1^λ, rather than one of a_m^μ.

Thus our procedure simply amounts to applying central Nielsen transformations to a one-relator generating set (a_1^r, a_m^s) of a group H of form (1), and then adjoining an n-th root $(n \geqslant 2)$ of one of the generators so obtained; the result is a group H_1 of the same type, with a corresponding one-relator generating set, and with H_1 not isomorphic to H. We note that the recipe of Meskin, Pietrowski and Steinberg [7] is the special case of this where the original one-relator generating set is just (a_1, a_m), and a single central Nielsen transformation is applied before a root is adjoined.

As a simple illustration of the above, the group

$$H = \langle a_1, a_2, a_3, a_4;\ a_1^{p_1} = a_2^3,\ a_2^2 = a_3^5,\ a_3^{13} = a_4^{q_3} \rangle$$

is one-relator on the generating set (a_1, a_4) for all choices of $p_1 \geqslant 2, q_3 \geqslant 2$. To see this we recall the observation of Zieschang [10] that if $p \geqslant 2$ and $q \geqslant 2$ then any generating set (a^r, b^s) of $\langle a, b;\ a^p = b^q \rangle$ with either $r = 1$ or $s = 1$ is a one-relator generating set. Now starting with the one-relator generating set (a_2, a_3^2) of $H_1 = \langle a_2, a_3;\ a_2^2 = a_3^5 \rangle$, we apply Nielsen transformations as follows:

$$(a_2, a_3^2) \longrightarrow (a_2^{-1} a_3^{10}, a_3^2) = (a_2^3, a_3^2) \longrightarrow (a_2^3, a_2^6 a_3^{-2}) = (a_2^3, a_3^{13}).$$

Thus (a_2^3, a_3^{13}) is a one-relator generating set of H_1, and H is obtained by adjoining roots a_1, a_4 of a_2, a_3 respectively, so that H is one-relator on (a_1, a_4) as claimed. We note that this example is not covered by Theorem 2 of [7], since 13 is not congruent to ± 1 modulo 15, and 3 is not congruent to ± 1 modulo 26.

We would conjecture that any one-relator group of the form (1) can be obtained by a number of applications of the above procedure, starting with a one-relator generating set (a^r, b^s) of a suitable group $H_1 = \langle a, b;\ a^p = b^q \rangle$. It would in fact then be sufficient to start with such a set with either $r = 1$ or $s = 1$, since Collins [4] has shown that any one-relator generating set (a^r, b^s) of H_1 is equivalent, by central Nielsen transformations, to one of this special form.

3. The Main Result.

In [3] Collins has shown that $(a_1^{r_1}, a_m^{s_1})$ is a generating set of the group (1) if and only if r_1 and s_1 are non-zero integers satisfying $(r_1, s_1) = (r, p_1 p_2 \dots p_{m-1}) = (s, q_1 q_2 \dots q_{m-1}) = 1$. Moreover if $r_1 \geqslant 1, s_1 \geqslant 1$ and $2r_1 > s_1 p_1 p_2 \dots p_{m-1}$, then $r =$

$|r_1 - sp_1p_2\dots p_{m-1}| < r_1$, and $(a_1^r, a_m^{s_1})$ is equivalent to $(a_1^{r_1}, a_m^{s_1})$ by the central Nielsen transformation $(a_1^{r_1}, a_m^{s_1}) \longrightarrow \left(a_1^{\varepsilon r_1} a_m^{-\varepsilon s_1 q_1 q_2 \dots q_{m-1}}, a_m^{s_1}\right)$, where $\varepsilon = \pm 1$ is chosen so that $r = \varepsilon(r_1 - s_1q_1q_2\dots q_{m-1})$. A similar observation applies if $2s_1 > r_1q_1q_2\dots q_{m-1}$. Thus it is clear that given a generating set $(a_1^{r_1}, a_m^{s_1})$ of (1) we can find, algorithmically, a series of central Nielsen transformations that, when applied to $(a_1^{r_1}, a_m^{s_1})$, yield a pair (a_1^r, a_m^s) satisfying $r \geqslant 1$, $s \geqslant 1$, $2r \leqslant sp_1p_2\dots p_{m-1}$ and $2s \leqslant rq_1q_2\dots q_{m-1}$. We shall describe this by saying that $(a_1^{r_1}, a_m^{s_1})$ *centrally reduces* to (a_1^r, a_m^s), and that (a_1^r, a_m^s) is *centrally reduced.* We can now state our main result as

THEOREM. *Let the group H have presentation*

$$(5) \qquad \langle a_1, \dots, a_m;\ a_1^{p_1} = a_2^{p_1}, \dots, a_{m-1}^{p_{m-1}} = a_m^{p_{m-1}} \rangle$$

where $m \geqslant 3$, $p_i \geqslant 2$ and $(p_i, p_j) = 1$ if $i \neq j$. If H is a one-relator group, then (a_1, a_m) is a one-relator generating set, and any one-relator generating set of the form (a_1^r, a_m^s) centrally reduces to (a_1, a_m). Moreover, the subgroup of H generated by (a_2, a_{m-1}) must be one-relator on the generating set $\left(a_2^{p_1}, a_{m-1}^{p_{m-1}}\right)$.

PROOF: Suppose that H is presented as above, and that H is a one-relator group. Then by the results of Collins alluded to above we have a one-relator centrally reduced generating set (a_1^r, a_m^s) of H, so that $r \geqslant 1$, $s \geqslant 1$, $2r \leqslant sp$, $2s \leqslant rp$ where $p = p_1p_2\dots p_{m-1}$, and $(r,s) = (r,p) = (s,p) = 1$. Now let $G = \langle x, y;\ R(x,y) = 1\rangle$ be isomorphic to H by an isomorphism $\phi : G \to H$ such that $\phi(x) = a_1^r$ and $\phi(y) = a_m^s$. We may assume $R(x,y)$ is of the form $R(x,y) = x^{m_1}y^{n_1}\dots x^{m_t}y^{n_t}$ for non-zero integers m_i, n_i $(1 \leqslant i \leqslant t)$. Here we will have $t \geqslant 2$, since $m \geqslant 3$. We then have $a_1^{rm_1}a_m^{sn_1}\dots a_1^{rm_t}a_m^{sn_t} = 1$ in H, and by Corollary 2.3 of [3] it follows that either $p_1 | m_i$ and $p_{m-1} | n_i$ for $1 \leqslant i \leqslant t$, or p divides some m_i or n_j.

We suppose firstly that $m_i = p_1 m_i'$ and $n_i = p_{m-1} n_i'$ for $1 \leqslant i \leqslant t$. We then have

$$G = \langle x, y;\ x^{p_1 m_1'} y^{p_{m-1} n_1'} \dots x^{p_1 m_t'} y^{p_{m-1} n_t'} = 1\rangle,$$

so that, if G_1 is given by

$$G_1 = \langle \alpha, \beta;\ \alpha^{m_1'}\beta^{n_1'} \dots \alpha^{m_t'}\beta^{n_t'} = 1\rangle,$$

then G can be described as the (repeated) free product with amalgamation

$$G = \langle x \rangle \underset{x^{p_1} = \alpha}{\star} G_1 \underset{\beta = y^{p_{m-1}}}{\star} \langle y \rangle.$$

Now $\phi(x^{p_1}) = a_1^{rp_1} = a_2^{rp_1}$ and $\phi(y^{p_{m-1}}) = a_m^{sp_{m-1}} = a_{m-1}^{sp_{m-1}}$, so that ϕ induces an isomorphism from G_1 to the subgroup of H generated by $\left(a_2^{rp_1}, a_{m-1}^{sp_{m-1}}\right)$. Now this subgroup is just the subgroup $\langle a_2, a_{m-1}\rangle$ generated by (a_2, a_{m-1}), and it follows that $\langle a_2, a_{m-1}\rangle$ is one-relator on the generating set $(a_2^{rp_1}, a_{m-1}^{sp_{m-1}})$. However $\langle a_2, a_{m-1}\rangle$ also has presentation

$$\langle a_2, \dots, a_{m-1};\ a_2^{p_2} = a_3^{p_2}, \dots, a_{m-2}^{p_{m-2}} = a_{m-1}^{p_{m-2}}\rangle \tag{6}$$

(in case $m = 3$ this is just the infinite cycle on a_2). Thus G_1 is isomorphic to the group given by (6), by an isomorphism taking α to $a_2^{rp_1}$ and β to $a_{m-1}^{sp_{m-1}}$. It follows from the description of G as a free product with amalgamation that G has presentation

$$\langle x, a_1, \dots, a_m, y;\ x^{p_1} = a_2^{rp_1},\ a_2^{p_2} = a_3^{p_2}, \dots, a_{m-2}^{p_{m-2}} = a_{m-1}^{p_{m-2}}, a_{m-1}^{sp_{m-1}} = y^{p_{m-1}}\rangle.$$

This contradicts the uniqueness result of [9] unless $r = s = 1$. This proves the theorem in the case under consideration.

We may now suppose that p divides some m_i or n_j. We will show that this is in fact impossible. Our arguments here use a number of standard results as developed, for example, in Murasugi [8], Baumslag and Taylor [1] and Collins [4]; the basic technique of course is due to Magnus (see Magnus, Karrass and Solitar [6]). We may suppose, without loss of generality, that $m_1 > 0$ and $p|m_1$, $m_1 = pm_1'$ say. We now define the group J by

$$J = \langle e, a_1, \dots a_m, d;\ e^r = a_1^r,\ a_1^{p_1} = a_2^{p_1}, \dots, a_{m-1}^{p_{m-1}} = a_m^{p_{m-1}}, a_m^s = d^s\rangle, \tag{7}$$

and we clearly have $J = \langle e, d;\ R(e^r, d^s) = 1\rangle$. Now $e^{rsp} = d^{rsp}$ in J, so that the exponent sums of e and d in $R(e^r, d^s)$ are equal in magnitude and opposite in sign. Introducing a new generator $z = ed^{-1}$ and eliminating $e = zd$, we obtain $J = \langle z, d;\ R((zd)^r, d^s) = 1\rangle$, where

$$R((zd)^r, d^s) \equiv (zd)^{rpm_1'} d^{sn_1} (zd)^{rm_2} d^{sn_2} \dots (zd)^{rm_t} d^{sn_t}$$

has exponent sum zero on d. It is well known in this context (see for example, [1]) that J is an extension of a free group F of finite rank, by the infinite cycle on d. Here F is generated by the conjugates $z_i = d^i z d^{-i}$ of z by powers of d, and if $R((zd)^r, d^s)$ is rewritten in terms of the z_i, as $R_0 = z_0 z_1 \dots z_{rpm_1'-1} w$ say, where w is the rewritten form of $d^{rpm_1' + sn_1}(zd)^{rm_2} d^{sn_2} \dots (zd)^{rm_t} d^{sn_t}$, then $z_0 z_1 \dots z_{rpm_1'-1}$ is a subword of R_0. Moreover if λ, μ are, respectively, the least and greatest subscripts on z occurring in R_0, then F has rank $r(F)$ given by $r(F) = \mu - \lambda$.

It follows immediately from the above observations that $r(F) \geqslant rpm_1' - 1 \geqslant rp - 1$. Now the rank of F can also be obtained from the presentation (7) of J by an Euler characteristic argument: see Karrass, Pietrowski and Solitar [5] and Proposition 7.3 of Chapter IX of Brown [2]. We obtain from this

$$r(F) = r + s + \left(\sum_{i=1}^{m-1} p_i\right) - m - 1,$$

so that

$$r + s + \sum_{i=1}^{m-1} p_i \geqslant rp + m. \tag{8}$$

We will show that (8) is impossible. First we note that $2s \leqslant rp - 1$ (since $2s = rp$ implies $s = 1$ and so $r = 1$ and $2 = p$, which is impossible since $m \geqslant 3$ implies $p = p_1 p_2 \dots p_{m-1} \geqslant 6$). We have therefore, from (8),

$$r + \sum_{i=1}^{m-1} p_i \;\geqslant\; \frac{1}{2} rp \;=\; \frac{1}{2}(2m + 1). \tag{9}$$

When $m = 3$ and $r = 1$, (9) becomes $p_1 + p_2 \geqslant p_1 p_2/2 + 5/2$, so that $p_1 p_2 - 2p_1 - 2p_2 + 5 \leqslant 0$, or $(p_1 - 2)(p_2 - 2) + 1 \leqslant 0$, which is impossible. We now use induction on r to establish that (9) is false for $m = 3$. Accordingly, we take the least $r > 1$ so that

$$r + p_1 + p_2 \geqslant \frac{r p_1 p_2}{2} + \frac{7}{2}.$$

We have then

$$r - 1 + p_1 + p_2 < (r - 1)\frac{p_1 p_2}{2} + \frac{7}{2}$$

and combining these yields $2 \geqslant p_1 p_2$, which is incorrect. Thus (9) is false for $m = 3$ and all r. We now use induction on m to show (9) is always false. Take $m \geqslant 4$ least so that (9) holds. We then have

$$r + \sum_{i=1}^{m-2} p_i < \frac{r p_1 p_2 \dots p_{m-2}}{2} + \frac{2m - 1}{2}. \tag{10}$$

We claim

$$p_{m-1} \;\leqslant\; (p_{m-1} - 1) r \frac{p_1 p_2 \cdots p_{m-2}}{2} + 1. \tag{11}$$

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Indeed if we put $\lambda = p_{m-1}$ and $\mu = rp_1p_2\cdots p_{m-2}$ then (11) is just $(\lambda-1)\mu/2-\lambda+1 \geqslant 0$, that is, $\lambda\mu-\mu-2\lambda+2 \geqslant 0$, or $(\lambda-2)(\mu-2)+\mu-2 \geqslant 0$, which is the case since $\lambda \geqslant 2$ and $\mu \geqslant 2$. Now adding (10) and (11) shows that (9) is false. This contradicts the existence of m such that (9) is true. Hence (9), and therefore (8), is always false. This proves the theorem. □

As an easy consequence, we have the

COROLLARY. *There is an algorithm to decide, given a presentation (5), whether or not the corresponding group is a one-relator group.*

PROOF: Let the group $H = H_m$ be given by (5). According to the theorem, if H is one-relator then the group $H_{m-2} = \langle a_2, a_{m-1}\rangle$ must be one-relator on the generating set $\left(a_2^{p_1}, a_{m-1}^{p_{m-1}}\right)$; conversely, if this is the case then clearly H is one-relator (on (a_1, a_m)). So the question reduces to deciding if H_{m-2} is one-relator on $\left(a_2^{p_1}, a_{m-1}^{p_{m-1}}\right)$. If $m = 3$ this is the case (and of course we have a special case of a result of [7] noted previously), while if $m = 4$ we may apply central Nielsen transformations to the generating set $(a_2^{p_1}, a_3^{p_3})$ of $H_2 = \langle a_2, a_3; a_2^{p_2} = a_3^{p_2}\rangle$ to obtain a Nielsen equivalent centrally reduced set (a_2^r, a_3^s), and the theorem of [4] tells us that H_2 is one-relator on $(a_2^{p_1}, a_3^{p_3})$ precisely if $r = 1$ or $s = 1$ holds. Thus we can decide the question in this case. If $m > 4$ then the theorem applies to H_{m-2}. Thus we centrally reduce $\left(a_2^{p_1}, a_{m-1}^{p_{m-1}}\right)$. If we do not obtain (a_2, a_{m-1}) then H is not one-relator. If we do obtain (a_2, a_{m-1}), then H is one-relator precisely if the same is true for H_{m-2}. This proves the result. □

We now consider the group $H = \langle a_1, a_2, a_3, a_4;\ a_1^2 = a_2^2, a_2^{p_2} = a_3^{p_2}, a_3^3 = a_4^3\rangle$, with $p_2 \geqslant 2$. If $(2, p_2) \neq 1$ or $(3, p_2) \neq 1$ then it follows from [7] that H is not one-relator, so we suppose $(2, p_2) = (3, p_2) = 1$. Now we know H is one-relator if, and only if, (a_2^2, a_3^3) is a one-relator generating set of $H_1 = \langle a_2, a_3; a_2^{p_2} = a_3^{p_2}\rangle$. Now it is clear that (a_2^2, a_3^3) is a centrally reduced generating set of H_1 (since $p_2 \geqslant 5$), and it follows that it is not a one-relator generating set of H_1. Hence H is not a one-relator group.

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