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Sets of Uniqueness for Univalent Functions

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Abstract. We observe that any set of uniqueness for the Dirichlet space $\mathcal{D}$ is a set of uniqueness for the class S of normalized univalent holomorphic functions.

The class S consists of the injective holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Obrock [7] and Duren [4] asked which sequences $Z = \{z_n\}$ of points in $\mathbb{D}$ are sets of uniqueness for S (that is, for any $f, g \in S$, if $f(z_n) = g(z_n)$ we have $f \equiv g$). Duren observed in [4] that the condition

$$\sum (1 - |z_n|) = \infty$$

is sufficient, and Lappan [5] found necessary conditions. And for Z a Blaschke sequence the necessary condition

$$\int_{-\pi}^{\pi} \log \frac{1}{\operatorname{dist}\bigl(Z, \exp(i\theta)\bigr)} \, d\theta = \infty$$

is implicit in work of Stegbuchner [9].

In this note we improve on Duren's sufficient condition. Suppose that Z is a sequence, $f, g \in S$ and $f(z_n) = g(z_n)$ for $z_n \in Z$, but $f \not\equiv g$. Consider the function

$$h(z) = \frac{1}{f(z)} - \frac{1}{g(z)}$$

in the unit disk. It is clearly holomorphic in $\mathbb{D}$, and is zero on Z, except possibly at the origin, but it is not identically zero. Now

$$h(z) = \sum_{n=0}^{\infty} (b_n - c_n) z^n,$$

where

$$1/f(1/\zeta) = \sum_{n=-1}^{\infty} b_n \zeta^{-n},$$

and the coefficients c_n are related to g in the same way. The Area Theorem of Gronwall, see p. 29 of [3], implies that

$$\sum_{n=1}^{\infty} n|b_n - c_n|^2 \le \sum_{n=1}^{\infty} 2n(|b_n|^2 + |c_n|^2) \le 4,$$

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and thus h is in the Dirichlet space $\mathcal{D}$ of holomorphic functions h in $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |h'(z)|^2 \, dx \, dy < \infty.$$

Hence there exists a nonzero function in $\mathcal{D}$ which is zero on Z, except possibly at the origin. By an elementary argument, see p. 221 of [8], there exists a function in $\mathcal{D}$ having simple zeros at the points of Z. Thus:

Any set of uniqueness for $\mathcal{D}$ is a set of uniqueness for S.

The sets of uniqueness for $\mathcal{D}$ have not been precisely characterized, but there is some work bearing on this problem, usually formulated in terms of zero sets for functions in $\mathcal{D}$. Shapiro and Shields [8] showed that for any continuous function h with $h(0) = 0$ and $h(t) > 0$ for $t > 0$ there exists a set of uniqueness for $\mathcal{D}$ satisfying

$$\sum \frac{1}{-\log(1 - |z_n|)} h(1 - |z_n|) < \infty.$$

Choosing in particular $h(x) = -x \log(x)$ near the origin, we deduce the existence of a set of uniqueness for S satisfying the Blaschke condition

$$\sum (1 - |z_n|) < \infty.$$

In fact, by a result originally due to Carleson, see [2], there is a set of uniqueness for $\mathcal{D}$ and thus for S satisfying the Blaschke condition and with $z_n \to 1$. It was shown by Nagel, Rudin and Shapiro [6] that for any sequence $\{r_n\}$ with $0 < r_n < 1$ and

$$\sum \frac{1}{-\log(1 - |r_n|)} = \infty,$$

there exists a sequence $\{\theta_n\}$ of angles such that $\{r_n \exp(i\theta_n)\}$ is a set of uniqueness for $\mathcal{D}$, hence for S. Bogdan [1] showed under the same conditions that if $\{\Theta_n\}$ are independent random variables uniformly distributed on $(-\pi, \pi]$ and $\{\theta_n\}$ are values of $\{\Theta_n\}$ then almost surely $\{r_n \exp(i\theta_n)\}$ is a set of uniqueness for $\mathcal{D}$, hence for S.

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