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Extensions by Simple C^*-Algebras: Quasidiagonal Extensions

Dedicated to Lawrence G. Brown on his 60th birthday

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Abstract. Let A be an amenable separable C^*-algebra and B be a non-unital but σ-unital simple C^*-algebra with continuous scale. We show that two essential extensions τ_1 and τ_2 of A by B are approximately unitarily equivalent if and only if

$$[\tau_1] = [\tau_2] \text{ in } KL(A, M(B)/B).$$

If A is assumed to satisfy the Universal Coefficient Theorem, there is a bijection from approximate unitary equivalence classes of the above mentioned extensions to $KL(A, M(B)/B)$. Using $KL(A, M(B)/B)$, we compute exactly when an essential extension is quasidiagonal. We show that quasidiagonal extensions may not be approximately trivial. We also study the approximately trivial extensions.

Introduction

The study of C^*-algebra extensions of $C(X)$ by compact operators was motivated by the understanding of essentially normal operators on an infinite dimensional Hilbert space. The Brown-Douglas-Fillmore Theory for essentially normal operators gives the classification of essentially normal operators up to unitary equivalence ([BDF1]). The original BDF-theory quickly developed into C^*-algebra extension theory ([BDF2, BDF3]) and the KK-theory of Kasparov. Applications of this development can be found not only in operator theory and operator algebras but also in both geometry and non-commutative geometry.

Let $0 \to B \to E \to A \to 0$ be an essential extension of A by B. This is determined by a monomorphism $\tau\colon A \to M(B)/B$. While KK-theory gives the classification of extensions up to stable unitary equivalence, it does not give much information on essential extensions when $B \neq \mathcal{K}$, where $\mathcal{K}$ is the compact operators on l^2. The example in 1.1 below shows that a non-trivial extension τ may have $[\tau] = 0$ in $KK^1(A, B)$. Other examples (see 4.8) show that there may be infinitely many non-equivalent trivial extensions. Extensions by simple C^*-algebras have been studied in a few special cases (see [Ln5], [Ln6], [Ln7] and [Ln8]).

In this paper, we study approximately unitary equivalence classes of essential extensions of separable amenable C^*-algebras by σ-unital simple C^*-algebras. One of the reasons that BDF-theory was successful is that the Calkin algebra $M(\mathcal{K})/\mathcal{K}$ is simple (and purely infinite). We will restrict ourselves to the case that $M(B)/B$ is

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simple. It is shown in [Ln1] and more recently in [Ln21] that, for a non-unital and σ-unital simple C^*-algebra $A \ncong \mathcal{K}$, $M(A)/A$ is simple if and only if A has a continuous scale. Furthermore, in [Ln21] it is shown that when $M(A)/A$ is simple it is necessarily purely infinite simple.

With the Busby invariant, to study essential extensions of A by B it is sufficient to study monomorphisms from A to $M(B)/B$. With recent development in classification of simple amenable C^*-algebras, we know a great deals concerning monomorphisms from one amenable (simple) C^*-algebra to a separable amenable purely infinite simple C^*-algebra (see for example, [Ph1, Ln19, Ln20]). However, $M(B)/B$ is not amenable and we do not assume that A is simple. One of the main results of this article is the following: Two essential extensions are approximately unitarily equivalent if they induce the same element in $KL(A, M(B)/B)$. If furthermore A satisfies the Universal Coefficient Theorem, then there is a bijection between $\mathbf{E}xt_{ap}(A, B)$, the approximately unitary equivalence classes of essential extensions, and $KL(A, M(B)/B)$.

However, unlike the classical case, the zero element in $KL(A, M(B)/B)$ does not in general give an approximately trivial extension. On the contrary, at least in some cases, $[\tau] = 0$ in $KL(A, M(B)/B)$ never gives an approximately trivial extension and only when $[\tau] \neq 0$ in $KL(A, M(B)/B)$ may approximately trivial extensions occur. To make matters worse, there may not be any essential trivial extensions of A by B even though we can use the above mentioned bijection to classify extensions. This leads us to study quasidiagonal extensions.

Quasidiagonality was defined by P. R. Halmos [H] in 1970. The C^*-algebra version soon appeared. L. G. Brown, R. G. Douglas and P. A. Fillmore [BDF2] first recognized that the study of quasidiagonal extensions might be approached by K-theory. They noticed that limits of trivial extensions correspond to the quasidiagonal extensions. L. G. Brown pursued this further in [Br2]. Further developments in the study of quasidiagonality can be found in [Sa, V1, V2]. C. L. Schochet proved that stable quasidiagonal extensions are the same as limits of stable trivial extensions and can be characterized by $\mathrm{Pext}(K_*(A), K_*(B))$ if A is assumed to be quasidiagonal relative to B and it satisfies the Universal Coefficient Theorem. These results might lead one to believe that quasidiagonal extensions are the same as limits of trivial extensions in greater generality. However, in this paper we show this fails when B is neither isomorphic to $\mathcal{K}$ nor purely infinite simple.

One should note that the existence of quasidiagonal extensions implies that B has at least one approximate identity consisting of projections. Suppose that B is a non-unital and σ-unital simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(B)$. If A is a quasidiagonal C^*-algebra, then there exists an essential quasidiagonal extension of A by B. This condition is necessary if we assume that B is also a quasidiagonal C^*-algebra. Using K-theory and the classification result mentioned above, we give a necessary and sufficient condition for essential extensions to be quasidiagonal for a large class of amenable quasidiagonal C^*-algebras A. We also give a necessary condition for essential extensions to be approximately trivial for amenable C^*-algebras which satisfy the UCT. As a consequence, a large class of quasidiagonal extensions are *not* the limits of trivial extensions.

The essential extensions of a separable amenable C^*-algebra A by B (where B is a non-unital and σ-unital C^*-algebra with a continuous scale) is proved in this paper to be determined by $KL(A, M(B)/B)$. However, to determine which elements in $KL(A, M(B)/B)$ give an approximately trivial extension is still a difficult task. As mentioned above, for example, the zero element in $KL(A, M(B)/B)$ does not usually give an approximately trivial extension. When A is stably finite, both $K_0(A)$ and $K_0(M(B))$ have nice order while $K_0(M(B)/B)$ has no useful order. Even if we know which homomorphism $\beta\colon K_0(A) \to K_0(M(B)/B)$ can be lifted to a homomorphism from $K_0(A)$ to $K_0(M(B))$, the lifting may not be positive. In this paper, at least for some special cases, we give a precise condition for an element in $KL(A, M(B)/B)$ to be represented by approximately trivial extensions.

This paper is organized as follows.

Section 1: Preliminaries. This section is a preparation for the rest of the paper which contains a computation of K-theory for $M(B)$ and $M(B)/B$ for σ-unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(B)$ and with a continuous scale. We also point out that $M(B)/B$ is simple (and purely infinite) if and only if B has a continuous scale (if $B \not\cong \mathcal{K}$).

Section 2: Monomorphisms from $A \otimes \mathcal{O}_2$ into a purely infinite simple C^*-algebra. This section studies homomorphisms from $A \otimes \mathcal{O}_2$ into a purely infinite simple C^*-algebra.

Section 3: Approximately unitarily equivalent extensions. We show that if B is a non-unital and σ-unital simple C^*-algebra with a continuous scale, two monomorphisms from A to $M(B)/B$ are approximately unitarily equivalent if and only if they induce the same element in $KL(A, M(B)/B)$.

Section 4: $\mathbf{Ext}_{ap}(A, B)$. In this section, under the assumption that A satisfies the UCT, we give a bijection $\Gamma\colon \mathbf{Ext}_{ap}(A, B) \to KL(A, M(B)/B)$.

Section 5: Examples. In this section, we present a few examples which show that the bijection Γ may not answer all questions about these extensions. For example, we show that the zero element in $KL(A, M(B)/B)$ does not represent an approximately trivial extension in general.

Section 6: Quasidiagonal extensions—general and infinite cases. This section discusses quasidiagonal extensions. Without assuming the UCT, we give a general K-theoretical necessary condition for an essential extension to be quasidiagonal. We also show that for any separable exact C^*-algebra A, there exist quasidiagonal extensions of A by any σ-unital purely infinite simple C^*-algebras.

Section 7: Quasidiagonal extensions—finite case. Let A be a separable amenable C^*-algebra and B be a σ-unital C^*-algebra admitting an approximate identity consisting of projections and having the property (SP). We show that if A is a quasidiagonal C^*-algebra, then there exists an essential quasidiagonal extension. If, in addition B is also assumed to be a quasidiagonal C^*-algebra, then the condition that A is quasidiagonal is also necessary. When B is a σ-unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(B)$ and with a continuous scale, we present a K-theoretical necessary and sufficient condition for an essential extension to be quasidiagonal for a class of separable quasidiagonal amenable C^*-algebras.

Section 8: Approximately trivial extensions. In the last section, we give a general K-theoretical necessary condition for essential extensions to be approximately trivial. Combining this condition with the results in section 7, we show that there are essential quasidiagonal extensions that are *not* approximately trivial extensions.

We also show how to use the bijection Γ to determine which essential extensions are approximately trivial at least in some special cases.

1 Preliminaries

Throughout this paper, we will use the following conventions:

(1) An ideal of a C^*-algebra is always a closed two-sided ideal.
(2) By a unital C^*-subalgebra C of a unital C^*-algebra A we mean $C \subset A$ and $1_C = 1_A$.
(3) If p and q are two projections in a C^*-algebra A, we say p is equivalent to q if there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.
(4) Let A and B be two C^*-algebras and $L_1, L_2 \colon A \to B$ be two maps. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a subset. We write

$$L_1 \approx_\varepsilon L_2 \text{ on } \mathcal{F}$$

if

$$\|L_1(a) - L_2(b)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Suppose that A and B are unital and $L_1(1)$ and $L_2(1)$ are projections. If there is an isometry $s \in B$ such that $s^*L_2(1)s = L_1(1)$, $sL_1(1)s^* = L_2(1)$ and

$$\operatorname{ad} s \circ L_2 \approx_\varepsilon L_1 \text{ on } \mathcal{F},$$

we will write

$$L_2 \sim_\varepsilon L_1 \text{ on } \mathcal{F}.$$

(5) A separable C^*-algebra A is said to be *amenable* (or nuclear), if for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there exists a finite dimensional C^*-algebra C and two contractive completely positive linear maps $L_1 \colon A \to C$ and $L_2 \colon C \to A$ such that

$$L_2 \circ L_1 \approx_\varepsilon \mathrm{id}_A \text{ on } \mathcal{F}.$$

(6) Let A be a separable amenable C^*-algebra. We say A satisfies the Universal Coefficient Theorem (UCT) and write $A \in \mathcal{N}$, if for any σ-unital C^*-algebra C, the map $\gamma \colon KK(A, C) \to \mathrm{Hom}(K_*(A), K_*(C))$ is surjective and the map $\kappa \colon \ker\gamma \to \mathit{ext}(K_*(A), K_*(C))$ is an isomorphism, *i.e.*, there is a short exact sequence:

$$0 \to \mathit{ext}(K_*(A), K_*(C)) \to KK(A, C) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(C)) \to 0.$$

If $h \colon A \to C$ is a homomorphism then h gives an element $[h]$ in $KK(A, C)$.
(7) An extension $0 \to B \to E \to A \to 0$ of C^*-algebras is said to be essential if

$$\{e \in E : eb = be = 0 \text{ for all } b \in B\} = \{0\}.$$

If E is an essential extension of A by B as above, then it is determined by a monomorphism $\tau \colon A \to M(B)/B$ and $E = \pi^{-1} \circ \tau(A)$, where $\pi \colon M(B) \to M(B)/B$ is the quotient map.

We start with the following essential extensions:

$$0 \to A \to E \to C(D) \to 0,$$

where D is the unit disk and $A = B \otimes \mathcal{K}$ and where B is a separable unital simple AF-algebra with a unique tracial state. For example B may be a UHF-algebra. Let I be the unique proper ideal of $M(A)$ which contains A (see [Ell1, 3.2]). Denote $J = \pi(I)$, where $\pi\colon M(A) \to M(A)/A$ is the quotient map. Let $p \in M(A)/A \setminus J$ be a projection such that $1 - p \in J$ is a non-zero projection. To see such a projection exists, one takes a projection $q \in M(A) \setminus A$ with finite trace. Then $q \in I$. Define $p = 1 - \pi(q)$. It follows from [Ln7, 1.17(4)] that $K_1(M(A)/I)) = \mathbb{R}$. It is known that $M(A)/I$ is purely infinite and simple (see [Zh1]). Thus there is a unitary $u \in p(M(A)/A)p$ such that $\bar{\pi}(u)$ is not in $U_0(M(A)/I)$, where $\bar{\pi}\colon M(A)/A \to M(A)/I$ is the quotient map. Let $y \in (1-p)M(A)/A(1-p)$ with the spectrum $sp(y) = D$. Set $x = u + y$. Define $\tau\colon C(D) \to M(A)/A$ by $\tau(f) = f(x)$ for $f \in C(D)$. It is easy to see that τ is not trivial nor it is approximately trivial. However, it is known that $Ext(C(D), A) = KK^1(C(D), A) = \{0\}$. So certainly in this case $KK^1(C(D), A)$ can not be used to understand extensions of $C(D)$ by A. Clearly the complicity of the extension is caused by the fact that $M(A)/A$ is not simple. One can easily imagine that when the ideal structure of $M(A)/A$ is more complicated, equivalent classes of extensions will be hard to compute if it is even possible to compute. The success of the BDF-theorem for the classification of extensions by $\mathcal{K}$ depends on the fact that $M(\mathcal{K})/\mathcal{K}$, the Calkin algebra, is simple. In this paper, we will therefore consider only those essential extensions by a simple C^*-algebra A such that $M(A)/A$ is simple.

So the question is: *When is $M(B)/B$ simple?*

Let B be a σ-unital simple C^*-algebra. Recall [Ln1] that B is said to have a *continuous scale* if for any approximate identity $\{e_n\}$ of B satisfying $e_{n+1}e_n = e_ne_{n+1} = e_n$ and any nonzero positive element $a \in B_+$, there exists an integer $N > 0$ such that

$$(e_m - e_n) \lesssim a, \ \text{ for } \ m > n \geq N$$

i.e., there exists a sequence of elements $r_n \in B$ such that $r_k^* a r_k \to e_m - e_n$ for all $m > n \geq N$. It should be noted that if p and q are projections and $p \lesssim q$, then p is equivalent to a projection $q' \leq q$.

It is proved in [Ln1] that, for non-unital separable simple C^*-algebra $B \ncong \mathcal{K}$, $M(B)/B$ is simple if B has a continuous scale. Recently we have proven the following:

Theorem 1.1 ([Ln21]) *Let $A \ncong \mathcal{K}$ be a non-unital and σ-unital simple C^*-algebra. The following are equivalent:*

(1) *A has a continuous scale;*
(2) *$M(A)/A$ is simple,*
(3) *$M(A)/A$ is a purely infinite simple C^*-algebra.*

Clearly every (non-unital) σ-unital purely infinite simple C^*-algebra has a continuous scale. Essential extensions of separable C^*-algebras A which satisfy the UCT by a non-unital separable purely infinite simple C^*-algebra B is classified by $KK^1(A,B)$ by Kirchberg's absorbing theorem [K1].

In this paper we will focus on essential extensions by a σ-unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated K_0 and a continuous scale.

Suppose that B is a non-unital separable simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(B)$. Fix any nonzero projection $e \in B$. Denote by T the set of those quasi-traces τ on B such that $\tau(e) = 1$. Note that T is (weak $*$-) compact convex set. Let $a \in M(B)_+$. Define $\widehat{a}(\tau) = \tau(a)$ for $\tau \in T$. Then $\widehat{a}$ is a lower semi-continuous affine function on T. If $a \in A$, then $\widehat{a}$ is continuous.

To see examples of simple C^*-algebras with continuous scale, we quote the following result [Ln21]. It also justifies the terminology "continuous scale".

Theorem 1.2 *Let A be a non-unital but σ-unital simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Let 1 be the identity of $M(A)$. Then A has a continuous scale if and only if $\widehat{1}(\tau) = \tau(1)$ for $\tau \in T$ is a continuous function on T.*

It is also proved in [Ln21] that given any separable simple C^*-algebra A, there is a non-unital hereditary C^*-subalgebra B such that B has a continuous scale. In particular, $M(B)/B$ is a purely infinite simple C^*-algebra. Note that $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$ and B may not have any non-trivial projections. Furthermore, B may contain both infinite and finite projections (given by Rørdam ([Ro5]).

Definition 1.3 Let T be a compact convex set. A function f defined on T is said to be affine if $f(a\xi + (1-a)\zeta) = af(\xi) + (1-a)f(\zeta)$ for all $\xi, \zeta \in T$ and $0 \le a \le 1$. We denote

$$\mathrm{Aff}(T) = \{f \in C(T) : f \text{ is affine}\}.$$

If $f, g \in \mathrm{Aff}(T)$ and $f(t) > g(t)$ for all $t \in T$, we will write $f \gg g$. Denote

$$\mathrm{Aff}(T)_{++} = \{f \in \mathrm{Aff}(T) : f \gg 0, \text{ or } f = 0\}.$$

Let B be a simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(B)$. Fix a nonzero projection $e \in B$. Denote by T the set of those traces τ defined on B such that $\tau(e) = 1$. Define

$$\rho_B \colon K_0(B) \to \mathrm{Aff}(T)$$

by $\rho_B([p])(\tau) = \tau(p)$ for projections $p \in M_m(B)$, $m = 1, 2, \dots$. It is known that ρ_B is a positive homomorphism from $(K_0(B), K_0(B)_+)$ to $(\mathrm{Aff}(T), \mathrm{Aff}(T)_{++})$ (see [BH]). In fact (by [BH]), $[p] \ge [q]$ and $[p] \ne [q]$ if and only if $\tau(p) > \tau(q)$ for all $\tau \in T$.

The following was first proved in [Ln2] in 1991.

Theorem 1.4 *Let $e \in B$ be a nonzero projection. Let*

$$T = \{\tau : \tau(e) = 1, \tau \text{ is trace defined on } B\}.$$

Then

(1) $(K_0(M(B)), K_0(M(B))_+) = (\mathrm{Aff}(T), \mathrm{Aff}(T)_{++})$;
(2) *two projections p and q in $M(B) \setminus B$ are equivalent if $\tau(p) = \tau(q)$ for all $\tau \in T$;*
(3) *for any $f \in \mathrm{Aff}(T)_{++}$, there is a projection $p \in M_k(M(B)) \setminus M_k(B)$ (for some $k \geq 1$) such that $\widehat{p} = f$; and*
(4) $K_1(M(B)) = \{0\}$ *and* $U(M(B)) = U_0(M(B))$.

Proof Since B has real rank zero, we obtain an approximate identity $\{e_n\}$ for B consisting of projections (with $e_0 = 0$). Let $p \in M(B)$ be a projection. It follows from [Zh2, Theorem 4.1] that we may assume that $p = \sum_{n=1}^{\infty} p_n$, where $p_n \leq e_{n+1} - e_n$. Since $\widehat{1}$ is continuous on T, by the Dini Theorem, $\rho_B(e_n)$ converges to $\widehat{1}$ uniformly on T. This implies that $\widehat{p}$ is also continuous. Define $\rho\colon K_0(M(B)) \to \mathrm{Aff}(T)$ by defining $\rho([p]) = \widehat{p}$. It is clear that ρ is a well-defined homomorphism.

We now prove (2). It follows from [Zh2, Theorem 4.1] again that we may assume that $p = \sum_{n=1}^{\infty} p_n$ and $q = \sum_{n=1}^{\infty} q_n$, where $p_n, q_n \leq e_{n+1} - e_n$ and the sum converges in the strict topology. Without loss of generality, we may assume that p_n and q_n are not zero. Since B is simple, we have $\widehat{p_1} \ll \widehat{p} = \widehat{q}$. Since $\sum_{n=1}^{\infty} \widehat{q_n}$ converges uniformly on T, there is $n_1 > 1$ such that $\sum_{j=1}^{n_1} \widehat{q_j} \gg \widehat{p_1}$. It follows from [BH, III2.2; III2.3] that there is a partial isometry $v_1 \in B$ such that

$$v_1^* v_1 = p_1 \quad \text{and} \quad v_1 v_1^* \leq \sum_{j=1}^{n_1} q_j.$$

There is $m_1 > 1$ such that

$$\tau(\sum_{j=2}^{m_1} p_j) > \tau(\sum_{j=1}^{n_1} q_j - v_1 v_1^*) \text{ for all } \tau \in T.$$

It follows that there is a partial isometry $u_1 \in B$ such that

$$u_1^* u_1 = \sum_{j=1}^{n_1} q_j - v_1 v_1^* \quad \text{and} \quad u_1 u_1^* \leq \sum_{j=2}^{m_1} p_j.$$

Put $w_1 = v_1 + u_1^*$. Then

$$\sum_{j=1}^{m_1} p_j > w_1^* w_1 \geq p_1 \quad \text{and} \quad \sum_{j=1}^{n_1} q_j = w_1 w_1^* \geq q_1.$$

By induction one constructs a sequence of partial isometries $w_k \in B$ ($w_0 = 0$) such that

$$\sum_{j=1}^{m_k} p_j - \sum_{j=1}^{k-1} w_j^* w_j > w_k^* w_k \geq \sum_{j=1}^{m_{k-1}} p_j - \sum_{j=1}^{k-1} w_j^* w_j$$

and

$$\sum_{j=1}^{n_k} q_j - \sum_{j=1}^{k-1} w_j w_j^* * = w_k w_k^* \geq \sum_{j=1}^{n_{k-1}} q_j - \sum_{j=1}^{k-1} w_j w_j^*,$$

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where $\{n_k\}$ and $\{m_k\}$ are increasing sequences of positive integers.

Set $W = \sum_{k=1}^{\infty} w_k$. One checks that the sum converges in the strict topology and W is a partial isometry in $M(B)$. One then verifies that

$$W^*W = p \quad \text{and} \quad WW^* = q.$$

This proves (2).

Note that (2) also implies that ρ is injective. In fact, if $p \in M(B) \setminus B$ and $q \in B$ are two projections such that $\tau(p) = \tau(q)$ for all $\tau \in T$, then $p \oplus 1$ and $q \oplus 1$ are both in $M(B) \setminus B$. So, by (2), $p \oplus 1$ and $q \oplus 1$ are equivalent. Therefore ρ is injective.

To see ρ is surjective, let $f \in \mathrm{Aff}(T)$. We need to show that f is in the image of ρ. It is clear that it suffices to prove the case for $f \gg 0$. So we may assume that $f \gg 0$. We claim that there exists a sequence of positive functions $\{f_n\}$ in $\rho_B(K_0(B))$ such that $f_n \ll f_{n+1}$ such that $f_n \to f$ uniformly on T.

Let $d_0 = \inf\{f(t) : t \in T\}$. Then $d_0 > 0$.

Let $d_0/2 > \varepsilon > 0$. Since $\rho_A(K_0(B))$ is dense in $\mathrm{Aff}(T)$ (see [BH]), there is $g_1 \in \rho_B(K_0(B)_+)$ such that

$$\|g_1 - (f - \varepsilon)\| < \varepsilon/4.$$

Therefore $(1 - \varepsilon/2)f \ll g_1 \ll f$. By applying the same argument to the function $f - g_1$, we obtain $g_2 \in \rho_B(K_0(B)_+)$ such that $(1 - \varepsilon/4)(f - g_1) \ll g_2 \ll f - g_1$. Note that $g_1 + g_2 \in \rho_B(K_0(B)_+)$. From this the claim follows.

Now we will show that f is in the image of ρ. By replacing f by $f - f_n$ if necessary, we may assume that $f \ll \widehat{e_1}$. There are projections $r_n \in B$ such that $\rho(r_n) = f_n - f_{n-1}$ (with $f_0 = 0$). Since $f \ll \widehat{e_1}$ and f_n converges to f uniformly on T, we may assume that

$$\sum_{k=1}^{n_1}(f_k - f_{k-1}) \ll \rho(e_1) \quad \text{and} \quad \sum_{k=n_1+1}^{\infty}(f_k - f_{k-1}) \ll \rho(e_2 - e_1).$$

Thus we may assume that

$$\sum_{k=1}^{n_1} r_k \le e_1 \quad \text{and} \quad \sum_{k=n_1+1}^{m} r_k \le e_2 - e_1$$

for all $m > n_1$. We obtain $n_2 > n_1$ such that

$$\sum_{k=n_2}^{\infty}(f_k - f_{k-1}) \ll \rho(e_3 - e_2).$$

Therefore we also assume that

$$\sum_{k=n_1+1}^{n_2} r_k \le e_2 - e_2 \quad \text{and} \quad \sum_{k=n_2+1}^{m} r_k \le e_3 - e_2$$

for all $m > n_2$. By induction, we obtain an increasing sequence of integers $\{n_k\}$ such that

$$\sum_{i=n_k+1}^{n_{k+1}} r_i \le e_{k+1} - e_k,\ k = 1, 2, \dots.$$

Note that this implies that $\sum_{n=1}^{\infty} r_n$ converges in the strict topology to a projection $p \in M(B)$. It is easy to see that $\widehat{p} = f$. This proves (1) as well as (3).

Finally we note pBp has real rank zero for all projections $p \in M(B)$ and by [Ln3] $\operatorname{cer}(pAp) \leq \pi$. Thus (4) has been proved in Lemma 3.3 in [Ln4]. ∎

Corollary 1.5 *Let B be as in Theorem 1.4. Then*

(1) $K_1(M(B)/B) = \ker \rho_B$, *and*
(2) *there is a short exact sequence*

$$0 \to \operatorname{Aff}(T)/\rho_A(K_0(B)) \to K_0(M(B)/B) \to K_1(B) \to 0.$$

Proof

From the following six-term exact sequence

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\rho_B} & K_0(M(B)) & \longrightarrow & K_0(M(B)/B) \\ \uparrow & & & & \downarrow \\ K_1(M(B)/B) & \longleftarrow & K_1(M(B)) & \longleftarrow & K_1(B) \end{array}$$

we obtain, by Theorem 1.4,

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\rho_B} & \operatorname{Aff}(T) & \longrightarrow & K_0(M(B)/B) \\ \uparrow & & & & \downarrow \\ K_1(M(B)/B) & \longleftarrow & 0 & \longleftarrow & K_1(B) \end{array}$$

This six-term exact sequence unsplices into

$$K_1(M(B)/B) = \ker \rho_B \text{ and } 0 \to \operatorname{Aff}(T)/\rho_B(K_0(B)) \to K_0(M(B)/B) \to K_1(B) \to 0.$$

∎

Remark 1.6 It should be noted that $\operatorname{Aff}(T)$ as an ordered group does not depend on the choice of the non-zero projection e. In what follows when we write $\operatorname{Aff}(T)$ it is understood that the projection e is fixed.

The following fact will be used in this paper.

Proposition 1.7 *Let G be a dense ordered subgroup of $\mathbb{R}$ containing 1 and let T be a Choquet simplex. Suppose that $h\colon G \to \operatorname{Aff}(T)$ is a positive homomorphism with $h(1) = a$. Then*

$$h(z) = za \quad \text{for all } z \in G.$$

Proof Since G is dense in $\mathbb{R}$, there exists a sequence $g_n > 0$ in G such that $g_n \to 0$. We may assume that $ng_n < 1$ for all n. Therefore

$$nh(g_n) \leq a, \quad n = 1, 2, \ldots.$$

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It follows that $h(g_n) \to 0$. Let $f_n > 0$ in G such that $f_n \to 0$. Thus, for each m, there exists $N(m)$ such that $f_n \leq g_m$ if $n \geq N(m)$. This implies that $h(f_n) \to 0$. Thus h is continuous. For each nonzero integer m, define $\tilde{h}(1/m) = a/m$. Then one checks that $\tilde{h}$ is a positive homomorphism from $\mathbb{Q}G$ to $\mathrm{Aff}(T)$. The same argument above shows that $\tilde{h}$ is also continuous. Fix $z \in G$. Suppose that $r_n \in \mathbb{Q}$ such that $r_n \to z$. Then $\tilde{h}(r_n) \to h(z)$, or $r_n a \to h(z)$. Therefore $h(z) = az$. ■

The following example shows that even in the case that $M(B)/B$ is simple, in general, $KK^1(A, B)$ can not be used to give a meaningful description of extensions of A by B.

Example 1.8 Let A be a unital simple AF-algebra and B be a σ-unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(B)$ and a continuous scale. Let r be an irrational number and $D_r = \{m + nr : m, n \in \mathbb{Z}\}$ $(= \mathbb{Z} \oplus \mathbb{Z}r)$. Suppose $K_0(A) = D_r \oplus \mathbb{Z}$. Define

$$K_0(A)_+ = \{x + z : x \in D_r, x > 0\} \cup \{(0, 0)\}.$$

Suppose that there is a group homomorphism $\theta\colon K_0(A) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ such that $\theta((0, 1)) \neq 0$, where $(0, 1)$ denotes the generator of the last summand $\mathbb{Z}$ of $K_0(A)$. This gives a group homomorphism $\alpha\colon K_0(A) \to K_0(M(B)/B)$ which maps $(0, 1)$ to r. Let $\Phi\colon\ \mathrm{Aff}(T) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ be the quotient map. Then $\theta = \Phi \circ \alpha$ gives one such homomorphism. Since $M(B)/B$ is a purely infinite simple C^*-algebra, it is easy to construct a homomorphism $\tau\colon A \to M(B)/B$ such that $\tau_{*0} = \theta$ (see for example Theorem 4.6). This τ gives an essential extension of A by B which gives an element in $KK^1(A, B)$. Let E be the C^*-algebra determined by τ. Then we have the following commutative diagram:

$$\begin{tikzcd}
K_0(B) \arrow[rr] \arrow[dr, "="'] & & K_0(E) \arrow[rr] \arrow[d] & & K_0(A) \arrow[dl, "\tau_{*0}"'] \arrow[ddd] \\
& K_0(B) \arrow[r] & K_0(M(B)) \arrow[r] & K_0(M(B)/B) \arrow[d] & \\
& K_1(M(B)/B) \arrow[u] & 0 \arrow[l] & K_1(B) \arrow[l] & \\
K_1(A) \arrow[uuu] \arrow[ur] & & K_1(E) \arrow[ll] \arrow[u] & & K_1(B) \arrow[ll] \arrow[ul, "="']
\end{tikzcd}$$

Since the image of τ_{*0} is in $\mathrm{Aff}(T)/\rho_B(K_0(B))$, by (2) in Corollary 1.5 and from the above diagram one concludes that the map from $K_0(A)$ to $K_1(B)$ is zero. Since $K_1(A) = \{0\}$, one further concludes that the map from $K_1(A)$ to $K_0(B)$ is also zero. By the Universal Coefficient Theorem, one computes that $[\tau] \in ext_{\mathbb{Z}}(K_0(A), K_0(B))$. Using the map α or using the fact that $K_0(A)$ is finitely generated free group, $[\tau] = 0$

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in $KK^1(A, B)$. However, there is no homomorphism $h\colon A \to M(B)$ such that $(\pi \circ h)_{*0} = \theta$. Otherwise, since h_{*0} is positive, it maps $\ker \rho_A = \mathbb{Z}$ into zero. That would imply that τ_{*0} maps $\ker \rho_A$ to zero. But we constructed otherwise. Therefore τ is not trivial. Furthermore it can not be even approximately trivial (see Theorem 8.1 below). This shows that even in the case that $M(B)/B$ is simple, $KK^1(A, B)$ cannot be used to give a good description of extensions of A by B.

2 Monomorphisms from $A \otimes \mathcal{O}_2$ into a Purely Infinite Simple C^*-algebra

Definition 2.1 Recall that a family ω of subsets of $\mathbb{N}$ is an *ultrafilter* if

(i) $X_1, \dots, X_n \in \omega$ implies $\bigcap_{i=1}^n X_i \in \omega$,
(ii) $\varnothing \notin \omega$,
(iii) if $X \in \omega$ and $X \subset Y$, then $Y \in \omega$ and
(iv) if $X \subset \mathbb{N}$ then either X or $\mathbb{N} \setminus X$ is in ω.

An ultrafilter is said to be *free*, if $\bigcap_{X \in \omega} X = \varnothing$. The set of free ultrafilters is identified with elements in $\beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Cech compactification of $\mathbb{N}$.

A sequence $\{x_n\}$ (in a normed space for example) is said to *converge to x_0 along ω*, written $\lim_\omega x_n = x_0$, if for any $\varepsilon > 0$ there exists $X \in \omega$ such that $\|x_n - x_0\| < \varepsilon$ for all $n \in X$.

Let $\{B_n\}$ be a sequence of C^*-algebras. We write $l^\infty(\{B_n\})$ for the C^*-algebra $\prod_{n=1}^\infty B_n$. Fix an ultrafilter ω. The ideal of $l^\infty(\{B_n\})$ which consists of those sequences $\{a_n\}$ in $l^\infty(\{B_n\})$ such that $\lim_\omega \|a_n\| = 0$ is denoted by $c_\omega(\{B_n\})$. Define

$$q_\omega(\{A_n\}) = l^\infty(\{B_n\})/c_\omega(\{B_n\}).$$

If $B_n = B, n = 1, 2, \dots$, we use $l^\infty(B)$ for $l^\infty(\{B_n\})$, $c_\omega(B)$ for $c_\omega(\{B_n\})$ and $q_\omega(A)$ for $q_\omega(\{A_n\})$, respectively.

Lemma 2.2 *Let A be a separable C^*-algebra and $\{B_n\}$ be a sequence of unital C^*-algebras. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Suppose that $\psi_m, \phi_m\colon A \to B_m$ are two bounded sequences of maps such that $\psi = \pi \circ \{\psi_m\}, \phi = \pi \circ \{\phi_m\}\colon A \to q_\omega(\{B_n\})$ are two homomorphisms, where $\pi\colon l^\infty(\{B_m\}) \to q_\omega(\{B_n\})$ is the quotient map.*

(1) *Suppose that there are isometries $u_n \in q_\omega(\{B_n\})$ such that*

$$\lim_{n\to\infty} \|u_n^* \psi(a) u_n - \phi(a)\| = 0 \text{ for all } a \in A.$$

Then there is an isometry $w \in q_\omega(\{B_n\})$ such that

$$w^* \psi(a) w = \phi(a) \text{ for all } a \in A.$$

(2) *Suppose that ψ and ϕ are approximately unitarily equivalent in $q_\omega(\{B_n\})$. Then they are unitarily equivalent.*

Proof Suppose that there is a sequence of isometries $u_n \in q_\omega(\{B_n\})$ such that

$$\lim_{n\to\infty} \|u_n^*\psi(a)u_n - \phi(a)\| = 0$$

for all $a \in A$. Let $\{a_n\}$ be a dense sequence of A. By passing to a subsequence if necessary, we may assume that

$$\|u_n^*\psi(a_j)u_n - \phi(a_j)\| < 1/n,\ j = 1, 2, \dots, n.$$

It follows from [Ro4, 6.2.4] that there exists, for each n, a sequence of isometries $u_m^{(n)} \in B_n$ such that $\pi(\{u_m^{(n)}\} = u_n$, where $\pi\colon l^\infty(\{B_n\}) \to q_\omega(\{B_n\})$ is the quotient map. For each n, there exists $X_n \in \omega$ such that for $m \in X_n$,

$$\|(u_m^{(n)})^*\psi_m(a_j)u_m^{(n)} - \phi_m(a_j)\| \le 1/n,\ j = 1, 2, \dots, n.$$

Since ω is free, there is for each j, $Y_j' \in \omega$ such that $j \notin Y_j'$. Let $Y_j = \bigcap_{1\le k\le j} Y_j'$. Then $Y_j \in \omega$ and $\{1, 2, \dots, j\} \cap Y_j = \varnothing$. Let $Z_k' = X_k \cap Y_k$. Then $Z_k' \in \omega$ and $\bigcap_{k\ge N} Z_k' = \varnothing$, $N = 1, 2, \dots$. Put $Z_k = \bigcap_{1\le j\le k} Z_j'$. Then $Z_k \in \omega$ and $Z_1 \supset Z_2 \supset \cdots \supset Z_k \supset \cdots$. Moreover, $Z_k \subset X_k$, $k = 1, 2, \dots$.

Define $l(m)$ as follows. If $m \in Z_k \setminus Z_{k+1}$, define $l(m) = k$, $k = 1, 2, \dots$; and if $m \notin Z_1$, define $l(m) = m$, $m = 1, 2, \dots$. Put $w_m = u_m^{(l(m))} \in B_m$ and $w = \pi(\{w_m\})$. Then, for any $\varepsilon > 0$ and j, let $k > 0$ be an integer such that $1/k < \varepsilon$ and $j \le k$. If $m \in Z_k$, then $m \in Z_{k'} \setminus Z_{k'+1}$ for some $k' \ge k$. Thus $w_m = u_m^{(k')}$ and $m \in X_{k'}$. Therefore

$$\|w_m^*\psi_m(a_j)w_m - \phi_m(a_j)\| = \|u_m^{(k')}\psi_m(a_j)u_m^{(k')} - \phi_m(a_j)\| < 1/k' \le 1/k < \varepsilon$$

for all $j = 1, 2, \dots, k$. This implies that

$$\lim_\omega \|w_m^*\psi_m(a_j)w_m - \phi_m(a_j)\| = 0$$

for all j. Hence

$$w^*\psi(a_j)w = \phi(a_j)\ j = 1, 2, \dots.$$

Since $\{a_n\}$ is dense in A,

$$w^*\psi(a)w = \phi(a) \text{ for all } a \in A.$$

This proves (1).

To prove (2), we note that if u_n are unitaries, so is w. ∎

Lemma 2.3 ([KP], Proposition 1.4) *Let A be a unital separable C^*-subalgebra of a unital purely infinite simple C^*-algebra B and let $\phi\colon A \to B$ be an amenable contractive completely positive linear map. Then for any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there is a non-unitary isometry $s \in B$ such that*

$$\|s^*as - \phi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

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Proof By the assumption that ϕ is amenable there are contractive completely positive linear maps $\sigma\colon A \to M_n(\mathbb{C})$ (for some integer $n > 0$) and $\eta\colon M_n(\mathbb{C}) \to B$ such that

$$\phi \approx_\varepsilon \eta \circ \sigma \text{ on } \mathcal{F}.$$

We may therefore assume that $\phi = \eta \circ \sigma$. It is well known (see for example, [Ln16, 2.3.5]) that there exists a contractive completely positive linear map $\tilde{\sigma}\colon B \to M_n(\mathbb{C})$ such that $\tilde{\sigma}|_A = \sigma$. Define $\tilde{\phi} = \eta \circ \tilde{\sigma}$. Since now B is purely infinite and $\tilde{\phi}$ is amenable, the lemma follows immediately from [KP, 1.4]. ■

Corollary 2.4 *Let B be a unital purely infinite simple C^*-algebra and A be a separable amenable C^*-algebra. Let $\phi, \psi\colon A \to B$ be two monomorphisms. Then there are two sequences of isometries s_n and w_n in B such that*

$$\lim_{n\to\infty} \|s_n^*\phi(a)s_n - \psi(a)\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|w_n^*\psi(a)w_n - \phi(a)\| = 0$$

for all $a \in A$.

The proof of the following proposition is exactly the same as [Ro4, 6.2.6].

Proposition 2.5 *Let B_n be a sequence of purely infinite simple C^*-algebras. Then $q_\omega(\{B_n\})$ is a purely infinite simple C^*-algebra for every free ultrafilter ω.*

Proposition 2.6 ([KP, 3.4]) *Let B_n be a sequence of unital purely infinite simple C^*-algebras and A be a unital separable amenable simple C^*-algebra. Suppose that $j\colon A \to q_\omega(\{B_n\})$ is a monomorphism. Then the relative commutant $j(A)'$ in $q_\omega(\{B_n\})$ is a unital purely infinite simple C^*-algebra.*

Proof Let $a \in j(A)'$ be a nonzero positive element with $\|a\| = 1$. It suffices to show that there is an isometry $s \in j(A)'$ such that $s^*as = 1$. Let $X = sp(a) \subset [0,1]$ and define two homomorphisms $\phi, \psi\colon C(X) \otimes A \to q_\omega(B)$ by

$$\phi(f \otimes b) = f(a)b \quad \text{and} \quad \psi(f \otimes b) = f(1)b$$

for $f \in C(X)$ and $b \in A$. Since A is an amenable simple C^*-algebra, ϕ is a monomorphism. It follows from Proposition 2.5 that $q_\omega(B)$ is a purely infinite simple C^*-algebra. Therefore by Lemma 2.3 there is a sequence of isometries $s_n \in q_\omega(B)$ such that

$$\lim_{n\to\infty} \|s_n^*\phi(x)s_n - \psi(x)\| = 0$$

for all $x \in C(X) \otimes A$. It follows from Lemma 2.2 that there is an isometry $s \in q_\omega(B)$ such that

$$s^*\phi(x)s = \psi(x) \text{ for all } x \in C(X) \otimes A.$$

In particular,

$$s^*as = s^*\phi(\imath \otimes 1)s = \psi(\imath \otimes 1) = 1,$$

where $\imath$ is the identity function $\imath(t) = t$. We also have

$$s^*bs = s^*\phi(1 \otimes b)s = \psi(1 \otimes b) = b.$$

Hence $s \in j(A)'$ by [Ro4, Lemma 6.3.6]. ■

Theorem 2.7 *Let B be a unital purely infinite simple C^*-algebra and A be a unital separable amenable C^*-algebra. Suppose that $\phi, \psi\colon A \otimes \mathcal{O}_2 \to B$ are two monomorphisms. Then ψ and ϕ are approximately unitarily equivalent.*

Proof Let $p_1 = \phi(1_{A\otimes\mathcal{O}_2})$ and $p_2 = \psi(1_{A\otimes\mathcal{O}_2})$. It follows from[Ro1, 3.6] that $\phi|_{1\otimes\mathcal{O}_2}$ and $\psi|_{1\otimes\mathcal{O}_2}$ are approximately unitarily equivalent. It follows that p_1 and q_1 are equivalent in B. Therefore we may assume, without loss of generality, that $p_1 = p_2$. By replacing B by p_1Bp_1, we may further assume that both ϕ and ψ are unital.

Let $\Psi, \Phi\colon A \otimes \mathcal{O}_2 \to l^\infty(B)$ be defined by $\Psi = \{\psi(a), \psi(a), \dots, \psi(a), \dots)$ and $\Phi(a) = \{\phi(a), \phi(a), \dots, \phi(a), \dots)$ for $a \in A$, respectively. Fix a free ultrafilter ω. Put $\bar\phi = \pi \circ \Phi$ and $\bar\psi = \pi \circ \Psi$, where $\pi\colon l^\infty(B) \to q_\omega(B)$ is the quotient map. It follows from [Ro1, 3.6] that $\bar\phi|_{1\otimes\mathcal{O}_2}$ and $\bar\psi|_{1\otimes\mathcal{O}_2}$ are approximately unitarily equivalent. It follows from Lemma 2.2 that they are unitarily equivalent in $q_\omega(B)$. Without loss of generality, we may assume that

$$\bar\phi|_{1\otimes\mathcal{O}_2} = \bar\psi|_{1\otimes\mathcal{O}_2}.$$

Let $D = \bar\phi(1 \otimes \mathcal{O}_2)$. Then $D \cong \mathcal{O}_2$. By [Ro2] $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. Let $\imath\colon \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be defined by $\imath(a) = a \otimes 1$ and let $\lambda\colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism. Then $\lambda \circ \imath$ and $\mathrm{id}_{\mathcal{O}_2}$ are approximately unitarily equivalent.

Let $\gamma\colon A \otimes \mathcal{O}_2 \to A \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \to A \otimes \mathcal{O}_2 \otimes 1$ be the homomorphism induced by $\lambda \circ \imath$ above. Then $\bar\phi \circ \gamma$ is approximately unitarily equivalent to $\bar\phi$ and $\bar\psi \circ \gamma$ is approximately unitarily equivalent to $\bar\psi$. To prove that $\bar\psi$ and $\bar\phi$ are approximately unitarily equivalent it suffices to show that $\bar\phi\circ\gamma$ and $\bar\psi\circ\gamma$ are approximately unitarily equivalent.

There is a unital C^*-subalgebra $D_1 \subset D$ of $q_\omega(B)$ which is isomorphic to $\mathcal{O}_2$ and its commutant contains both the images of $\bar\phi \circ \gamma$ and $\bar\psi \circ \gamma$.

Let D_1' be the commutant in $q_\omega(B)$. Then by Proposition 2.6 D' is purely infinite simple. It follows from Corollary 2.4 that there are isometries $s_n, w_n \in D_1'$ such that

$$\lim_{n\to\infty} \|s_n^*\bar\psi \circ \gamma(a)s_n - \bar\phi(a)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|w_n^*\bar\phi \circ \gamma(a)w_n - \bar\psi(a)\| = 0$$

for all $a \in A$. Since $((D_1)')'$ contains a unital subalgebra D_1 which is isomorphic to $\mathcal{O}_2$, by [Ro4, Lemma 6.3.7], $\bar\phi\circ\gamma$ and $\bar\psi\circ\gamma$ are approximately unitarily equivalent. It follows that $\bar\phi$ and $\bar\psi$ are approximately unitarily equivalent in $q_\omega(B)$. It follows from [Ro4, 6.2.5] that ϕ and ψ are approximately unitarily equivalent. ∎

3 Approximately Unitarily Equivalent Extensions

The purpose of this section is to prove Theorem 3.7 and Theorem 3.9. The statements have been proved for the case that the target C^*-algebra is a separable amenable purely infinite simple C^*-algebra. The problem we deal with in this section is to show a certain absorption property in the absence of "approximate divisibility" for $M(B)/B$.

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Lemma 3.1 *Let B be a non-unital and σ-unital C^*-algebra and A be a separable C^*-algebra. Let $\tau\colon A \to M(B)/B$ be a homomorphism. Then there is a sequence of non-zero mutually orthogonal elements $a_n \in \tau(A)'$, where $\tau(A)'$ is the commutant of $\tau(A)$ in $M(B)/B$.*

Proof Let D be a separable C^*-algebra in $M(B)$ such that $\tau(A) \subset \pi(D)$, where $\pi\colon M(B) \to M(B)/B$ is the quotient map. It follows from [Ln21, Lemma 3.1] that there exists an approximate identity $\{e_n\}$ such that $e_{n+1}e_n = e_n e_{n+1} = e_n$, $n = 1, 2, \dots$, and

$$\|e_k d - d e_k\| \to 0 \text{ as } k \to \infty$$

for all $d \in D$. Fix a subsequence $X \subset \mathbb{N}$, then $a_X = \sum_{n\in X}(e_{n+1} - e_n)$ is a positive element in $M(A)$. Since $\lim_{k\to\infty} \|e_k d - d e_k\| = 0$ for each $d \in A$, $\pi(a_X)\pi(d) = \pi(d)\pi(a_X)$. In other words $a_X \in \tau(A)'$. Suppose that X and Y are two disjoint subsets of $\mathbb{N}$ such that for any $n \in x$ and $m \in Y$, $|n - m| \geq 2$. By the assumption that $e_{n+1}e_n = e_n e_{n+1} = e_n$, we conclude that $a_X a_Y = a_Y a_X = 0$. From this it is easy to see that there exists a sequence of nonzero mutually orthogonal elements in $\tau(A)'$. ■

Lemma 3.2 *Let A be a unital separable amenable C^*-algebra, B be a non-unital but σ-unital simple C^*-algebra with a continuous scale and let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Suppose that $\tau\colon A \to M(B)/B$ is an essential unital extension. Let $\tau_\infty\colon A \to l^\infty(M(B)/B)$ be defined by $\tau_\infty(a) = (\tau(a), \tau(a), \dots)$ and let $\psi = \Phi \circ \tau_\infty$, where $\Phi\colon l^\infty(M(B)/B) \to q_\omega(M(B)/B)$. Then there is a unital C^*-subalgebra $C \cong \mathcal{O}_\infty$ in the commutant of $\psi(A)$ in $q_\omega(M(B)/B)$.*

Proof Let $J\colon M(B)/B \to q_\omega(M(B)/B)$ be defined by $J(b) = \Phi((b, b, \dots, b, \dots))$ for $b \in B$. By (the proof of) [Ln19, 7.4], there exists a unital separable purely infinite simple C^*-algebra $D \subset M(B)/B$ such that $\tau(A) \subset D$. It follows from Lemma 3.1 that there is a sequence of nonzero mutually orthogonal positive elements $\{a_n\}$ in D', the commutant of D in $M(B)/B$. Let $X = sp(a_1)$. Without loss of generality we may assume that $\|a_1\| = 1$ and $1 \in X$. Define $L_1, L_2\colon A \otimes C(X) \to q_\omega(M(B)/B)$ by $L_1(x \otimes f) = \psi(x) f(J(a))$ and $L_2(x \otimes f) = \psi(x) f(1)$ for $x \in A$ and $f \in C(X)$. Define $L_3\colon D \otimes C(X) \to q_\omega(M(B)/B)$ by $L_3(y \otimes f) = y f(J(a))$ for $y \in D$. Since ψ is injective and D is purely infinite simple, one concludes that L_3 is injective. Consequently, L_1 is injective. Now we apply an argument in [KP]. Since $A \otimes C(X)$ is amenable, the Choi-Effros lifting theorem provides unital completely positive lifting $\rho, \sigma\colon A \otimes C(X) \to l^\infty(B)$ of L_1 and L_2 [CE]. Write

$$\rho(a) = (\rho_1(a), \rho_2(a), \dots, \rho_n(a), \dots) \quad \text{and} \quad \sigma(a) = (\sigma_1(a), \sigma_2(a), \dots, \sigma_n(a), \dots)$$

for $a \in A \otimes C(X)$, where ρ_k and σ_k are unital, completely positive maps from $A \otimes C(X)$ into B_k. It follows from [Ro4, 6.3.5 (iii)] that there are non-unitary isometries $s_k \in B_k$ $(k = 1, 2, \dots)$ such that

$$\lim_{n\to\infty} \|s_n^* \rho_n(a) s_n - \sigma_n(a)\| = 0 \quad \text{for all} \quad a \in A \otimes C(X).$$

Put $t_1 = \pi(s_1, s_2, \dots, s_n, \dots) \in l^\infty(B)$. Then t_1 is a non-unitary isometry. It follows that

$$t_1^* \psi(a) t_1 = t_1^* L_1(a \otimes 1) t_1 = L_2(a \otimes 1) = \psi(a)$$

for all $a \in A$. It follows from [Ro4, Lemma 6.3.6] that $t_1 \in (\psi(A))'$. Furthermore,

$$t_1^* J(a_1) t_1 = t_1^* L_1(1 \otimes \imath) t_1 = L_2(1 \otimes \imath) = 1,$$

where $\imath$ is the function $\imath(t) = t$. Let $t_1 t_1^* = q_1$. Then $q_1 \in \psi(A)'$ and $q_1 \in \overline{J(a_1)\psi(A)' J(a_1)}$. We repeat the above argument for $a_2, a_3, \dots$. Then we obtain a sequence of isometries $t_1, t_2, \dots$. in $\psi(A)'$ such that $t_n^* t_n = 1$ and

$$\sum_{i=1}^n t_i t_i^* \in \overline{(\sum_{i=1}^n J(a_i))\psi(A)'(\sum_{i=1}^n J(a_i))}.$$

It follows that $\sum_{i=1}^n a_i \leq 1$. Therefore we obtain a unital C^*-subalgebra $C \cong \mathcal{O}_\infty$ in $\psi(A)'$. ■

Lemma 3.3 *Let A be a unital separable amenable C^*-algebra and C be a unital separable amenable purely infinite simple C^*-algebra. Suppose that $A \otimes C$ is a unital C^*-subalgebra of a unital C^*-algebra B. Then there is an embedding $j\colon A \otimes C \to A \otimes \mathcal{O}_2 \to B$ satisfying the following: for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A \otimes C$ and any integer $n > 0$, there exists a partial isometry $u \in M_{n+1}(B)$ such that $u^* u = 1$, $uu^* = 1 \oplus j(1_{A\otimes C}) \oplus j(1_{A\otimes C}) \oplus \cdots \oplus j(1_{A\otimes C})$ (where $j(1_{A\otimes C})$ repeats n times) and*

$$u^*(\mathrm{id}_{A\otimes C} \oplus j \oplus j \oplus \cdots \oplus j) u \approx_\varepsilon \mathrm{id}_{A\otimes C} \text{ on } \mathcal{F},$$

where j repeats n times.

Proof Let $\varepsilon > 0$ and $\mathcal{G} \subset C$ be a finite subset. It follows from [KP] that there is homomorphism $\imath\colon C \to \mathcal{O}_2 \to C$ satisfying the following: for any $\varepsilon > 0$, any finite subset $\mathcal{G} \subset C$ and any integer n there exists a partial isometry $w \in M_{n+1}(C)$ such that $w^* w = 1_C$, $ww^* = p = 1_C \oplus \imath(1_C) \oplus \imath(1_C) \oplus \cdots \oplus \imath(1_C)$ (where $\imath(1_C)$ repeats n times) and

$$w^*(\mathrm{id}_C \oplus \imath \oplus \imath \oplus \cdots \oplus \imath) w \approx_{\varepsilon/2} \mathrm{id}_C \text{ on } \mathcal{G},$$

where $\imath$ repeats n times. Let $u = 1 \otimes w$ and define $j\colon A \otimes C \to B$ by $j(a \otimes b) = a \otimes \imath(b))$ for $a \in A$ and $b \in C$. One checks that the lemma follows. ■

Lemma 3.4 *Let A be a unital separable amenable C^*-algebra and B be a non-unital but σ-unital simple C^*-algebra with a continuous scale. Suppose that $h\colon A \to M(B)/B$ is a unital injective homomorphism and $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a free ultrafilter. Let*

$$\pi\colon l^\infty(M(B)/B) \to q_\omega(M(B)/B)$$

be the quotient map. Define $H_0\colon A \to l^\infty(M(B)/B)$ by

$$H_0(a) = (h(a), h(a), \dots, h(a), \dots)$$

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and $H = \pi \circ H_0$. Then, there exists an injective homomorphism $j\colon A \to A \otimes \mathcal{O}_2 \to q_\omega(M(B)/B)$ satisfying the following: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any integer $n > 0$ there exist an isometry $u \in M_{n+1}(q_\omega(M(B)/B)$ with $u^*u = 1$, $uu^* = 1 \oplus j(1_A) \oplus \cdots \oplus j(1_A)$ (where $j(1_A)$ repeats n times) such that

$$u^*(H \oplus j \oplus j \oplus \cdots \oplus j)u \approx_\varepsilon H \text{ on } \mathcal{F},$$

where j repeats n times. Moreover, there is $q \in q_\omega(M(B)/B)$ such that $[q] = [H(1_A)]$ and $qj(a) = j(a)q$ for all $a \in A$ and qjq is an injective full homomorphism.

Proof Fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. We identify $h(A)$ with A. It follows from Lemma 3.2 that $H(A)'$ contains a unital C^*-subalgebra $C \cong \mathcal{O}_\infty$. Thus we obtain an injective homomorphism $\Psi\colon A \otimes \mathcal{O}_\infty \to q_\omega(M(B)/B)$. Thus the first part of the lemma follows from this and Lemma 3.3.

To prove the very last part of the lemma, we may assume that $[\imath(1_C)] \neq [1_C]$, where $\imath$ is as in Lemma 3.3. There is a projection $q \in C$ such that $q \leq \imath(1_C)$ and $[q] = [1_C]$. Then $qj(a) = j(a)q$ for all $a \in A$. Since Ψ is injective, for any nonzero element $a \in A$ and $b \in C$, $ab = 0$ implies that $b = 0$. Thus qjq is injective. To see that qjq is full we note that $q_\omega(M(B)/B)$ is purely infinite (see [Ro4, 6.26]). ■

Definition 3.5 Let A be a separable amenable C^*-algebra and B be a σ-unital C^*-algebra. Then $KL(A,B)$ is defined to be $KK(A,B)/\mathcal{T}(A,B)$, where $\mathcal{T}(A,B)$ is the subgroup of stable approximately trivial extensions (see [Ln18]). When A is in $\mathcal{N}$, then $KL(A,B) = KK(A,B)/\operatorname{Pext}(K_*(A), K_*(B))$ (see [Ro3]).

Let C_n be a commutative C^*-algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Then set $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$ (see [Sc1]). One has the following six-term exact sequence (see [Sc1]):

$$\begin{tikzcd}
K_0(A) \arrow[r] & K_0(A, \mathbb{Z}/k\mathbb{Z}) \arrow[r] & K_1(A) \arrow[d, "\mathbf{k}"'] \\
K_0(A) \arrow[u, "\mathbf{k}"'] & K_0(A, \mathbb{Z}/k\mathbb{Z}) \arrow[l] & K_1(A) \arrow[l]
\end{tikzcd}$$

In [DL], $K_i(A, \mathbb{Z}/n\mathbb{Z})$ is identified with $KK^i(\mathbb{I}_n, A)$ for $i = 0, 1$.

As in [DL], we use the notation

$$\underline{K}(A) = \bigoplus_{i=0,1,n\in\mathbb{Z}_+} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-called Bockstein operations (see [DL]). It follows from the definition in [DL] that if $x \in KK(A,B)$, then the Kasparov product $KK^i(\mathbb{I}_n, A) \times x$ gives an element in $KK^i(\mathbb{I}_n, B)$ which we identify with $\operatorname{Hom}(K_i(A, \mathbb{Z}/n\mathbb{Z}), K_0(B, \mathbb{Z}/n\mathbb{Z}))$. Thus one obtains a map

$$\Gamma\colon KK(A,B) \to \operatorname{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big).$$

It is shown by Dadarlat and Loring [DL] that if A is in $\mathcal{N}$ then, for any σ-unital C^*-algebra B, the map Γ is surjective and $\ker\Gamma = \mathrm{Pext}(K_*(A), K_*(B))$. In particular,

$$\Gamma\colon KL(A,B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

is an isomorphism.

We will use the following theorem. It is a consequence of the uniqueness theorem in [Ln16, 5.6.4] which first appeared in [Ln10]. It is proved in [Ln18].

Theorem 3.6 ([Ln18, Theorem 3.9]) *Let A be a separable unital amenable C^*-algebra and let B a unital C^*-algebra. Suppose that $h_1, h_2\colon A \to B$ are two unital homomorphisms such that*

$$[h_1] = [h_2] \text{ in } KL(A,B).$$

Suppose that $h_0\colon A \to B$ is a full unital monomorphism. Then, for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there is an integer n and a unitary $W \in U(M_{n+1}(B))$ such that

$$\|W^*\mathrm{diag}(h_1(a), h_0(a), \dots, h_0(a))W - \mathrm{diag}(h_2(a), h_0(a), \dots, h_0(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Theorem 3.7 *Let A be a separable amenable C^*-algebra and B be a non-unital and σ-unital simple C^*-algebra with continuous scale. Suppose that $\tau_1, \tau_2\colon A \to M(B)/B$ be two essential extensions. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \text{ in } KL(A, M(B)/B).$$

Proof Fix an ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Let $\pi\colon l^\infty(M(B)/B) \to q_\omega(M(B)/B)$ denote the quotient map. Define $\Psi_i\colon A \to l^\infty(M(B)/B)$ by $\Psi_i(A) = (\tau_i(a), \tau_i(a), \dots)$ for $a \in A$. Set $H_i = \pi \circ \Psi_i$. We will show that for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there is a unitary $w \in q_\omega(M(B)/B)$ such that

$$\mathrm{ad}\, w \circ H_1 \approx_{\varepsilon/2} H_2 \text{ on } \mathcal{F}.$$

There are unitaries $u_n \in M(B)/B$ such that $\pi((u_1, u_2, \dots, u_n, \dots)) = w$. Therefore

$$\liminf_{\omega} \|u_n^* h_1(a) u_n - h_2(a)\| \le \varepsilon/2$$

for $a \in \mathcal{F}$. Hence, there exists a subset $X \subset \omega$ such that for all $n \in X$,

$$\|u_n^* h_1(a) u_n - h_2(a)\| < \varepsilon \quad \text{for all} \quad a \in \mathcal{F}.$$

Then the theorem follows.

Let j_1 and q be as in Lemma 3.4 associated with H_1 and j_2 be as in Lemma 3.4 associated with H_2. It follows from Theorem 3.6 that there is a unitary

$$z \in M_{K+1}(q_\omega(M(B)/B))$$

for some integer $K > 0$ such that

$$\mathrm{ad}\, z \circ (H_1 \oplus qj_1q \oplus qj_1q \oplus \cdots \oplus qj_1q) \approx_{\varepsilon/5} H_2 \oplus qj_1q \oplus qj_1q \oplus \cdots \oplus qj_1q \text{ on } \mathcal{F}.$$

Therefore (by adding $(1-q)j_1(1-q) \oplus \cdots \oplus (1-q)j_1(1-q)$) there is a unitary $v \in M_{K+1}(q_\omega(M(B)/B))$ such that

$$\mathrm{ad}\, v \circ (H_1 \oplus j_1 \oplus j_1 \oplus \cdots \oplus j_1) \approx_{\varepsilon/5} H_2 \oplus j_1 \oplus j_1 \oplus \cdots \oplus j_1 \text{ on } \mathcal{F}.$$

In particular, we may assume that $v^*(1 \oplus j(1_A) \oplus \cdots \oplus j(1_A))v = 1$. It follows from Lemma 3.4 and Theorem 2.7 that

$$H_1 \oplus j_1 \oplus j_1 \oplus \cdots \oplus j_1 \sim_{\varepsilon/5} H_1 \text{ on } \mathcal{F}.$$

By Theorem 2.7,

$$j_1 \sim_{\varepsilon/5} j_2.$$

By Lemma 3.4 and Theorem 2.7 again,

$$H_2 \oplus j_1 \oplus j_1 \oplus \cdots \oplus j_1 \sim_{2\varepsilon/5} H_2 \text{ on } \mathcal{F}.$$

Combining these inequalities, we obtain a unitary $w \in q_\omega(M(B)/B)$ such that

$$\mathrm{ad} \circ w H_1 \approx_\varepsilon H_2 \text{ on } \mathcal{F}. \qquad \blacksquare$$

When A satisfies the UCT, we have the following approximate version of Theorem 3.7. This statement is very close to that of [Ln19, Theorem 6.3].

Theorem 3.8 *Let A be a separable amenable C^*-algebra in $\mathcal{N}$. For any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there exists a finite subset $\mathcal{P} \subset \underline{K}(A)$ satisfying the following: if $h_1, h_2, h_3 \colon A \to C$ are three homomorphisms where C is a unital purely infinite simple C^*-algebra such that*

$$[h_1]|_{\mathcal{P}} = [h_2]|_{\mathcal{P}},$$

then there is an integer $n > 0$ and a unitary $u \in M_{n+1}(C)$ such that

$$\mathrm{ad} \circ (h_1 \oplus h_3 \oplus h_3 \oplus \cdots h_3) \approx_\varepsilon h_2 \oplus h_3 \oplus h_3 \oplus \cdots h_3) \text{ on } \mathcal{F},$$

where h_3 repeats n times.

Proof Let $\{\mathcal{P}_n\}$ be a sequence of finite subsets of $\underline{K}(A)$ such that $\bigcup_{n=1}^\infty \mathcal{P}_n = \underline{K}(A)$. Suppose that there are three sequences of homomorphisms $\phi_n, \psi_n, f_n \colon A \to C_n$, where C_n is a sequence of unital purely infinite simple C^*-algebras such that

$$[\phi_n]|_{\mathcal{P}_n} = [\psi_n]|_{\mathcal{P}_n}, \quad n = 1, 2, \ldots.$$

It suffices to show that there exists $N > 0$ and $K > 0$ such that when $n \geq N$ there are unitaries $u_n \in M_{K+1}(C_n)$ satisfying the following:

$$\mathrm{ad}\, u_n(\phi_n \oplus f_n \oplus f_n \oplus \cdots \oplus f_n) \approx_\varepsilon (\psi_n \oplus f_n \oplus f_n \oplus \cdots \oplus f_n) \text{ on } \mathcal{F},$$

where f_n repeats K times. Let $H_1 = \{\phi_n\}, H_2 = \{\psi_n\}$ and $H_3 = \{f_n\}$ be homomorphisms from A into $l^\infty(\{C_n\})$ and let $\bar{H}_i = \pi \circ H_i$, where $\pi\colon l^\infty(\{C_n\}) \to l^\infty(\{C_n\})/c_0(\{C_n\})$ is the quotient map, $i = 1, 2, 3$. So it suffices to show that there exists $K > 0$ such that there is a unitary $U \in M_{K+1}(l^\infty(\{C_n\})/c_0(\{C_n\})$ such that

$$\operatorname{ad} U(\bar{H}_1 \oplus \bar{H}_3 \oplus \bar{H}_3 \oplus \cdots \oplus \bar{H}_3) \approx_\varepsilon \bar{H}_2 \oplus \bar{H}_3 \oplus \bar{H}_3 \oplus \cdots \oplus \bar{H}_3 \text{ on } \mathcal{F},$$

where $\bar{H}_3$ repeats K times.

Since each C_n is a unital purely infinite simple C^*-algebra, it follows from [Ln18, 6.5] that $\bar{H}_3\colon A \to l^\infty(\{C_n\})/c_0(\{C_n\})$ is full. So the theorem follows from Theorem 3.6 if we can show that

$$[\bar{H}_1] = [\bar{H}_2] \text{ in } KL(A, l^\infty(\{C_n\})/c_0(\{C_n\}).$$

It follows from [GL, Corollary 2.1] that, if each C_n is purely infinite and simple,

$$K_i(l^\infty(\{C_n\})) = \prod_n K_i(C_n),\ i = 0, 1,$$

$$K_i(l^\infty(\{C_n\}), \mathbb{Z}/k\mathbb{Z})) \subset \prod_n K_i(C_n, \mathbb{Z}/k\mathbb{Z}), i = 0, 1, k = 2, 3, \ldots$$

and

$$K_i(l^\infty(\{C_n\})/c_0(\{C_n\}) = \prod_n K_i(C_n)/\oplus_n K_i(C_n),\ i = 0, 1,$$

$$K_i(l^\infty(\{C_n\})/c_0(\{C_n\}), \mathbb{Z}/k\mathbb{Z})) \subset \prod_n K_i(C_n, \mathbb{Z}/k\mathbb{Z})/\oplus_n K_i(C_n, \mathbb{Z}/k\mathbb{Z}),\ k = 2, 3, \ldots.$$

Thus, since $[H_1]|_{\mathcal{P}_n} = [H_2]|_{\mathcal{P}_n}$ for each n, we conclude from the above computation that

$$[\bar{H}_1] = [\bar{H}_2] \text{ in } \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}\big(l^\infty(\{C_n\})/c_0(\{C_n\})\big).$$

Therefore the theorem follows. ■

The following is an approximate version of Theorem 3.7.

Theorem 3.9 *Let A be a unital separable amenable C^*-algebra in $\mathcal{N}$ and B be a non-unital but σ-unital simple C^*-algebra with a continuous scale. Suppose that*

$$h_1, h_2\colon A \to M(B)/B$$

are two monomorphisms. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a finite subset $\mathcal{P} \subset \underline{K}(A)$ satisfying the following: if

$$[h_1]|_{\mathcal{P}} = [h_2]|_{\mathcal{P}}$$

then there exists a unitary $u \in M(B)/B$ such that

$$\operatorname{ad} \circ h_1 \approx_\varepsilon h_2 \text{ on } \mathcal{F}.$$

Proof The proof is exactly the same as that of Theorem!3.7 but applying Theorem 3.8 instead of Theorem 3.6. ■

Corollary 3.10 *Let A be a separable amenable C^*-algebra in $\mathcal{N}$ and B be a non-unital but σ-unital simple C^*-algebra with a continuous scale. Let $\tau_1, \tau_2 \colon A \to M(B)/B$ be two essential extensions. Then there exists a sequence of unitaries $u_n \in M(B)/B$ such that*

$$\lim_{n\to\infty} \operatorname{ad} u_n \circ \tau_1(a) = \tau_2(a) \quad \textit{for all} \quad a \in A$$

if and only if $[\tau_1] = [\tau_2]$ in $KL(A, M(B)/B)$.

4 $\mathbf{Ext}_{ap}(A, B)$

Definition 4.1 Let A be a separable C^*-algebra and C be a non-unital and σ-unital C^*-algebra. Let $\tau_1, \tau_2 \colon A \to M(C)/C$ be two essential extensions. We say τ_1 and τ_2 are *strongly approximately unitarily equivalent* if there exists a sequence of unitaries $U_n \in M(C)$ such that

$$\lim_{n\to\infty} \pi(U_n)^*\tau_1(a)\pi(U_n) = \tau_2(a) \quad \text{for all} \quad a \in A.$$

An essential extension $\tau \colon A \to M(C)/C$ is said to be *approximately trivial* if there is a sequence of trivial extensions $\tau_n \colon A \to M(C)/C$ such that $\tau(a) = \lim_{n\to\infty} \tau_n(a)$ for all $a \in A$. Denote by $\mathbf{Ext}_{ap}(A, B)$ the set of approximately unitarily equivalent classes of essential extensions.

Let B be a non-unital but σ-unital simple C^*-algebra with a continuous scale and $A \in \mathcal{N}$. In this section we will classify essential extensions of A by B:

$$0 \to B \to E \to A \to 0$$

up to strong approximately unitary equivalence.

Lemma 4.2 *Let B be a unital purely infinite simple C^*-algebra and let G_i be a countable subgroup of $K_i(B)$ $(i = 0, 1)$. There exists a unital separable purely infinite simple C^*-algebra $B_0 \subset B$ such that $K_i(B_0) \supset G_i$ and $j_{*i} = \mathrm{id}_{K_i(B_0)}$, where $j \colon B_0 \to B$ is the embedding.*

Proof Since B is purely infinite, all elements in $K_0(B)$ and in $K_1(B)$ can be represented by projections and unitaries in B, respectively. Let $p_1, \dots, p_n, \dots$ be projections in B and $u_1, u_2, \dots, u_n, \dots$ be unitaries in B such that $\{p_n\}$ and $\{u_n\}$ generate G_0 and G_1, respectively. Let B_1 be a unital separable purely infinite simple C^*-algebra containing $\{p_n\}$ and $\{u_n\}$ (see the proof of 7.4 in [Ln19]). Note that $K_i(B_1)$ is countable. The embedding $j_1 \colon B_1 \to B$ gives homomorphisms

$$(j_1)_{*i} \colon K_0(B_1) \to K_i(B).$$

Let $F_{1,i}$ be the subgroup of $K_0(B_1)$ generated by $\{p_n\}$ and $\{u_n\}$, respectively. It is clear that $(j_1)_{*i}$ is injective on $F_{1,i}$, $i = 0, 1$. In particular, the image of $(j_1)_{*i}$ contains G_i, $i = 0, 1$. Let $N'_{1,i} = \ker(j_1)_{*i}$ and let $N_{1,i}$ be the set of all projections (if $i = 0$), or unitaries (if $i = 1$) in B_1 which have images in $N'_{1,i}$. Let $\{p_{1,n}\}$ be a dense subset of $N_{1,0}$ and $\{u_{1,n}\}$ be a dense subset of $N_{1,1}$, respectively. Fix a nonzero projection $e \in B_1$ such that $[e] = 0$ in $K_0(B)$. For each $p_{1,n}$, there exists a partial isometry $w_{1,n} \in B$ such that $e = w_{1,n}^* w_{1,n}$ and $w_{1,n} w_{1,n}^* = p_{1,n}$, $n = 1, 2, \ldots$. For each $u_{1,n}$, there are unitaries $z_{1,n,k} \in B$, $k = 1, 2, \ldots, m(n)$ such that

$$\|z_{1,n,1} - 1\| < 1/2, \|z_{1,n,m(n)} - u_{1,n}\| < 1/2 \quad \text{and} \quad \|z_{1,n,k} - z_{1,n,k+1}\| < 1/2,$$

$k = 1, 2, \ldots, m(n)$, $n = 1, 2, \ldots$. Let B_2 be a separable unital purely infinite simple C^*-algebra containing B_1 and all $\{w_{1,n}\}$ and $\{z_{1,n,k}\}$. Note that if $p \in B_1$ is a projection and $[p] \in N_{1,0}$, then $[p] = 0$ in $K_0(B_2)$. Similarly, if $u \in B_1$ and $[u] \in N_{1,1}$, then $[u] = 0$ in B_2. Suppose that B_l has been constructed. Let $j_l \colon B_l \to B$ be the embedding. Let $N_{l,i} = \ker(j_l)_{*i}$, $i = 0, 1$. As before, we obtain a unital separable purely finite simple C^*-algebra $B_{l+1} \supset B_l$ such that every projection $p \in B_l$ with $[p] \in N_{l,0}$ has the property that $[p] = 0$ in $K_0(B_{l+1})$, and every unitary $u \in B_l$ with $[u] \in N_{l,1}$ has the property that $[u] = 0$ in $K_1(B_{l+1})$. Let B_0 be the closure of $\bigcup_{l=1}^\infty B_l$. Since each B_l is purely infinite and simple, so is B_0. Note also that B_0 is separable. Let $j \colon B_0 \to B$ be the embedding.

We claim that j_{*i} is injective. Suppose that $p \in B_0$ is a projection such that $[p] \in \ker j_{*0}$ and $[p] \neq 0$ in B_0. Without loss of generality, we may assume that $p \in B_l$ for some large integer l. Then $[p]$ must be in the $\ker(j_l)_{*0}$. By the construction, $[p] = 0$ in $K_0(B_{l+1})$. This would imply that $[p] = 0$ in $K_0(B_0)$. Thus j_{*0} is injective. Exactly the same argument shows that j_{*1} is also injective. The lemma then follows. ■

Lemma 4.3 *Let B be a unital purely infinite simple C^*-algebra. Suppose that $G_i \subset K_i(B)$ and $F_i(k) \subset K_i(B, \mathbb{Z}/k\mathbb{Z})$ are countable subgroups such that the image of $F_i(k) \subset K_i(B, \mathbb{Z}/k\mathbb{Z})$ in $K_{i-1}(B)$ is contained in G_{i-1} ($i = 0, 1, k = 2, 3, \ldots$). Then there exists a separable unital purely infinite simple C^*-algebra $C \subset B$ such that $K_i(C) \supset G_i$, $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset F_i(k)$ and the embedding $j \colon C \to B$ induces an injective map $j_{*i} \colon K_0(C) \to K_i(B)$ and an injective map $j_* \colon K_i(C, \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z})$, $k = 2, 3, \ldots$.*

Proof It follows from Lemma 4.2 that there is a separable unital purely infinite simple C^*-algebra C_1 such that $K_0(C_1) \supset G_0$, $K_1(C_1) \supset G_1$ and j induces an identity map on $K_0(C_1)$ and $K_1(C_1)$, where $j \colon C \to B$ is the embedding. Let $K_i(C_1) = \{g_1^{(i)}, g_2^{(i)}, \ldots, \}$. Suppose that $\{s_1^{(i)} i, s_2^{(i)}, \ldots, \}$ be a subset of $K_{i-1}(B)$ such that the map from $K_i(B, \mathbb{Z}/k\mathbb{Z})$ to $K_{i-1}(B)$ maps $s_j^{(i)}$ to $g_j^{(i)}$. For each $z^{(i)} \in K_i(C_1, \mathbb{Z}/k\mathbb{Z})$, there is $s_j^{(i)}$ such that $z^{(i)} - s_j^{(i)} \in K_i(B)/kK_i(B)$. Since $K_i(C_1)$ is countable, the set of all possible $z^{(i)} - s_j^{(i)}$ is countable. Thus one obtains a countable subgroup $G_i^{(')}$ which contains $K_i(C_1)$ such that $G_i^{(')}/kK_i(B)$ contains the above the mentioned countable set as well as $F_i(k) \cap (K_i(B)/kK_i(B))$ for each k. Since the union of countably many countable sets is still countable, we obtain a countable subgroup $G_i^{(2)} \subset K_i(B)$ such

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that $G_i^{(2)}$ contains $G_i^{(')}$ and $kK_i(B) \cap G_i^{(2)} = kG_i^{(2)}$, $k = 1, 2, \dots$, and $i = 0, 1$. Note also $F_i(k) \cap (K_i(B)/kK_i(B)) \subset G_i^{(2)}/kK_i(B)$. By applying Lemma 4.2, we obtain a separable purely infinite simple C^*-algebra $C_2 \supset C_1$ such that $K_i(C_2) \supset G_i^{(2)}$ and embedding from C_2 to B gives an injective map on $K_i(C_2)$, $i = 0, 1$. Repeating what we have done above, we obtain an increasing sequence of countable subgroups $G_i^{(n)} \subset K_i(B)$ such that $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)}$ for all k and $i = 0, 1$ and an increasing sequence of separable purely infinite simple C^*-subalgebras C_n such that $K_i(C_n) \supset G_i^{(n)}$, and the embedding from C_n into B gives an injective map on $K_i(C_n)$, $i = 0, 1$, and $n = 1, 2, \dots$. Moreover $F_i^{(k)} \cap (K_i(B)/kK_i(B)) \subset K_i(C_n)/kK_i(B)$. Let C denote the closure of $\bigcup_n C_n$ and $j\colon C \to B$ be the embedding. Then C is a separable purely infinite simple C^*-algebra and j_{*i} is an injective map, $i = 0, 1$. We claim that $K_i(C) \cap kK_i(B) = kK_i(C)$, $k = 1, 2, \dots$, and $i = 0, 1$. Note that $K_i(C) = \bigcup_n G_i^{(n)}$. Since $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)} \subset kK_i(C)$, we see that $K_i(C) \cap kK_i(B) = kK_i(C)$, $i = 0, 1$. Thus $K_i(C)/kK_i(C) = K_i(C)/kK_i(B)$. Since $K_i(C)/kK_i(B) \supset F_i^{(k)} \cap (K_i(B)/kK_0(B))$, we conclude also that $K_i(C, \mathbb{Z}/k\mathbb{Z})$ contains $F_i(k)$. Since j_{*0} is injective, j induces an injective map from $K_0(C)/kK_0(C)$ into $K_0(B)/kK_0(B)$ for all integer $k \geq 1$. Using this fact and the fact that $j_{*i}\colon K_i(C) \to K_i(B)$ is injective by chasing the following commutative diagram,

$$
\begin{tikzcd}
K_0(C) \arrow[rr] \arrow[dr, "j_{*0}"'] & & K_0(C, \mathbf{Z}/k\mathbf{Z}) \arrow[rr] \arrow[d, "j_*"'] & & K_1(C) \arrow[dl, "j_{*1}"'] \arrow[ddd] \\
& K_0(B) \arrow[r] & K_0(B, \mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(B) \arrow[d] & \\
& K_0(B) \arrow[u] & K_1(B, \mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(B) \arrow[l] & \\
K_0(C) \arrow[uuu] \arrow[ur, "j_{*0}"'] & & K_1(C, \mathbf{Z}/k\mathbf{Z}) \arrow[ll] \arrow[u, "j_*"'] & & K_1(C) \arrow[ll] \arrow[ul, "j_{*1}"']
\end{tikzcd}
$$

one sees that j induces an injective map from $K_i(C, \mathbb{Z}/k\mathbb{Z})$ to $K_i(B, \mathbb{Z}/k\mathbb{Z})$. ■

Theorem 4.4 *Let A be a unital separable amenable C^*-algebra in $\mathcal{N}$ and B be a non-unital but σ-unital simple C^*-algebra with a continuous scale. Then, for any $x \in KL(A, M(B)/B)$, there exists a monomorphism $h\colon A \to M(B)/B$ such that $[h] = x$.*

Proof Put $Q = M(B)/B$. Since A satisfies the UCT, we may view x as an element in $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q))$. Note that $K_i(A)$ is a countable abelian group ($i = 0, 1$). Let $G_0^{(i)} = \gamma(x)(K_i(A))$, $i = 0, 1$, where $\gamma\colon \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q)) \to \mathrm{Hom}(K_*(A), K_*(Q))$ is the surjective map. Then $G_0^{(i)}$ is a countable subgroup of $K_i(Q)$, $i = 0, 1$. Consider

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the following commutative diagram:

$$\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr, "\gamma(x)"] & & K_0(A, \mathbf{Z}/k\mathbf{Z}) \arrow[rr] \arrow[d, "\times x"] & & K_1(A) \arrow[dl, "\gamma(x)"'] \arrow[ddd] \\
& K_0(Q) \arrow[r] & K_0(Q, \mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(Q) \arrow[d] & \\
& K_0(Q) \arrow[u] & K_1(Q, \mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(Q) \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur, "\gamma(x)"] & & K_1(A, \mathbf{Z}/k\mathbf{Z}) \arrow[ll] \arrow[u, "\times x"'] & & K_1(A) \arrow[ll] \arrow[ul, "\gamma(x)"']
\end{tikzcd}$$

It follows from Lemma 4.3 that there is a unital purely infinite simple C^*-algebra $C \subset Q$ such that $K_i(C) \subset G_0^{(i)}$, $K_i(C) \cap kK_i(Q) = kK_i(C)$, $k = 1, 2, \dots,$ and $i = 0, 1$, and the embedding $j \colon C \to Q$ induces injective maps on $K_i(C)$ as well as on $K_i(C, \mathbb{Z}/k\mathbb{Z})$ for all k and $i = 0, 1$. Moreover $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset (\times x)(K_i(A, \mathbb{Z}/k\mathbb{Z}))$ for $k = 1, 2, \dots.$ and $i = 0, 1$. We have the following commutative diagram:

$$\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr, "\gamma(x)"] & & K_0(A, \mathbf{Z}/k\mathbf{Z}) \arrow[rr] & & K_1(A) \arrow[dl, "\gamma(x)"'] \arrow[ddd] \\
& K_0(C) \arrow[r] & K_0(C, \mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(C) \arrow[d] & \\
& K_0(C) \arrow[u] & K_1(C, \mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(C) \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur, "\gamma(x)"] & & K_1(A, \mathbf{Z}/k\mathbf{Z}) \arrow[ll] & & K_1(A) \arrow[ll] \arrow[ul, "\gamma(x)"']
\end{tikzcd}$$

We will add two more maps on the above diagram. From the fact that the image of $K_i(A, \mathbb{Z}/k\mathbb{Z})$ under $\times x$ is contained in $K_i(C, \mathbb{Z}/k\mathbb{Z})$, $(k = 2, 3, \dots, i = 0, 1)$, we obtain two maps $\beta_i \colon K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_i(C, \mathbb{Z}/k\mathbb{Z})$, $k = 2, 3, \dots, i = 0, 1$ such that $j_* \circ \beta_i = \times x$ and obtain the following commutative diagram:

$$\begin{tikzcd}
K_0(A) \arrow[rr] \arrow[dr, "\gamma(x)"'] & & K_0(A, \mathbf{Z}/k\mathbf{Z}) \arrow[rr] \arrow[d, "\beta_0"'] & & K_1(A) \arrow[dl, "\gamma(x)"'] \arrow[ddd] \\
& K_0(Q) \arrow[r] & K_0(C, \mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(C) \arrow[d] & \\
& K_0(C) \arrow[u] & K_1(C, \mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(C) \arrow[l] & \\
K_0(A) \arrow[uuu] \arrow[ur, "\gamma(x)"'] & & K_1(A, \mathbf{Z}/k\mathbf{Z}) \arrow[ll] \arrow[u, "\beta_1"'] & & K_1(A) \arrow[ll] \arrow[ul, "\gamma(x)"]
\end{tikzcd}$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(A,\mathbb{Z}/mn\mathbb{Z}) & \longrightarrow & K_i(A,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & K_{i-1}(A,\mathbb{Z}/m\mathbb{Z}) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K_i(Q,\mathbb{Z}/mn\mathbb{Z}) & \longrightarrow & K_i(Q,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & K_{i-1}(Q,\mathbb{Z}/m\mathbb{Z}) & \longrightarrow
\end{array}$$

Since $j_* \circ \beta_i = \times x$ and all vertical maps in the following diagram are injective

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C,\mathbb{Z}/mn\mathbb{Z}) & \longrightarrow & K_i(C,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & K_{i-1}(C,\mathbb{Z}/m\mathbb{Z}) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K_i(Q,\mathbb{Z}/mn\mathbb{Z}) & \longrightarrow & K_i(Q,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & K_{i-1}(Q,\mathbb{Z}/m\mathbb{Z}) & \longrightarrow
\end{array}$$

we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(A,\mathbb{Z}/mn\mathbb{Z}) & \longrightarrow & K_i(A,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & K_{i-1}(A,\mathbb{Z}/m\mathbb{Z}) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K_i(C,\mathbb{Z}/mn\mathbb{Z}) & \longrightarrow & K_i(C,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & K_{i-1}(C,\mathbb{Z}/m\mathbb{Z}) & \longrightarrow
\end{array}$$

Thus we obtain an element $y \in KL(A,C)$ such that $y \times [j] = x$. Since A satisfies the UCT, one checks that $KL(A,C) = KL(A \otimes \mathcal{O}_\infty, C)$. It follows from [Ln20, 6.6 and 6.7] that there exists a homomorphism $\phi\colon A \otimes \mathcal{O}_\infty \to C \otimes \mathcal{K}$ such that $[\phi] = y$. Define $\psi = \phi|_{A\otimes 1}$. It is then easy to check that $[\psi] = y$. Since A is unital, we may assume that the image of ψ is in $M_m(C)$ for some integer $m \geq 1$. Since C is a unital purely infinite simple C^*-algebra, 1_m is equivalent to a projection in C. Thus we may further assume that ψ maps A into C. Put $h_1 = j \circ \psi$. To obtain a monomorphism, we note that there is an embedding $\imath\colon A \to \mathcal{O}_2$ (see [KP, Theorem 2.8]). Since $M(B)/B$ is purely infinite, we obtain a monomorphism $\psi\colon \mathcal{O}_2 \to M(B)/B$. Let $e = \psi(1_{\mathcal{O}_2})$. There is a partial isometry $w \in M_2(M(B)/B)$ such that $w^*w = 1_{M(B)/B}$ and $ww^* = 1 \oplus e$. Define $h = w^*(h_1 \oplus \psi \circ \imath)w$. One checks that $[h] = [h_1]$ and h is a monomorphism. ■

Theorem 4.5 *Let A be a unital separable amenable C^*-algebra in $\mathcal{N}$ and B be a non-unital but σ-unital simple C^*-algebra with a continuous scale. Let $\tau_1, \tau_2\colon A \to M(B)$ be two unital essential extensions of A by B. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \ \text{ in } KL(A, M(B)/B).$$

Proof We only need to prove the "if" part of the statement. Suppose that $[\tau_1] = [\tau_2]$ in $KL(A, M(B)/B)$. It follows from Corollary 3.10 that there is a sequence of unitaries $w_n \in M(B)/B$ such that

$$\lim_{n\to\infty} \|w_n^*\tau_1(a)w_n - \tau_2(a)\| = 0 \quad \text{for all} \quad a \in A. \qquad ■$$

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Theorem 4.6 *Let A be a separable amenable C^*-algebra in $\mathcal{N}$ and B be a non-unital but σ-unital simple C^*-algebra with a continuous scale. Then there is a bijection:*

$$\Gamma\colon \mathbf{Ext}_{ap}(A,B) \to KL(A, M(B)/B).$$

Proof This follows immediately from Theorem 4.5 and Corollary 3.10. ∎

Corollary 4.7 *Let A be a unital separable amenable C^*-algebra satisfying the UCT and B be a non-unital but σ-unital simple C^*-algebra with a continuous scale. Let τ be a unital essential extension and $\psi\colon A \to M(B)$ be a contractive completely positive linear map such that $\pi \circ \psi = \tau$. Suppose that $[\tau] = [t]$ in $KL(A, M(B))$ for some trivial extension t. Then, there exists a sequence of monomorphisms $h_n\colon A \to M(B)$ such that*

$$\lim_{\to\infty} \pi \circ \big(h_n(a) - \psi(a)\big) = 0$$

for all $a \in A$.

Example 4.8 Let B be a non-unital separable simple C^*-algebra with finite trace and $K_0(B) = \mathbb{Q}$. So B has a continuous scale and $K_0(M(B)/B) = \mathbb{R}/\mathbb{Q}$. Let $\xi \neq 0, 1$ in $\mathbb{R}/\mathbb{Q}$. Suppose that A is a unital separable amenable C^*-algebra which satisfies the UCT and suppose that there are two nonzero elements g_1, g_2 in $K_0(A)$ such that $[1_A] = g_1$ and the subgroup generated g_1 and g_2 is not cyclic. Since $\mathbb{R}/\mathbb{Q}$ is divisible, there is a group homomorphism $\alpha\colon K_0(A) \to K_0(M(B)/B)$ such that $\alpha(g_1) = 1$ and $\alpha(g_2) = \xi$. It follows from Theorem 4.6 that there is an essential unital extension $\tau_\xi\colon A \to M(B)/B$ such that $(\tau_\xi)_{*0} = \alpha$ and $(\tau_\xi)_{*1} = 0$. Since $K_0(B) = \mathbb{Q}$ and $K_1(B) = 0$, we compute that $[\tau_\xi] = 0$ in $KK(A, B)$ for any such ξ. However, $[\tau_\xi] \neq [\tau_{\xi'}]$ in $KL(A, M(B)/B)$ if $\xi \neq \xi'$. This shows that there are uncountably many non-equivalent essential extensions which represent the same element in $KK^1(A, B)$. This example shows how $KL(A, M(B)/B))$ can be used to compute $\mathbf{Ext}_{ap}(A, B)$, while $KK^1(A, B)$ fails.

5 Examples

Theorem 4.6 provides a complete classification of $\mathbf{Ext}_{ap}(A, B)$. However, it is not immediately clear which elements in $KL(A, M(B)/B)$ give an approximate trivial extension or a quasidiagonal extension. It turns out it is rather a complicated problem. First of all from item (1) below, it could be the case that there are no essential extensions which are approximately trivial. Second, item (2) and item (3) below show that that $[\tau] = 0$ in $KL(A, M(B)/B)$ does not imply that τ is an approximately trivial extension. In this section we will discuss these problems.

In this section B is a non-unital and σ-unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(B)$ and with a continuous scale.

We will show the following:

(1) There are A and B such that there are no trivial essential extensions of A by B.
(2) There are essential extensions τ such that $[\tau] = 0$ in $KL(A, M(B)/B)$ which are not limits of trivial extensions.

(3) For the same A and B as in (2), there are trivial essential extensions τ such that $[\tau] \neq 0$.

Example 5.1 Let A be a unital separable amenable C^*-algebra and B be a non-unital but σ-unital simple C^*-algebra (with a continuous scale). It is possible that there are no essential trivial extensions of the form:

$$0 \to B \to E \to A \to 0.$$

For example, let $A = \mathcal{O}_n$ $(n \geq 2)$ and B be any non-unital AF-algebra with a continuous scale. There are many extensions of A by B. This is because $M(B)/B$ is purely infinite simple and one can easily find monomorphisms from $\mathcal{O}_n$ into $M(B)/B$. But none of them are splitting. In fact there is no monomorphism $h \colon A \to M(B)$. Since $M(B)$ admits a tracial state, $h(A)$ would have a tracial state too. But this is impossible.

From this example, one sees clearly that for many C^*-algebras A there is no single essential trivial extension of A by B. Therefore some restriction on A is needed to guarantee that there are trivial essential extensions.

Lemma 5.2 *Let A be a unital AF-algebra such that there is a positive homomorphism $\alpha \colon K_0(A) \to \mathrm{Aff}(T)$. Then there exists a homomorphism $h \colon A \to M(B)$ such that $h_{*0} = \alpha$ and $h(A) \cap B = \{0\}$.*

Proof It is easy to see and well known that the lemma holds for the case that A is finite dimensional. Let F be a finite dimensional C^*-algebra. Suppose that $h_1, h_2 \colon F \to M(B)$ are two homomorphisms such that $h_i(F) \cap B = \{0\}$. Suppose also that $(h_1)_{*0} = (h_2)_{*0}$. Then by (2) in Theorem 1.4, h_1 and h_2 are unitarily equivalent.

Now let A be the closure of $\bigcup_{n=1}^{\infty} A_n$, where $A_n \subset A_{n+1}$ and $\dim A_n < \infty$. Denote by j_n the embedding from A_n to A. Let $\alpha_n = \alpha \circ (j_n)_{*0}$. Let $h_1 \colon A_1 \to M(A)$ be such that $h_1(A_1) \cap B = \{0\}$ and $(h_1)_{*0} = \alpha_1$. Suppose that $h_m \colon A_m \to M(B)$ has been defined such that $h_m|_{A_j} = h_j$ for $j < m$, $h_m(A_m) \cap B = \{0\}$ and $(h_m)_{*0} = \alpha_m$. Let $\phi_{m+1} \colon A_{m+1} \to M(B)$ be such that $\phi_{m+1}(A_{m+1}) \cap M(B) = \{0\}$ and $(\phi_{m+1})_{*0} = \alpha_{m+1}$. Let $\imath_m \colon A_m \to A_{m+1}$ be the embedding. Then we have $\alpha_m = (h_m)_{*0} = (\phi_{m+1} \circ \imath_m)$. From what we have shown, there is a unitary $u_{m+1} \in M(B)$ such that

$$\mathrm{ad} \circ \phi_{m+1} \circ \imath_m = h_m.$$

Put $h_{m+1} = \mathrm{ad} \circ \phi_{m+1}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
A1 \arrow[r] \arrow[d, "h_1"] & A2 \arrow[r] \arrow[d, "h_2"] & A3 \arrow[r] \arrow[d, "h_3"] & \cdots & A \\
M(B) \arrow[r, "id_{M(B)}"] & M(B) \arrow[r, "id_{M(B)}"] & M(B) \arrow[r, "id_{M(B)}"] & \cdots & M(B)
\end{tikzcd}$$

It follows that there is a monomorphism $h \colon A \to M(B)$ such that $h(A) \cap B = \{0\}$. ∎

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Theorem 5.3 *Let A be a separable amenable C^*-algebra satisfying the UCT. Suppose that A can be embedded into a unital simple AF-algebra. Then for any B there exists an essential trivial extension τ of A by B.*

Proof Suppose that C is a unital simple AF-algebra and $j\colon A \to C$ is an embedding. Let t be a normalized trace on C. Define $\beta\colon K_0(C) \to \mathrm{Aff}(T)$ by $\beta([p]) = t(p)[1_{M(B)/B}]$ for projection $p \in C$. Then β is a positive homomorphism. It follows from Lemma 5.2 that there is a monomorphism $h\colon C \to M(B)$ such that $h_{*0} = \beta$ and $h(C) \cap B = \{0\}$. Define $\phi\colon A \to M(B)$ by $\phi = h \circ j$. One sees that ϕ give an essential trivial extension of A by B. ■

Suppose that there are trivial essential extensions of A by B. One would like to know when an extension is trivial, or when an extension is the limit of trivial extensions.

Example 5.4 There are essential extensions τ which are not approximately trivial but $[\tau] = 0$ in $KL(A, M(B)/B)$.

Let A be a unital separable amenable C^*-algebra and let τ be an essential extension of A by B such that $[\tau] = 0$ in $KL(A, M(B)/B)$. Such τ exists (by Theorem 4.6 or by first mapping A to $\mathcal{O}_2$ and then mapping $\mathcal{O}_2$ into $M(B)/B$).

To be more precise, we let A be the unital simple AF-algebra with $K_0(A) = D_\theta$, where θ is an irrational number and

$$D_\theta = \{m + n\theta : m, n \in \mathbb{Z}\}.$$

with the usual order inherited from $\mathbb{R}$. We may assume that $[1_A] = 1$. Let B be a non-unital (non-zero) hereditary C^*-subalgebra of the UHF-algebra with $K_0(B) = \mathbb{Z}[1/2]$. Note that B has a unique normalized trace, so it has a continuous scale. We further assume that $[1_{M(B)/B}] = 0$. So there is an essential extension τ of A by B such that $[\tau] = 0$. However, there is no (non-zero) positive homomorphism $\alpha\colon K_0(A) \to K_0(B)$. If there is a trivial extension τ of A by B with $[\tau] = 0$ in $KL(A, M(B)/B)$, then $\tau_{*0}\colon K_0(A) \to K_0(M(B)/B)$ is zero. It follows Lemma 5.2 that

$$K_0(M(B)/B) = \mathrm{Aff}(T(B))/K_0(B) = \mathbb{R}/\mathbb{Z}[1/2].$$

If τ were trivial, there would be a monomorphism $h\colon A \to M(B)$ such that $h|_{*0}$ maps $K_0(A)$ to $K_0(B) \subset \mathrm{Aff}(T(B))$ positively. However there is no positive homomorphism from D_θ into $\mathbb{Z}[1/2]$. In fact any positive homomorphism from D_θ into $\mathbb{R}$ has to be the form (see Corollary 1.7)

$$h_{*0}(r) = (h)_{*0}(1)r \text{ for all } r \in D_\theta.$$

So τ can never be trivial. Furthermore, τ cannot be approximately trivial. To see this, assume that $\tau_n\colon A \to M(B)/B$ are trivial extensions such that

$$\lim_{n\to\infty} \tau_n(a) = \tau(a)$$

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for all $a \in A$. Let $G_0 \subset K_0(A)$ which contains 1 and θ. Thus, for all large n, $(\tau_n)_{*0}(\theta) = 0$. Suppose that $h_n\colon A \to M(B)$ such that $\pi \circ h_n = \tau_n$, $n = 1, 2, \dots$, where $\pi\colon M(B) \to M(B)/B$ is the quotient map. Thus $(h_n)_{*0}$ is a positive homomorphism from D_θ into $\mathbb{R}$. From the above expression of h_{*0} (see Corollary 1.7) we see that $(h_n)_{*0}$ cannot map both 1 and θ into rational numbers. In other words, such an h_n does not exist. Hence τ is not approximately trivial.

Example 5.5 Nevertheless, there are essential trivial extensions of A by B such that $[\tau] \neq 0$ in $KL(A, M(B)/B)$.

Let s be the unique normalized trace on B. Suppose that $[1_A] = 1$ in D_θ. Let $\beta\colon D_\theta \to \mathbb{R}$ the usual embedding. It follows from Lemma 5.2 that there is a monomorphism $h\colon A \to M(B)$ such that $h_{*0} = \beta$ and $h(A) \cap B = \{0\}$. Let $\tau = \pi \circ h$, where $\pi\colon M(B) \to M(B)/B$ is the quotient map. Then τ is a trivial essential extension. However, $\tau_{*0}\colon \mathbb{Q} \to \mathbb{R}/\mathbb{Z}[1/2] (\cong K_0(M(B)/B))$ is not zero. Therefore $[\tau] \neq 0$ in $KL(A, M(B)/B)$.

6 Quasidiagonal Extensions—General and Infinite Cases

Definition 6.1 Let A be a separable C^*-algebra, C be a non-unital but σ-unital C^*-algebra and $\tau\colon A \to M(C)/C$ be an essential extension. Let $\pi\colon M(C) \to M(C)/C$ be the quotient map. Set $E = \pi^{-1}(\tau(A))$. The extension τ is said to be *quasidiagonal* if there exists an approximate identity $\{e_n\}$ of C consisting of projections such that

$$\lim_{n\to\infty} \|e_n b - b e_n\| = 0$$

for all $b \in E$.

Suppose that there is a bounded linear map $L\colon A \to M(B)$ such that $\pi \circ L = \tau$. Then

$$\|e_n L(a) - L(a) e_n\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$.

In this section and the next, we will study quasidiagonal extensions. The first question is when quasidiagonal extensions exist.

Theorem 6.2 *Let A be a separable amenable C^*-algebra and B be a non-unital and σ-unital C^*-algebra. Suppose that $\tau\colon A \to M(B)/B$ is an essential quasidiagonal extension. Then for each finitely generated subgroup G of $\underline{K}(A)$ there exists a homomorphism $\alpha\colon G \to \underline{K}(M(B))$ such that*

$$\pi_* \circ \alpha|_G = (\tau_*)|_G,$$

where $\pi\colon M(B) \to M(B)/B$ is the quotient map.

Proof Suppose that $\tau\colon A \to M(B)/B$ is a quasidiagonal essential extension of A by B. Let $L\colon A \to M(B)$ be a contractive completely positive linear map such that

$\pi \circ L = \tau$. Since τ is quasidiagonal, there exists an approximate identity $\{e_n\}$ for B such that

$$\|e_n L(a) - L(a)e_n\| \to 0, \text{ as } n \to \infty$$

for all $a \in A$. Define $L_n \colon A \to M(A)$ by $L_n(a) = (1-e_n)L(a)(1-e_n)$ for $a \in A$. Then $\{L_n\}$ is a sequence of asymptotically multiplicative contractive completely positive linear maps. It follows that for each finitely generated subgroup G of $\underline{K}(A)$, there exists $N > 0$ such that $\{L_n\}$ gives a homomorphism $\alpha_n \colon G \to \underline{K}(M(B))$ for all $n \geq N$. Since $\pi \circ L_n = \tau$ for all n, it follows that $\pi_* \circ \alpha|_G = (\tau_*)|_G$. ∎

Corollary 6.3 *Let A be a separable amenable C^*-algebra and B be a σ-unital and stable C^*-algebra. Suppose that $\tau \colon A \to M(B)/B$ is an essential quasidiagonal extension. Then τ induces the zero map from $\underline{K}(A)$ to $\underline{K}(M(B)/B)$. Furthermore the six-term exact sequence in K-theory associated with the extension splits into two pure extensions of groups:*

$$0 \to K_i(B) \to K_i(E) \to K_i(A) \to 0 \qquad i = 0, 1.$$

Proof This follows from the fact that when B is stable, $K_i(M(B)) = 0$ ($i = 0, 1$) and Theorem 6.2. ∎

Lemma 6.4 *Let A be a separable amenable C^*-algebra and C be a non-unital but σ-unital C^*-algebra. Suppose that $\tau \colon A \to M(C)/C$ is an essential extension such that there exists a sequence of quasidiagonal extensions $\tau_n \colon A \to M(C)/C$ such that*

$$\lim_{n\to\infty} \tau_n(a) = \tau(a)$$

for all $a \in A$. Then τ is quasidiagonal.

Proof Let $\{a_n\}$ be a dense sequence in the unit ball of A. Suppose that

$$\|\tau_n(a) - \tau(a)\| < 1/2^{n+3}$$

for all $a \in \{a_1, a_2, \dots, a_n\}$, $n = 1, 2, \dots$. Let $L_n \colon A \to M(B)$ be a contractive completely positive linear map such that $\pi \circ L_n = \tau_n$. There exists an approximate identity $\{e_k^{(n)}\}$ for B consisting of projections such that

$$\lim_{n\to\infty} \|e_k^{(n)} L_n(a) - L_n(a)e_k^{(n)}\| = 0$$

for all $a \in A$. Let $L \colon A \to M(B)$ be a contractive completely positive linear map such that $\pi \circ L = \tau$. Suppose that b is a strictly positive element for B. We may assume that

$$\|(1 - e_1^{(1)})(L_1(a) - L(a))\| < 1/2^3 \quad \text{and} \quad \|(L_1(a) - L(a))(1 - e_1^{(1)})\| < 1/2^3$$

for $a = a_1$. Put $q_1 = e_1^{(1)}$. Note that

$$\|q_1 L(a_1) - L(a_1)q_1\| < 1/2.$$

By changing notation if necessary, we may assume that

$$\|(1 - e_2^{(2)})(L_2(a) - L(a))\| < 1/2^{2+2} \quad \text{and} \quad \|(L_2(a) - L(a))(1 - e_2^{(2)})\| < 1/2^{2+2}$$

for $a \in \{a_1, a_2\}$ as well as

$$\|(1 - e_2^{(2)})q_1\| < 1/2^{2+3} \quad \text{and} \quad \|(1 - e_2^{(2)})b\| < 1/2^3.$$

There is a projection $q_2 \geq q_1$ such that

$$\|e_2^{(2)} - q_2\| < 1/2^3.$$

Note also

$$\|q_2 L(a) - L(a)q_2\| < 1/2^2 \ \text{ for } \ a \in \{a_1, a_2\}.$$

We also have

$$\|(1 - q_2)b\| < 1/2^2.$$

We may assume that

$$\|(1 - e_3^{(3)})(L_3(a) - L(a))\| < 1/2^{3+2} \quad \text{and} \quad \|(L_3(a) - L(a))(1 - e_3^{(3)}\| < 1/2^{3+2}$$

for $a \in \{a_1, a_2, a_3\}$ as well as

$$\|(1 - e_3^{(3)})q_i\| < 1/2^{3+3}, i = 1, 2 \quad \text{and} \quad \|(1 - e_3^{(3)})b\| < 1/2^4.$$

There exists a projection $q_3 \geq q_2$ such that

$$\|e_3^{(3)} - q_3\| < 1/2^4.$$

Thus we have

$$\|q_3 L(a) - L(a)q_3\| < 1/2^3 \ \text{ for } \ a \in \{a_1, a_2, a_3\}.$$

We also have

$$\|(1 - q_3)b\| < 1/2^3.$$

We continue in this fashion. It follows that we obtain an increasing sequence of projections $\{q_n\}$ in B such that

$$\|q_n L(a) - L(a)q_n\| < 1/2^n \, a \in \{a_1, a_2, \ldots, a_n\}$$

and

$$\|(1 - q_n)b\| < 1/2^n,$$

$n = 1, 2, \ldots$. It remains to show that $\{q_n\}$ is an approximate identity for B. Since

$$\lim_{n\to\infty} \|(1 - q_n)b\| = 0,$$

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one concludes that for any positive function $f \in C_0((0, \|b\|])$,

$$\lim_{n\to\infty} \|(1-q_n)f(b)\| = 0.$$

For any $a \in A$ and $\varepsilon > 0$ there exists a positive function $f \in C_0((0, \|b\|])$ such that

$$\|f(b)a - a\| < \varepsilon/3.$$

Choose $N > 0$ such that

$$\|(1-q_n)f(b)\| < \varepsilon/3(\|a\|+1) \text{ for all } n \geq N.$$

Then, for $n \geq N$,

$$\|(1-q_n)a\| \leq \|q_n a - q_n f(b)a\| + \|q_n f(b)a - f(b)a\| + \|f(b)a - a\| < \varepsilon.$$

It follows that $\{q_n\}$ is an approximate identity for B. ■

Results in this paper can be also used to prove the following.

Theorem 6.5 (Brown-Salinas-Schochet) *Let A be a separable amenable C^*-algebra in $\mathcal{N}$ and B be a σ-unital stable C^*-algebra. Suppose that there exists an essential quasidiagonal extension of A by B. The zero element in $KL(A, M(B)/B)$ corresponds to the set of stably quasidiagonal extensions as well as stably approximately trivial extensions.*

Proof It is proved in [Ln18] that stably approximately trivial extensions correspond to the zero element in $KL(A, M(B)/B)$ without assuming A satisfies the UCT. Corollary 6.3 proves that quasidiagonal extensions give a zero element in $KL(A, M(B)/B)$. It follows from [Ln18, 3.9] that extensions which represent the same element in $KL(A, M(B)/B)$ are stably approximately unitarily equivalent. Then by Lemma 6.4 every (stably) approximately trivial extension is stably quasidiagonal. ■

Theorem 6.6 *Let A be a separable exact C^*-algebra and B be a σ-unital purely infinite simple C^*-algebra. Then there are essential quasidiagonal extensions.*

Proof Let $e \in B$ be a nonzero projection such that $[e] = 0$ in $K_0(B)$. Then by [Br1] $eBe \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. It follows from a result of S. Zhang that $B \cong B \otimes \mathcal{K}$ [Zh3]. Thus we obtain an approximate identity $\{e_n\}$ of B such that each e_n is a projection and $[e_n] = 0$. Since A is exact, by [KP, Theorem 2.8], there exists a monomorphism $\imath\colon A \to \mathcal{O}_2$. Since $e_n B e_n$ is purely infinite and $[e_n] = 0$, there is an embedding $\phi_n\colon \mathcal{O}_2 \to e_n B e_n$. Now define

$$\psi(a) = \sum_{n=1}^{\infty} \phi_n \circ \imath(a) \text{ for } a \in A.$$

Then ψ is an injective homomorphism from A into $M(B)$ such that $\psi(A) \cap B = \{0\}$. Let $\tau = \pi \circ \psi$. Then τ is an essential quasidiagonal extension of A by B. ■

Corollary 6.7 follows from Theorem 6.6, [Sc2, Theorem 1.4] and Kirchberg's absorbing theorem [K1]. It also follows from Theorem 6.6 and Corollary 6.3.

Corollary 6.7 *Let A be a separable amenable C^*-algebra in $\mathcal{N}$ and B be a non-unital and σ-unital purely finite simple C^*-algebra. Suppose that $\tau\colon A \to M(B)/B$ is an essential extension. Then τ is a quasidiagonal extension if and only if it is an approximately trivial extension, and, if and only if τ induces a zero element in $KL(A, M(B)/B)$.*

In the next section we will discuss the case that B is not purely infinite.

7 Quasidiagonal Extensions—Finite Case

Definition 7.1 Recall that a separable C^*-algebra is said to be quasidiagonal if there exists a faithful representation $\phi\colon A \to B(H)$ for some separable Hilbert space H such that

$$\|p_n\phi(a) - \phi(a)p_n\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$, where $\{p_n\}$ is an approximate identity of $\mathcal{K}$ consisting of finite rank projections.

All AF-algebras are quasidiagonal. All commutative C^*-algebras are quasidiagonal. All AH-algebras are quasidiagonal. All residually finite dimensional C^*-algebras are quasidiagonal. Inductive limits of quasidiagonal C^*-algebras are quasidiagonal.

Recall that a C^*-algebra A is said to have the property (SP) if every non-zero hereditary C^*-subalgebra contains a nonzero projection. One should note that every C^*-algebra with real rank zero has the property (SP) but the converse is not true.

Theorem 7.2 *Let A be a separable quasidiagonal amenable C^*-algebra and C be a non-unital but σ-unital simple C^*-algebra which admits an approximate identity consisting of projections and has the property* (SP). *Then there exists an (essential) quasidiagonal extension $\tau\colon A \to M(C)/C$.*

Proof We may assume that $C \neq \mathcal{K}$. There is a sequence of contractive completely positive linear maps $L_n\colon A \to F_n$, where F_n are finite dimensional C^*-algebras, such that

$$\|L_n(ab) - L_n(a)L_n(b)\| \to 0, \text{ as } n \to \infty$$

for all $a, b \in A$. Let $\{a_n\}$ be a dense sequence in the unit ball of A. By passing to a subsequence, we may assume that

$$\|L_n(ab) - L_n(a)L_n(b)\| < 1/2^{n+1}$$

for all $a, b \in \{a_1, \dots, a_n\}$.

Let $\{e_n\}$ be an approximate identity for B consisting of projections. We may assume that $e_{n+1} - e_n \neq 0$ for all n. It is known (see [Ln16, 3.5.7]) that there is a monomorphism $j_n\colon F_n \to (e_{n+1} - e_n)C(e_{n+1} - e_n)$. Define

$$L(a) = \sum_{n=1}^{\infty} j_n \circ L_n(a) \text{ for } a \in A.$$

Note that the sum converges in the strict topology. One checks that $L\colon A \to M(C)$ is a (completely) positive linear contraction. Note also, for any $a, b \in \{a_1, a_2, \ldots, a_n\}$,

$$\Big\| \sum_{k=n}^{n+m} \big(L_n(ab) - L_n(a)L_n(b) \big) \Big\| < 1/2^n \text{ for all } m > 0.$$

This implies

$$L(ab) - L(a)L(b) = \sum_{n=1}^{\infty} j_n \circ \big(L_n(ab) - L_n(a)L_n(b) \big) \in C$$

for all $a, b \in A$. Let $\tau = \pi \circ L$, where $\pi\colon M(C) \to M(C)/C$ is the quotient map. Then τ is an essential quasidiagonal extension. ∎

Theorem 7.3 *Let A be a separable amenable C^*-algebra and let C be as in Theorem 7.2. Suppose that, in addition, C is also a quasidiagonal C^*-algebra. Then there is an (essential) quasidiagonal extension of A by B if and only if A is quasidiagonal.*

Proof It suffices to show the "only if" part. Suppose that $L\colon A \to M(C)$ is a bounded linear map such that $\pi \circ L\colon A \to M(C)/C$ is a monomorphism and

$$\lim_{n\to\infty} \|e_n L(a) - L(a)e_n\| = 0 \text{ for all } a \in A,$$

where $\{e_n\}$ is an approximate identity consisting of projections. Let $\{a_n\}$ be a dense sequence of A. We may assume that

$$\|e_n L(a) - L(a)e_n\| < 1/2^n \text{ for } a \in \{a_1, a_2, \ldots, a_n\},$$

$n = 1, 2, \ldots$. Since $\pi \circ L$ is a monomorphism, we may further assume that

$$\|e_n L(a)e_n\| \geq \|a\| - 1/2^n \text{ for } a \in \{a_1, a_2, \ldots, a_n\},$$

$n = 1, 2, \ldots$. Since C is quasidiagonal, it follows that there exists a finite dimensional C^*-algebra F_n and a contractive completely positive linear map $\phi_n\colon e_nCe_n \to F_n$ such that

$$\|\phi_n(b)\| \geq \|b\| - 1/2^n \quad \text{and} \quad \|\phi_n(bc) - \phi_n(b)\phi_n(c)\| < 1/2^n$$

for $b \in e_nL(a_i)e_n$, $i = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$. Define $L_n\colon A \to F_n$ by $L_n(a) = \phi_n(e_nL(a)e_n)$ for $a \in A$. Then

$$\|L_n(a_i)\| \geq \|a_i\| - 1/2^{n-1}, i = 1, 2, \ldots, n \quad \text{and} \quad \lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0$$

for all $a, b \in A$. It follows from Theorem 1 in [V2] that A is quasidiagonal. ∎

For the rest of this section B is always a non-unital but σ-unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(B)$ and a continuous scale.

Lemma 7.4 *Let A be a finite dimensional C^*-algebra. Let $\tau\colon A \to M(B)/B$ be a monomorphism such that* $\operatorname{im}\tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$*. Then τ is trivial and there is a monomorphism $h\colon A \to M(B)$ such that $\pi \circ h = \tau$.*

Proof Suppose that $A = M_{r(1)} \oplus \cdots \oplus M_{r(k)}$, so $K_0(A)$ is k copies of $\mathbb{Z}$. Let e_i be a minimal projection in $M_{r(i)}$, $i = 1, 2, \dots, k$. There are $x_i \in \operatorname{Aff}(T(B))/\rho(K_0(B))$ such that $[\tau(e_i)] = x_i$, $i = 1, 2, \dots, k$ and $\sum_{i=1}^{k} r(i)x_i = [\tau(1_A)]$. It follows from [Ln8, Lemma 1.3] that there are projections $q_i \in M(B)$ such that $[\pi(q_i)] = r(i)[\tau(e_i)]$. Thus we obtain a positive homomorphism $\alpha\colon K_0(A) \to \operatorname{Aff}(T)$ such that $\pi_{*0} \circ \alpha = \tau_{*0}$. It follows from Lemma 5.2 that there is a monomorphism $h\colon A \to M(B)$ such that $h(A) \cap B = \{0\}$ and $h_{*0} = \alpha$. It then follows that $[\pi \circ h] = [\tau]$ in $KL(A, M(B)/B)$. Since A is finite dimensional, it follows that $\pi \circ h$ is strongly unitarily equivalent to τ. In other words, τ is trivial. ∎

Lemma 7.5 *Let A be a separable amenable C^*-algebra. Suppose that*

$$\tau\colon A \to M(B)/B$$

is an essential quasidiagonal extension. Then $\tau_{*1} = 0$, $\operatorname{im}\tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$ *and $[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0$ for $i = 0, 1$ and for all $k \geq 2$.*

Proof Since $K_1(M(B)) = \{0\}$ (see Theorem 1.4), by Theorem 6.2, $\tau_{*1} = 0$. By Theorem 6.2 and Definition 1.5, $\operatorname{im}\tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$. Since $K_0(M(B)) = \operatorname{Aff}(T)$ is torsion free, $\tau_{*0}|_{\operatorname{tor}(K_0(A))} = 0$. Let C_k be as in Definition 3.5. Then $L \otimes \mathrm{id}_{C(C_k)}\colon A \otimes C(C_k) \to M(B) \otimes C(C_k)$ lifts $\tau \otimes \mathrm{id}_{C(C_k)}$. Since $K_0(M(B)) = \operatorname{Aff}(T)$ is divisible and $K_1(M(B)) = 0$, $K_0(M(B), \mathbb{Z}/k\mathbb{Z}) = \{0\}$ for all $k \geq 2$. It follows from Theorem 6.2 that $[\tau]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})} = 0$.

Note also that since $K_0(M(B))$ is torsion free and $K_1(M(B)) = 0$,

$$K_1(M(B), \mathbb{Z}/k\mathbb{Z}) = \{0\}.$$

The same argument above also shows that $[\tau]|_{K_1(A,\mathbb{Z}/k\mathbb{Z})} = 0$, $k = 2, 3, \dots$. ∎

Remark 7.6 It should be noted that, since $\operatorname{Aff}(T)/\rho_B(K_0(B))$ is divisible,

$$K_0(M(B)/B)/kK_0(M(B)/B) = K_0(B)/kK_0(M(B)/B).$$

Therefore one sees that, for any nonzero homomorphism

$$\gamma\colon K_0(A) \to \operatorname{Aff}(T)/\rho_B(K_0(B)),$$

there is $\alpha \in \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(M(B)/B))$ such that $\alpha|_{K_0(A)} = \gamma$ but $[\alpha]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})} = 0$, $k = 1, 2, \dots$. Furthermore, if $K_1(B)$ is also divisible (or $K_1(B) = \{0\}$), one computes that $\tau_{*1} = 0$ and $\operatorname{im}\tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$ imply that

$$[\tau]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})} = 0, \quad k = 2, 3, \dots,$$

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by using the six-term exact sequence in Definition 3.5. One should note that $[\tau]|_{K_1(A,\mathbb{Z}/k\mathbb{Z})} = 0$ for $k = 2, 3, \ldots$, implies that $\tau_{*0}(\mathrm{tor}(K_0(A))) = 0$. On the other hand, if $\tau_{*0}(\mathrm{tor}(K_0(A)) = 0$ and $\ker\rho_B$ is divisible (or $\ker \rho_B(K_0(B)) = \{0\}$), then

$$[\tau]|_{K_1(A,\mathbb{Z}/k\mathbb{Z})} = 0, \quad k = 2, 3, \ldots.$$

Proposition 7.7 *Let A be the closure of $\bigcup_{n=1}^{\infty} A_n$, where each A_n is a separable amenable C^*-algebra in $\mathcal{N}$ and let $j_n\colon A_n \to A$ be the embedding. Suppose that $\tau\colon A \to M(B)/B$ is an essential extension such that $\tau \circ j_n$ is a quasidiagonal extension for each n. Then τ is also a quasidiagonal extension.*

Proof The proof is almost the exactly the same as that of Lemma 6.4. ■

Definition 7.8 Denote by $\mathcal{C}_{\mathrm{afem}}$ the class of separable C^*-algebras A satisfying the following: there is an embedding $j\colon A \to C$ such that $j_{*0}\colon K_0(A)/\mathrm{tor}(K_0(A)) \to K_0(C)$ is injective, where C is a unital AF-algebra.

Clearly every AF-algebra is in $\mathcal{C}_{\mathrm{afem}}$. It is easy to see that C^*-algebras of the form $C(X) \otimes M_n$ are in $\mathcal{C}_{\mathrm{afem}}$, where X is a finite CW complex. But much more is true.

Recall that a C^*-algebra A is called *residually finite dimensional* if there is a separating family Π of finite dimensional irreducible representations of A, *i.e.*, for any $a \in A$, there is $\phi \in \Pi$ such that $\phi(a) \neq 0$.

The following is a modification of Dadarlat's construction.

Theorem 7.9 *Let A be a separable amenable residually finite dimensional C^*-algebra in $\mathcal{N}$. Then there exists a separable unital simple AF-algebra C and an embedding $j\colon A \to C$ such that j_{*0} induces an injective map from $K_0(A)/\mathrm{tor}(K_0(A))$ into $K_0(C)$. In particular, $A \in \mathcal{N} \cap \mathcal{C}_{afem}$.*

Proof Fix a separating sequence of finite dimensional irreducible representations $\{t_n\}$. For convenience, we assume that each t_n repeats infinitely many times in the sequence. Suppose that $t_n(A)$ has rank $k(n)$. For each n, define $\psi_n\colon A \to M_{k(n)}$ by the composition: $A \xrightarrow{t_n} M_{k(n)} \xrightarrow{\mathrm{id}\otimes 1_A} M_{k(n)}(A)$. We define a homomorphism $h_1\colon A \to M_{I(2)}(A)$, where $I(2) = 1 + k(1)$, by

$$h_1(a) = \mathrm{diag}(a, \psi_1(a)) \ \text{ for } \ a \in A.$$

Suppose that $h_m\colon M_{I(m)}(A) \to M_{I(m+1)}(A)$ is defined. Define $h_{m+1}\colon M_{I(m+1)}(A) \to M_{I(m+2)}(A)$ by

$$h_{m+1}(a) = \mathrm{diag}(a, \bar{\psi}_1(a), \bar{\psi}_2(a), \ldots, \bar{\psi}_{m+1}(a)) \ \text{ for } a \in M_{I(m+1)}(A),$$

where $I(m+2) = I(m+1)(1+\sum_{i=1}^{m+1} k(i))$ and $\bar{\psi}_i = \psi_i \otimes \mathrm{id}_{I(m+1)}$, $i = 1, 2, \ldots, m+1$. Set $B = \lim_{m\to\infty}(M_{I(m)}(A), h_m)$. It is shown (see [Ln16, 3.7.8; 3.7.9]) that B is a unital separable amenable simple C^*-algebra with $TR(B) = 0$. Since each $M_{I(m)}(A)$ satisfies the UCT, so does B. It follows from [Ln17] that B is isomorphic to a unital simple AH-algebra with real rank zero and with no dimension growth. Let C be a unital simple AF-algebra with $K_0(C) = K_0(B)/\mathrm{tor}(K_0(B))$. It follows from [EG] that there exists a monomorphism ϕ from B into C such that ϕ_{*0} is the quotient map from $K_0(B)$ onto $K_0(B)/\mathrm{tor}(K_0(B)) = K_0(C)$.

Thus it remains to show that $h_{1,\infty}\colon A \to B$ induces an injective map $(h_{1,\infty})_{*0}$ on $K_0(A)$.

It suffices to show that $(h_m)_{*0}$ is injective. Suppose that p and q are two projections in $M_{I(m)}(A)$ such that $h_m(p)$ and $h_m(q)$ are equivalent. Then $\bar\psi_j \circ h_m(p)$ and $\bar\psi_j \circ h_m(q)$ are equivalent (in a matrix algebra) for each j. Therefore

$$\mathrm{diag}(\bar\psi_1(p), \dots, \bar\psi_{m+1}(p), \bar\psi_1(p), \dots, \bar\psi_{m+1}(p))$$

and

$$\mathrm{diag}(\bar\psi_1(q), \dots, \bar\psi_{m+1}(q), \bar\psi_1(q), \dots, \bar\psi_{m+1}(q))$$

are equivalent. Let t be any finite dimensional representation of $M_{I(m)}(A)$. Then $(t \oplus t) \circ h_m(p)$ and $(t \oplus t) \circ h_m(q)$ are equivalent (in a matrix algebra). From the above, it follows that $t(p) \oplus t(p)$ and $t(q) \oplus t(q)$ are equivalent in a matrix algebra. Thus $t(p)$ and $t(q)$ are equivalent in the matrix algebra. This in turn implies that

$$\mathrm{diag}(0, \bar\psi_1(p), \bar\psi_2(p), \dots, \bar\psi_m(p)) \quad \text{and} \quad \mathrm{diag}(0, \bar\psi_1(q), \bar\psi_2(q), \dots, \bar\psi_m(q))$$

are equivalent. Consequently

$$[p] = [q] \text{ in } K_0\big(M_{I(m)}(A)\big).$$

This implies that $(h_m)_{*0}$ is injective for each m. ∎

Theorem 7.10 *Let A be a separable amenable C^*-algebra in $\mathcal{N} \cap \mathcal{C}_{afem}$. Suppose that τ is an essential extension.*

Then τ is quasidiagonal if and only if $\tau_{*1} = 0$, $\mathrm{im}\,\tau_{*0} \subset \mathrm{Aff}(T(B))/\rho(K_0(B))$ and $[\tau]|_{K_i(\mathbb{Z}/k\mathbb{Z})} = 0$, $i = 0, 1$ and $k = 2, 3, \dots$.

Proof The "if only" part follows from Lemma 7.5. For the "if" part, we first assume that A is an AF-algebra. We may write $A = \overline{\bigcup_{n=1}^\infty A_n}$, where $A_n \subset A_{n+1}$ and each A_n is a finite dimensional C^*-algebra. Let $j_n\colon A_n \to A$ be the embedding. It follows from Proposition 7.7 that it suffices to show that $\tau \circ j_n$ are quasidiagonal. Note that $(\tau \circ j_n)_{*0} \subset \mathrm{Aff}(T)/\rho_B(K_0(B))$. It follows from Lemma 7.4 that each $\tau \circ j_n$ is in fact trivial and therefore quasidiagonal (since A_n is finite dimensional).

For the general case, let C be a unital AF-algebra and $j\colon A \to C$ be an embedding such that j_{*0} induces an injective homomorphism from $K_0(A)/\mathrm{tor}(K_0(A))$ into $K_0(C)$. Let τ be as in the theorem. Since $\mathrm{Aff}(T)/\rho_B(K_0(B))$ is divisible, there exists a homomorphism $\alpha\colon K_0(C) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ such that $\alpha \circ j_{*0} = \tau_{*0}$. It follows from Theorem 4.6 that there is an essential extension $t\colon C \to M(B)/B$ such that $t_{*0} = \alpha$. From what we have shown, t is quasidiagonal. Let $\tau_0 = t \circ j$.

Since C is an AF-algebra, $K_0(C, \mathbb{Z}/k\mathbb{Z}) = K_0(C)/kK_0(C)$, $k = 2, 3, \dots$. On the other hand, $\mathrm{im}\, t_{*0} \subset \mathrm{Aff}(T)/\rho_B(K_0(B))$ and $\mathrm{Aff}(T)/\rho_B(K_0(B))$ is divisible, so one computes that $[t]|_{K_0(C,\mathbb{Z}/k\mathbb{Z})} = 0$ for all k, by using the six-term exact sequence in Definition 3.5. Thus $[\tau_0]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})} = 0$ for all k. We also have $(\tau_0)_{*1} = 0$.

Since C is an AF-algebra, $K_1(C, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ for $k = 2, 3, \ldots$. Since τ_0 factors through C, we conclude that $[\tau_0]|K_1(A, \mathbb{Z}/k\mathbb{Z}) = 0$. Furthermore $(\tau_0)_{*0} = \tau_{*0}$. We then conclude that

$$[\tau] = [\tau_0] \text{ in } KL(A, M(B)/B).$$

Therefore, by Theorem 4.5, τ and τ_0 are strongly approximately unitarily equivalent. We have shown that t is a quasidiagonal extension. So is τ_0, by Proposition 7.7. It follows that τ is a quasidiagonal extension. ■

For the last theorem in this section, one should note that every strong NF-algebra is an inductive limit of amenable residually finite dimensional C^*-algebras (see [BK, 6.16]).

Theorem 7.11 *Let A be the closure of $\bigcup_{n=1}^{\infty} A_n$, where each A_n is a separable amenable residually finite dimensional C^*-algebra in $\mathcal{N}$. Let $\tau\colon A \to M(B)/B$ be an essential extension. Then τ is quasidiagonal if and only if $\tau_{*1} = 0$, $\operatorname{im} \tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$ and $[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0$, $i = 0, 1$ and $k = 2, 3, \ldots$.*

Proof It follows from Lemma 7.5 that we only need to prove the "if" part of the theorem. Fix an integer $n \geq 1$. Let $\phi_n\colon A_n \to A$ be the embedding. Put $\tau_n = \tau \circ \phi_n$. So τ_n is an essential extension. Then $(\tau_n)_{*1} = 0$, $\operatorname{im}(\tau_n)_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$ and $[\tau_n]|_{K_i(A_n,\mathbb{Z}/k\mathbb{Z})} = 0$ for $i = 0, 1$ and for $k = 2, 3, \ldots$. It follows from Theorem 7.10 that τ_n is quasidiagonal. Therefore the theorem follows from Proposition 7.7. ■

8 Approximately Trivial Extensions

Let

$$0 \to \mathcal{K} \to E \to A \to 0$$

be an essential extension for a amenable quasidiagonal C^*-algebra A. It is shown that the extension is quasidiagonal if and only if it is approximately trivial (see [Br2] and [Sc2]).

In this section, we will show that there are quasidiagonal extensions that are not approximately trivial. The obstruction of a quasidiagonal extension to be approximately trivial can be computed. We will also discuss when an essential extension is approximately trivial.

Throughout this section B is always a non-unital but σ-unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(B)$ and a continuous scale.

If A is a unital separable quasidiagonal C^*-algebra then A admits at least one tracial state. Let T_A denote the tracial state space. Let $\rho_A\colon K_0(A) \to \operatorname{Aff}(T_A)$ be defined by $\rho_A([p])(t) = t(p)$ for projection $p \in M_k(A)$, $k = 1, 2, \ldots$. This map ρ_A is a positive homomorphism. In general, $\ker \rho_A$ is not zero.

Theorem 8.1 *Let A be a unital separable amenable C^*-algebra in $\mathcal{N}$. Let $\tau\colon A \to M(B)/B$ be an essential extension of A by B. If τ is approximately trivial, then $\tau_{*1} = 0$,* im $\tau_{*0} \subset \mathrm{Aff}(T)/\rho_B(K_0(B))$, $\tau_{*0}|_{\ker\rho_A} = 0$ *and* $[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0$ *for* $i = 0, 1$ *and for* $k = 2, 3, \ldots$.

Proof Suppose that there are monomorphisms $t_n \colon A \to M(B)$ such that

$$\lim_{n\to\infty} \pi \circ t_n(a) = \tau(a) \text{ for all } a \in A,$$

where $\pi \colon M(B) \to M(B)/B$ is the quotient map. We obtain a positive homomorphism from $K_0(A)$ into $\mathrm{Aff}(T)$. This implies that

$$(t_n)_{*0}\big(\ker\rho_A)\big) = 0.$$

Since $K_0(B) = \mathrm{Aff}(T)$ is a torsion free divisible group and $K_1(M(B)) = 0$, we compute that, by using the six-term exact sequence in Definition 3.5,

$$K_i(M(B), \mathbb{Z}/k\mathbb{Z}) = \{0\}, \quad i = 0, 1, k = 2, 3, \ldots.$$

Thus

$$(t_n)_{*1} = 0 \quad \text{and} \quad [t_n]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0, \quad i = 0, 1, \quad k = 2, 3, \ldots, \quad n = 1, 2, \ldots.$$

For any finite subset $\mathcal{P} \subset \underline{K}(A)$, there is an integer $n_0 \geq 1$ such that

$$[\tau_n]|_{\mathcal{P}} = [\tau]|_{\mathcal{P}} \text{ for all } n \geq n_0.$$

Thus

$$(\pi \circ t_n)_{*0}(\ker\rho_A) = 0, \; (\pi \circ t_n)_{*1} = 0$$

and

$$[\pi \circ t_n]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0, \quad i = 0, 1, \quad k = 2, 3, \ldots, \quad n = 1, 2, \ldots.$$

Therefore

$$\tau_{*0}(\ker\rho_A) = 0, \; \tau_{*1} = 0$$

and

$$[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0, \quad i = 0, 1, \quad k = 2, 3, \ldots, \quad n = 1, 2, \ldots.$$

We also have $\mathrm{im}(\pi \circ t_n{*0}) \subset \mathrm{Aff}(T)/\rho_B(K_0(B))$. It follows that

$$\mathrm{im}\,\tau_{*0} \subset \mathrm{Aff}(T)/\rho_B(K_0(B)). \qquad \blacksquare$$

Corollary 8.2 *Let A be a unital separable AF-algebra such that $\ker\rho_A \neq 0$. Then there are quasidiagonal extensions of A by B that are not approximately trivial.*

Proof Let $g_0 \in \ker \rho_A$ be a nonzero element. Take any nonzero

$$x \in \operatorname{Aff}(T)/\rho_B(K_0(B))$$

and define a nonzero homomorphism $\alpha_0\colon \mathbb{Z}g_0 \to \mathbb{Z}x \in \operatorname{Aff}(T)/\rho_B(K_0(B))$ by

$$\alpha_0(mg_0) = mx \text{ for } m \in \mathbb{Z}.$$

Since $\operatorname{Aff}(T)/\rho_B(K_0(B))$ is divisible, we obtain a homomorphism $\alpha\colon K_0(A) \to \operatorname{Aff}(T)/\rho_B(K_0(B))$ such that $\alpha(g_0) = \alpha_0(g_0)$. It follows from Lemma 5.2 that there is a monomorphism $\tau\colon A \to M(B)/B$ such that $\tau_{*0} = \alpha$. It follows from Theorem 7.10 that τ is quasidiagonal. But by Theorem 8.1 τ is not approximately trivial. ∎

Remark 8.3 From Corollary 8.2, one sees that it is typical rather than unusual that quasidiagonal extensions are different from approximately trivial extensions. The assumption that A is AF is certainly not necessary.

Lemma 8.4 *Let G be an unperforated ordered group with the Riesz interpolation property. Suppose that $G_0 \subset G$ is a countable ordered subgroup. Then there exists a countable subgroup $G_1 \supset G_0$ which satisfies the Riesz interpolation property and is unperforated. If G is simple, we may further assume that G_1 is also simple.*

Proof Since G_0 is countable, there exists a countable ordered subgroup F_1 of G such that $G_0 \subset F_1$ and if $g_1, g_2, g_3, g_4 \in G_0$ with $g_1, g_2 \leq g_3, g_4$ then there is $g \in F_1$ such that

$$g_1, g_2 \leq h \leq g_3, g_4.$$

If a countable ordered subgroup F_n has been constructed, we have a countable ordered subgroup F_{n+1} such that if $x_1, x_2 \leq y_1, y_2$ are in F_n there exists $g \in F_{n+1}$ such that

$$x_1, x_2 \leq g \leq y_1, y_2.$$

Set $G_1 = \bigcup_{n=1}^{\infty} F_n$. Then G_1 is a countable ordered subgroup of G containing G_0. From the construction, it is also clear that G_1 has the Riesz interpolation property.

Now we further assume that G is simple. Let $g \in (G_1)_+$ be a nonzero positive element and $f \in G_1$. Since G is simple there is an integer $n \geq 1$ such that $ng \geq f$. This implies that G_1 is also simple. Since G is unperforated, it follows that G_1 is also unperforated. ∎

Lemma 8.5 *Let A be a unital separable commutative C^*-algebra. Then there exists a monomorphism $h\colon A \to M(B)$ such that $\pi \circ h$ is an essential extension and* $\operatorname{im} h_{*0} \subset \rho_B(K_0(B))$.

Proof Let $G \subset \rho_B(K_0(B))$ be a countable simple ordered group with the Riesz interpolation property. We may also assume that $G \not\cong \mathbb{Z}$. It follows [EHS] that there is a unital non-elementary simple AF-algebra C such that $K_0(C)$ is order isomorphic to G. Thus we obtain an order isomorphism $\alpha\colon K_0(C) \to G \subset \rho_B(K_0(B)) \subset \operatorname{Aff}(T)$.

It follows from [AS, p. 67] that there is monomorphism $j\colon A \to C$. It follows from Lemma 5.2 that there is a homomorphism $\phi\colon C \to M(B)$ such that $\phi(A) \cap B = \{0\}$ and $\phi_{*0} = \alpha$. Define $h = \phi \circ j$. ■

Theorem 8.6 *Let A be a separable unital commutative C^*-algebra. Then $\tau\colon A \to M(B)/B$ is approximately trivial if and only if $\tau_{*1} = 0$, $\tau_{*0}(\ker \rho_A) = 0$, $\operatorname{im} \tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$ and $[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0$, $i = 0, 1$, $k = 2, 3, \dots$.*

Proof We may write $A = C(X)$, where X is a compact metric space. There are finite CW complexes X_n such that $X = \lim_{\leftarrow n} X_n$. We also write $C(X) = \lim_{n\to\infty} C(X_n)$ and $\psi_n\colon C(X_n) \to C(X)$ is the induced homomorphism. We will show that if $\tau_{*1} = 0$, $\tau_{*0}(\ker \rho_A) = 0$, $\operatorname{im} \tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$ and $[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0$, $i = 0, 1$ and $k = 2, 3, \dots$, then τ is approximately trivial. Let $e_1^{(n)}, e_2^{(n)}, \dots, e_{k(n)}^{(n)}$ be projections corresponding to each summand of $C(X_n)$ which corresponds to each connected component of X_n. So $K_0(C(X_n))/\ker \rho_C$ is generated by $\{e_1^{(n)}, e_2^{(n)}, \dots, e_{k(n)}^{(n)}\}$. Denote by G_n the subgroup of $K_0(C(X))$ generated by $[\psi_n(e_1^{(n)})], [\psi_n(e_2^{(n)})], \dots, [\psi_n(e_{k(n)}^{(n)})]$. Let $f_n\colon X \to X_n$ denote the continuous map induced by the inverse inductive limit $X = \lim_{\leftarrow n} X_n$. Let $\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_{k(n)}^{(n)}$ be points in X_n which lie in different components. Let $x_1^{(n)}, x_2^{(n)}, \dots, x_{k(n)}^{(n)}$ be points in X such that $f_n(x_i^{(n)}) = \xi_i^{(n)}$. Since $\rho_B(K_0(B))$ is dense in $\operatorname{Aff}(T)$, we can find mutually orthogonal projections $p_1^{(n)}, p_2^{(n)}, \dots, p_{k(n)}^{(n)}$ in $M(B)$ such that $\sum_{i=1}^{k(n)} p_i^{(n)} \leq 1_{M(B)}$, $1_{M(B)} - \sum_{i=1}^{k(n)} p_i^{(n)} \notin B$ and $[\pi(p_i^{(n)})] = [\tau \circ \psi_n(p_i^{(n)})]$, $i = 1, 2, \dots, k(n)$. Define a homomorphism $\phi_n\colon C(X) \to M(B)$ by

$$\phi_n(f) = \sum_{i=1}^{k(n)} f(x_i^{(n)}) p_i^{(n)} \ \text{ for } \ f \in C(X).$$

Let $P_n \leq 1_{M(B)} - \sum_{i=1}^{k(n)} p_i^{(n)}$ be a projection in $M(B) \setminus B$ such that $[P_n] \in \rho_B(K_0(B))$. It follows from Lemma 8.5 that there is a monomorphism $h_n^{(0)}\colon C(X) \to P_n M(B) P_n$ such that $\operatorname{im}(h_n^{(0)})_{*0} \subset \rho_B(K_0(B))$ and $\pi \circ h_n^{(0)}$ is injective. Now define $h_n = \phi_n + h_n^{(0)}$. Then $h_n\colon C(X) \to M(B)$ give an essential trivial extension of $C(X)$ by B. Note that $(\pi \circ h_n)_{*1} = 0$, $[\pi \circ h_n]|_{K_i(\mathbb{Z}/k\mathbb{Z})} = 0$, $i = 0, 1$, $k = 2, 3, \dots$, and

$$(\pi \circ h_n)_{*0}|_{G_n} = \tau_{*0}|_{G_n}, n = 1, 2, \dots,$$

and $\operatorname{im}(\pi \circ h_n)_{*0}) \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$. Since $\bigcup_{n=1}^{\infty} G_n = K_0(C(X))$, one checks that, for any finite subset $\mathcal{P} \subset \underline{K}(C(X))$, there exists an integer n, such that

$$[\pi \circ h_n]|_{\mathcal{P}} = [\tau]_{\mathcal{P}}.$$

It follows from Theorem 3.9 that there exists a sequence of unitaries $u_n \in M(B)/B$ such that

$$\lim_{n\to\infty} \operatorname{ad}\, u_n \circ \pi \circ h_n(a) = \tau(a) \quad a \in C(X).$$

This implies that τ is approximately trivial. ■

Definition 8.7 Let $\mathrm{Pl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\right)$ be the set of those elements α in $\mathrm{Hom}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))$ such that there exists a positive homomorphism $\beta\colon K_0(A) \to \mathrm{Aff}(T)$ such that $\Phi \circ \beta = \alpha$, where $\Phi\colon\ \mathrm{Aff}(T) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ is the quotient map.

Denote by $\mathrm{Apl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\right)$ the set of those elements α in

$$\mathrm{Hom}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))$$

satisfying the following: there exists an increasing sequence of finitely generated subgroups $\{G_n\} \subset K_0(A)$ such that $\bigcup_{n=1}^{\infty} G_n = K_0(A)$ and a sequence of homomorphism $\alpha_n \in \mathrm{Pl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)))$ such that

$$(\alpha_n)|_{G_n} = \alpha|_{G_n}, \quad n = 1, 2, \dots.$$

One should note that if $\alpha \in \mathrm{Apl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)\right)$ then $\alpha|_{\ker \rho_A} = 0$. In fact, if $x \in \ker \rho_A$, then $x \in G_n$ for some n. But $\alpha_n(x) = 0$ for all n. Therefore $\alpha(x) = 0$.

Proposition 8.8 *Let A be a separable amenable C^*-algebra and let $\tau\colon A \to M(B)/B$ be an essential approximately trivial extension. Then*

$$\tau_{*1} = 0, [\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0$$

and

$$\tau_{*0} \in \mathrm{Apl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\right).$$

Proof Suppose that $t_n\colon A \to M(B)/B$ be a sequence of trivial extensions such that

$$\lim_{n\to\infty} t_n(a) = \tau(a) \text{ for all } a \in A.$$

There is a sequence of monomorphism $h_n\colon A \to M(B)$ such that $\pi \circ h_n = t_n$, where $\pi\colon M(B) \to M(B)/B$ is the quotient map, $n = 1, 2, \dots$. Since $K_1(M(B)) = 0$ and $K_0(M(B))$ has no torsion, we conclude (by using the six-term exact sequence in Definition 3.5) that

$$[t_n]|_{K_1(A)} = 0 \quad \text{and} \quad [t_n]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0 \quad i = 0, 1,\ k = 1, 2, \dots.$$

It follows that

$$[\tau]|_{K_1(A)} = 0 \quad \text{and} \quad [t]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0, \quad i = 0, 1,\ k = 1, 2, \dots.$$

Let $\{G_n\}$ be a sequence of finitely generated groups of $K_0(A)$ such that $\bigcup_{n=1}^{\infty} G_n = K_0(A)$. For each n, there is $m(n)$ such that

$$(t_m)|_{G_n} = \tau|_{G_n}$$

for all $m \geq m(n)$, since $\lim_{n\to\infty} t_n(a) = \tau(a)$ for all $a \in A$. However, $(t_m)_{*0} \in \mathrm{Pl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\right)$. Therefore $\tau_{*0} \in \mathrm{Apl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\right)$. ∎

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Theorem 8.9 *Let A be a separable amenable C^*-algebra in $\mathcal{N}$ which can be embedded into a unital AF-algebra C such that $K_0(A)/\ker\rho_A = K_0(C)/\ker\rho_C$ (as ordered groups). Let $\tau\colon A \to M(B)/B$ be an essential extension. Then τ is approximately trivial if and only if $\tau_{*1} = 0$, $[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0$, $i = 0, 1$, $k = 1, 2, \ldots$, and $\tau_{*0} \in \mathrm{Apl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)))$.*

Proof The "only if" part follows from Proposition 8.8. Suppose that

$$\alpha_n \in \mathrm{Pl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\right)$$

is such that

$$(\alpha_n)|_{G_n} = \tau_{*0}|_{G_n},$$

where $\bigcup_{n=1}^{\infty} G_n = K_0(A)$ and G_n is finitely generated. Suppose also $h_n\colon K_0(A) \to K_0(M(B)) = \mathrm{Aff}(T)$ is a positive homomorphism such that $\Phi \circ h_n = \alpha_n$. Note that $h_n|_{\ker\rho_A} = 0$. Let $\tilde{h}_n\colon K_0(A)/\ker\rho_A \to K_0(M(B))$ denote the induced positive homomorphism. Let C be the unital simple AF-algebra such that there exists an embedding $j\colon A \to C$ such that j_{*0} induces an order isomorphism from $K_0(A)/\ker\rho_A$ onto $K_0(C)/\ker\rho_C$ with $K_0(C) = K_0(A)/\ker\rho_A$ and $[1_C] = [1_A]$. There is a homomorphism $\psi_n\colon C \to M(B)$ such that $(\psi_n)_{*0} = \tilde{h}_n$. Put $\phi_n = \psi_n \circ j$. Let $t_n = \pi \circ \phi_n$. It is clear that $[t_n]_{K_1(A)} = 0$ and $(t_n)_{*0} = \alpha_n$. We note that since $K_0(M(B)) = \mathrm{Aff}(T)$ is divisible, $K_0(M(B))/kK_0(M(B)) = 0$. We also have $K_1(M(B)) = 0$. This implies that $K_0(M(B), \mathbb{Z}/k\mathbb{Z}) = 0$ for all k. Therefore

$$[t_n]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})} = 0 \quad k = 1, 2, \ldots.$$

Since $(\phi_n)_{*1} = 0$ and $K_0(M(B))$ has no torsion, we compute that

$$[t_n]|_{K_1(A)} = 0 \quad \text{and} \quad [t_n]|_{K_1(A,\mathbb{Z}/k\mathbb{Z})} = 0$$

for all k. For τ, we note that $\tau_{*0} \in A\,\mathrm{Pl}\big(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\big)$ implies that $\tau_{*0}|_{\ker\rho_A} = 0$. In particular, $\tau_{*0}|_{\mathrm{tor}(K_0(A))} = 0$. Since $\tau_{*1} = 0$, it follows that

$$[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0 \quad i = 0, 1 \quad k = 1, 2, \ldots.$$

Therefore, for any finite subset $\mathcal{P} \subset \underline{K}(A)$, there exists n such that

$$[\tau]|_{\mathcal{P}} = [t_m]|_{\mathcal{P}}$$

for all $m \geq n$. It follows from Theorem 3.9 that there are unitaries $u_n \in M(B)/B$ such that

$$\lim_{n\to\infty} \mathrm{ad}\, u_n \circ t_n(a) = \tau(a)$$

for all $a \in A$. ∎

Example 8.10 There are many examples where a homomorphism $\alpha\colon K_0(A) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ can not be lifted to a positive homomorphism $\beta\colon K_0(A) \to \mathrm{Aff}(T)$. In Example 5.4, where, even if $\alpha = 0$, one can not get a nonzero positive homomorphism $\beta\colon K_0(A) \to \mathrm{Aff}(T)$ such that $\Phi \circ \beta = \alpha$. To show other complications, let $\rho_B(K_0(B)) = \mathbb{Z}[1/2]$ and let $K_0(A) = \mathbb{Z} \oplus \mathbb{Z}\sqrt{2}$. Define a homomorphism $\alpha\colon K_0(A) \to \mathrm{Aff}(T)/\rho_B(K_0(B)) = \mathbb{R}/\mathbb{Z}[1/2]$ so that $\alpha(1) = 1$ and $\alpha(\sqrt{2}) = \pi$. If there is a nonzero homomorphism $\beta\colon K_0(A) \to \mathbb{R}$ such that $\Phi \circ \beta(1) = \alpha(1)$ and $\Phi \circ \beta(\sqrt{2}) = \alpha(\sqrt{2})$, then $\beta(1) = x$ and $\beta(\sqrt{2}) = y$ with $x \in \mathbb{Z}[1/2]$ and $y = \pi + z$, where $z \in \mathbb{Z}[1/2]$. If, in addition, β were positive, then $\beta(\sqrt{2}) = x\sqrt{2}$. But that would imply that $x\sqrt{2} = \pi + z$. This is impossible since $x, z \in \mathbb{Z}[1/2]$. Therefore $\alpha \notin \mathrm{Apl}\left(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\right)$.

This example also shows that there are very few elements in

$$A\,\mathrm{Pl}\big(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\big)$$

or in $\mathrm{Pl}\big(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\big)$.

Nevertheless, we have the following:

Theorem 8.11 *Let A be a unital separable C^*-algebra with $K_0(A) = \mathbb{Z}[1/p]$, where p is a prime number. Suppose that $\tau_{*0}\colon K_0(A) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ is a homomorphism. Then the following hold:*

(1) $\tau_{*0} \in \mathrm{Apl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))$ *if* $\tau_{*0} \neq 0$,

(2) *if $\rho_B(K_0(B))$ is divisible by p, then every such nonzero τ_{*0} is in*

$$\mathrm{Pl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)).$$

(3) *if $\rho_B(K_0(B))$ is finitely generated (such as $\mathbb{Z} \oplus \mathbb{Z}_\theta$ for some irrational number θ), then*

$$\mathrm{Apl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)) \neq \mathrm{Pl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)),$$

(4) $\tau_{*0} = 0$ *is in* $\mathrm{Apl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)))$ *if and only if there is a sequence of nonzero positive elements $\eta_n \in \rho_B(K_0(B))$ such that $\eta_n \leq \widehat{1_{M(B)}}$ and η_n is divisible by p^n, $n = 1, 2, \dots$.*

Proof To prove (1), let $\tau_{*0}\colon K_0(A) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ be a nonzero homomorphism. There exists N such that $\xi_n = \tau_{*0}(1/p^n) \neq 0$ for all $n \geq N$. Since $\rho_B(K_0(B))$ is dense in $\mathrm{Aff}(T)$, one can choose $r_n \gg 0$ in $\mathrm{Aff}(T)$ such that $\Phi(r_n) = \tau_{*0}(1/p^n)$, where $\Phi\colon \mathrm{Aff}(T) \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ is the quotient map. Since $\rho_B(K_0(B))$ is dense, it is easy to find $r_n \gg 0$ in $\mathrm{Aff}(T)$ such that $p^n r_n \leq 1_{M(B)}$. Define $\beta_n(z) = p^n r_n z$ for $z \in K_0(A)$ $(n \geq N)$. Here we identify z with the real number z (so $rz \in \mathrm{Aff}(T)$). Write

$$G_n = \{m/p^{n+N} : m \in \mathbb{Z}\}, \quad n = 1, 2, \dots.$$

Then G_n is finitely generated and $\bigcup_{n=1}^{\infty} G_n = K_0(A)$. Moreover,

$$(\Phi \circ \beta_n)|_{G_n} = \tau_{*0}|_{G_n}, \quad n = 1, 2, \dots.$$

This proves (1).

To prove (2), we first note that $\mathrm{Aff}(T)/\rho_B(K_0(B))$ has no p-torsion. Suppose $x \in \mathrm{Aff}(T)/\rho_B(K_0(B))$ is a nonzero element so that $px = 0$. Let $y \in \mathrm{Aff}(T)$ so that $\Phi(y) = x$. Then $py \in \rho_B(K_0(B))$. Since $\rho_B(K_0(B))$ is divisible by p, there is $z \in \rho_B(K_0(B))$ such that $p(y - z) = 0$. Since $\mathrm{Aff}(T)$ is torsion free, $y = z$, or $x = 0$.

We will show that $\Phi(p_N^N z) = \tau_{*0}(z)$ for $z \in K_0(A) = \mathbb{Z}[1/p]$. Note we have shown above that for any $z \in \mathbb{Z}(1/p^N)$, $\Phi(p^N r_N z) = \tau_{*0}(z)$. Suppose that $x = \tau_{*0}(1/p^{N+1})$. Then $px = \tau_{*0}(1/p^N) = \Phi(r_N)$. Thus $px = p\Phi \circ \beta_N(1/p^{N+1})$. This implies that $p(x - \xi_N) = 0$ in $\mathrm{Aff}(T)/\rho_B(K_0(B))$. Since $\mathrm{Aff}(T)/\rho_B(K_0(B))$ has no p-torsion, $x = \xi_N$. Therefore $\Phi \circ \beta_N(z) = \tau_{*0}(z)$ for all $z \in \mathbb{Z}(1/p^{N+1})$. By induction, we verify that $\Phi \circ \beta_N = \tau_{*0}$. It is clear that β_N is positive.

For (3), we note that $\mathrm{ext}_{\mathbb{Z}}(\mathbb{Z}[1/p], \mathbb{Z}) \neq \{0\}$ (see [Fu, Theorem 99.1; p 179], and also [Rot]). Therefore, since $\rho_B(K_0(B))$ is a finite sum of $\mathbb{Z}$, we conclude that $\mathrm{ext}_{\mathbb{Z}}(\mathbb{Z}[1/p], \rho_B(K_0(B))) \neq \{0\}$. Consider the following exact sequence:

$$\begin{aligned}\cdots \mathrm{Hom}(\mathbb{Z}[1/p], \mathrm{Aff}(T)) &\to \mathrm{Hom}(\mathbb{Z}[1/p], \mathrm{Aff}(T)/\rho_B(K_0(B)))\\ &\to \mathrm{ext}(\mathbb{Z}[1/p], \rho_B(K_0(B))) \to \mathrm{ext}(\mathbb{Z}[1/p], \mathrm{Aff}(T))\\ &\to \mathrm{ext}(\mathbb{Z}[1/p], \mathrm{Aff}(T)/\rho_B(K_0(B))) \to \cdots.\end{aligned}$$

Since $\mathrm{Aff}(T)$ is divisible, $\mathrm{ext}(\mathbb{Z}[1/p], \mathrm{Aff}(T)) = \{0\}$. This implies that the map from $\mathrm{Hom}(\mathbb{Z}[1/p], \mathrm{Aff}(T)/\rho_B(K_0(B)))$ to $\mathrm{ext}(\mathbb{Z}[1/p], \rho_B(K_0(B)))$ is surjective. Thus we obtain $\tau_{*0} \colon \mathbb{Z}[1/p] \to \mathrm{Aff}(T)/\rho_B(K_0(B))$ such that it gives a non-splitting extension of $\mathbb{Z}[1/p]$ by $\rho_B(K_0(B))$. This τ_{*0} cannot be lifted to a homomorphism from $K_0(A)$ into $\mathrm{Aff}(T)$. In particular $\tau_{*0} \notin \mathrm{Pl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))$. It follows from (1) that $\tau_{*0} \in \mathrm{Apl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))$. Thus (3) follows.

To see (4), suppose that there exists a sequence of nonzero positive elements $\eta_n \in \rho_B(K_0(B))$ such that $\eta_n \leq \widehat{1_{M(B)}}$ and $\eta_n = p^n \zeta_n$ for some $\zeta_n \in \rho_B(K_0(B))$, $n = 1, 2, \dots$. Since $\rho_B(K_0(B))$ is weakly unperforated, $\zeta_n \gg 0$. Fix n, define

$$\beta_n \colon \mathbb{Z}[1/p] \to \mathrm{Aff}(T)$$

by $\beta_n(z) = p^n z \zeta_n$ for all $z \in \mathbb{Z}[1/p]$ ($\subset \mathbb{R}$). Then β_n is a positive homomorphism. Let G_n be as in the proof of (1). Then $\beta_n(G_n) \subset \rho_B(K_0(B))$, $n = 1, 2, \dots$. So

$$(\Phi \circ \beta_n)|_{G_n} = 0, \quad n = 1, 2, \dots.$$

Therefore $\tau_{*0} = 0$ is in $\mathrm{Apl}\big(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\big)$.

Conversely, if $\tau_{*0} = 0$ is in $\mathrm{Apl}\big(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B))\big)$. Suppose that $F_n \subset \mathbb{Z}[1/p]$ is an increasing sequence of finitely generated subgroups such that $\bigcup_{n=1}^{\infty} F_n = \mathbb{Z}[1/p]$ and there is a sequence of positive homomorphisms $\alpha_n \colon \mathbb{Z}[1/p] \to \mathrm{Aff}(T)$ such that $\alpha_n(F_n) \subset \rho_B(K_0(B))$, $n = 1, 2, \dots$. Replacing α_n by $t_n\alpha_n$ for some positive

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real numbers t_n, we may assume that $\alpha_n(1) \leq \widehat{1_{M(B)}}$, $n = 1, 2, \ldots$. Without loss of generality, we may also assume that $1, 1/p^n \subset F_n$, $n = 1, 2, \ldots$. Let $\eta_n = \alpha_n(1)$. Since $\alpha_n(1/p^n) \subset \rho_B(K_0(B))$, η_n is divisible by p^n, $n = 1, 2, \ldots$. ■

Remark 8.12 Note that $\rho_B(K_0(B))$ may not have any nonzero elements to be divided by p. In these cases, the condition in (4) in the previous theorem and the next corollary never holds. In other words, when $[\tau] = 0$ in $KL(A, M(B)/B)$, τ is never approximately trivial.

Corollary 8.13 *Let A be a unital separable amenable C^*-algebra in $\mathcal{N}$. Suppose that there is a monomorphism $j\colon A \to C$ for some unital simple AF-algebra C with $C = \mathbb{Z}[1/p]$. Suppose also that j_{*0} maps $K_0(A)/\ker \rho_A$ injectively to $\mathbb{Z}[1/p]$. Let $\tau\colon A \to M(B)/B$ be an essential extension.*

(1) *If $[\tau] \neq 0$, $\tau_{*1} = 0$, $\tau_{*0}|_{\ker \rho_A} = 0$, $\operatorname{im} \tau_{*0} \subset \operatorname{Aff}(T)/\rho_B(K_0(B))$ and*

$$[\tau]|_{K_i(A,\mathbb{Z}/k\mathbb{Z})} = 0, \quad i = 0, 1$$

and $k = 2, 3, \ldots$. Then τ is approximately trivial.

(2) *If there is no positive homomorphism from $K_0(A)$ into $\rho_B(K_0(B))$, then no essential extension with $[\tau] = 0$ in $KL(A, M(B)/B)$ can be trivial. Furthermore, if $A = C$, then there exists an essential trivial extension τ with $[\tau] = 0$ in $KL(A, M(B)/B)$ if and only if there is a positive homomorphism $\alpha\colon K_0(A) \to \rho_B(K_0(B))$.*

(3) *Suppose further that $K_0(A)/\ker \rho_A = \mathbb{Z}[1/p]$. Then $[\tau] = 0$ implies that τ is approximately trivial if and only if there exists a sequence of nonzero positive elements $\eta_n \in \rho_B(K_0(B))$ such that $\eta_n \leq \widehat{1_{M(B)}}$ and η_n is divisible by p^n, $n = 1, 2, \ldots$.*

Proof Suppose $j\colon A \to C$ is the embedding. Since $\operatorname{Aff}(T)/\rho_B(K_0(B))$ is divisible, there exists a homomorphism $\alpha\colon K_0(C) \to \operatorname{Aff}(T)/\rho_B(K_0(B))$ such that

$$\alpha|_{j_{*0}(K_0(A))} = \tau_{*0}$$

which is nonzero. Let $\psi\colon C \to M(B)/B$ be an essential extension such that $\psi_{*0} = \alpha$. Since $\operatorname{Aff}(T)/\rho_B(K_0(B))$ is divisible and $K_1(C) = \{0\}$, one computes that

$$[\tau]|_{K_0(C,\mathbb{Z}/k\mathbb{Z})} = 0 \text{ for all } k.$$

Since $K_0(C)$ has no torsion and $K_1(C) = 0$, $[\tau]|_{K_1(C,\mathbb{Z}/k\mathbb{Z})} = 0$ for all k. It follows that

$$[\tau] = [\psi \circ j] \text{ in } KL(A, M(B)/B).$$

So τ and $\psi \circ j$ are strongly approximately unitarily equivalent. Thus it suffices to show that $\psi \circ j$ is approximately trivial. It follows from Theorem 8.11 that

$$\psi_{*0} \subset \operatorname{Apl}(K_0(C), \operatorname{Aff}(T)/\rho_B(K_0(B))).$$

It follows from Theorem 8.9 that ψ is approximately trivial. Hence $\psi \circ j$ is approximately trivial.

For (2), we assume that $[\tau] = 0$. Suppose that τ is trivial and $h\colon A \to M(B)$ is a monomorphism such that $\pi \circ h = \tau$. Then h_{*0} gives a positive homomorphism from $K_0(A)$ into $\rho_B(K_0(B))$.

Suppose now that $A = C$. If there is a $\alpha\colon K_0(A) \to \rho_B(K_0(B)) \subset \mathrm{Aff}(T)$, then by Lemma 5.2 there exists a monomorphism $h\colon A \to M(B)$ such that $h_{*0} = \alpha$ and $h(A) \cap B = \{0\}$. Thus $\tau = \pi \circ h$ is trivial and $[\tau] = 0$.

Now consider (3). Suppose that $[\tau] = 0$ in $KL(A, M(B)/B)$ and suppose also that there is a homomorphism $h_n\colon A \to M(B)$ such that

$$\lim_{n\to\infty} \pi \circ h_n(a) = \tau(a) \text{ for all } a \in A.$$

Then $\tau_{*0} \in \mathrm{Apl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)))$. Thus the "if only" part follows from (4) in Theorem 8.11. On the other hand, if those η_n exists, by (4) in Theorem 8.11, $\tau_{*0} \in \mathrm{Apl}(K_0(A), \mathrm{Aff}(T)/\rho_B(K_0(B)))$. Thus the "if" part follows from Theorem 8.9. ■

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