



Twistor spaces of algebraic dimension two associated to a connected sum of projective planes

Akira Fujiki

ABSTRACT

For any integer $m \geq 4$ we construct twistor spaces of algebraic dimension $a = 2$ associated to some self-dual structures on $m\mathbf{P}^2$, the connected sum of m complex projective planes \mathbf{P}^2 . Together with previously known results this implies that for any $m \geq 5$ all the possible values for algebraic dimension $0 \leq a \leq 3$ are attained for some such twistor spaces. Our method is to construct such twistor spaces by small deformations of Joyce twistor spaces.

1. Introduction

Let m be a positive integer and $M := m\mathbf{P}^2$ the smooth connected sum of m copies of the complex projective plane \mathbf{P}^2 . Suppose that we are given a self-dual conformal class $[g]$ on M . Assume that $[g]$ is of *positive type* in the sense that the conformal class $[g]$ contains a Riemannian metric whose scalar curvature is a positive constant. Let Z be the twistor space associated to this self-dual manifold $(M, [g])$. Then Z is a three-dimensional compact complex manifold which fibers smoothly over M with fiber isomorphic to the complex projective line \mathbf{P} . We are interested in the algebraic dimension $a(Z)$ of Z , which is by definition the transcendence degree of the field of meromorphic functions on Z , and which can take one of the values 0, 1, 2 or 3. As for the problem as to which of these values can actually be attained for some twistor spaces as above, the following results are known:

- 1) We have $a(Z) = 3$ if $m \leq 3$, and $a(Z) \geq 1$ if $m = 4$.
- 2) For each $m \geq 1$ LeBrun [LeB91] and Joyce [Joy95] have explicitly constructed families of self-dual metrics on $m\mathbf{P}^2$ whose twistor spaces turn out to be Moishezon, that is, $a(Z) = 3$ (cf. [LeB91, Joy95, Fuj00]).
- 3) For each $m \geq 5$ there exist twistor spaces Z with $a(Z) = 0$ (see [DF89, LP92, Cam91]), and similarly for each $m \geq 4$ there exist twistor spaces Z with $a(Z) = 1$ (see [LP92, Cam94, Poo92, Hon99]).
- 4) For $m = 4$ there exist twistor spaces Z with $a(Z) = 2$ (see [CK99]).

A natural problem thus arises as to whether there exists a twistor space Z with $a(Z) = 2$ in the case $m \geq 5$. The purpose of this paper is to show that the answer to this problem is affirmative. Namely we prove the following Main Theorem.

MAIN THEOREM. *For any $m \geq 4$ there exist twistor spaces Z associated to a self-dual conformal class of positive type on $m\mathbf{P}^2$ with $a(Z) = 2$.*

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The examples in results 3 and 4 are all obtained as small deformations of a ‘general’ LeBrun twistor space in [LeB91] and the main line of arguments here is basically the same, except that we use small deformations of Joyce twistor spaces studied in detail in [Fuj00], which includes as a special case the ‘degenerate’ case of the LeBrun twistor spaces. We note that if $a(Z) = 2$ for a twistor space Z of a compact self-dual manifold $(M, [g])$ in general, then M is homeomorphic either to $m\mathbf{P}^2$ or to $(S^1 \times S^3) \# m\mathbf{P}^2$ and $[g]$ is of positive type (cf. [Fuj02] for more details).

After some preliminaries concerning symmetric toric surfaces in § 2, we construct in § 3 some projective nonsingular rational surfaces with anti-Kodaira dimension one as a local deformation of a symmetric toric surface. In § 4 we basically show that a twistor space is of algebraic dimension two if it contains as a fundamental divisor one such rational surface. In § 5 we prove a general unobstructedness result for local deformations of a pair consisting of a twistor space and a smooth fundamental divisor on it. Finally in § 6 we consider a pair of a Joyce twistor space and a smooth fundamental divisor on it which is a symmetric toric surface. We apply the result of the previous section to such a pair and show that a certain deformation of the pair gives rise to a twistor space which contains one of the rational surfaces in § 3 as a fundamental divisor. The Main Theorem then follows.

Notations. For a line bundle L on a complex manifold X we denote by $|L|$ the corresponding linear system of divisors on X . For a vector bundle E on X we write $h^i(E) = h^i(X, E) = \dim H^i(X, E)$ and $\chi(X, E) = \sum_i (-1)^i h^i(E)$.

2. Preliminaries

2.1 Let B be a cycle of nonsingular rational curves, i.e. B is a compact curve whose irreducible components B_1, \dots, B_l are nonsingular rational curves such that B_i and B_{i+1} intersect transversally at a single point p_i for $1 \leq i \leq l$, where $B_{l+1} = B_1$, and that there exist no other intersections among the irreducible components of B . The natural map $H^1(B, \mathbf{C}^*) \rightarrow H^1(B, \mathcal{O}_B^*)$ induces an isomorphism $H^1(B, \mathbf{C}^*) \cong \text{Pic}_0 B$, the identity component of the Picard group $\text{Pic } B$ of B . Moreover, we have $H^1(B, \mathbf{C}^*) \cong \mathbf{C}^*$.

We fix a generator g of the fundamental group $\pi_1 \cong \mathbf{Z}$ of B . Suppose that there exists an anti-holomorphic involution σ of B . It induces an involution σ_* of π_1 and we have $\sigma_*(g) = g^\varepsilon$, where $\varepsilon = 1$ or -1 . Also σ induces an anti-holomorphic involution of $\text{Pic } B$ which is an automorphism of a group. Denote by $\text{Pic}^R B$ the fixed subgroup of this action and set $\text{Pic}_0^R B = \text{Pic}^R B \cap \text{Pic}_0 B$. A line bundle L on B defines a point of $\text{Pic}^R B$ if and only if σ lifts to an anti-holomorphic bundle automorphism $\tilde{\sigma}$ of L . (A priori $\tilde{\sigma}$ may not be an involution.)

LEMMA 2.1. *Suppose that $\varepsilon = 1$ (respectively -1). Then $\text{Pic}_0^R B \cong \mathbf{R}^*$ (respectively S^1) with respect to a natural isomorphism $\text{Pic}_0 B \cong \mathbf{C}^*$. Moreover, for any line bundle L in $\text{Pic}_0^R B$ the lift $\tilde{\sigma}$ as above can be taken to be an involution.*

Proof. First note that an anti-holomorphic involution of \mathbf{C}^* as a group is either $t \rightarrow \bar{t}$ or $t \rightarrow \bar{t}^{-1}$, where the bar denotes the complex conjugation; in the former (respectively latter) case the real part is isomorphic to \mathbf{R}^* (respectively S^1). Now let $p : \tilde{B} \rightarrow B$ be the universal covering map, whose covering transformation group is identified with π_1 . We fix a lift $\tilde{\sigma}$ of σ to an anti-holomorphic automorphism of \tilde{B} . Then we have the equality $\tilde{\sigma}g\tilde{\sigma}^{-1} = g^\varepsilon$ of biholomorphic automorphisms of \tilde{B} . For any element $b \in \mathbf{C}^*$ we define the action of π_1 on the product $\tilde{B} \times \mathbf{C}$ by $\hat{g}((\tilde{x}, \zeta)) = (g(\tilde{x}), b\zeta)$, $\tilde{x} \in \tilde{B}$, $\zeta \in \mathbf{C}$. Then the quotient $L_b := (\tilde{B} \times \mathbf{C})/\pi_1$ is a flat line bundle on B and the map $\mathbf{C}^* \rightarrow \text{Pic}_0 B, b \rightarrow L_b$, gives a group isomorphism. Let the anti-holomorphic automorphism $\tilde{\sigma}$ be lifted to an anti-holomorphic bundle automorphism $\hat{\sigma}$ of $\tilde{B} \times \mathbf{C}$ via $\hat{\sigma}(\tilde{x}, \zeta) = (\tilde{\sigma}(\tilde{x}), \bar{\zeta})$. This automorphism then descends to one of L_b on B if the equality $\hat{\sigma}\hat{g}\hat{\sigma}^{-1} = \hat{g}^\varepsilon$ for $\hat{g} = \hat{g}_b$ holds, which is equivalent to

the condition $b = \bar{b}$ (respectively $b = \bar{b}^{-1}$). Together with the remark at the beginning of the proof, this implies the first assertion of the lemma. The second assertion is clear from the above proof. \square

Suppose now that $l = 2m$ is even and that for any $1 \leq i \leq m$ the anti-holomorphic involution σ of B interchanges B_i and B_{i+m} . Suppose further that B is embedded in a smooth complex surface T and σ is induced by an anti-holomorphic involution on T (still denoted by σ). Let $\text{Pic}_c B$ be the connected component of $\text{Pic } B$ consisting of the line bundles which have the same Chern class with $K_T^{-1}|B$, where K_T is the canonical bundle of T . Set $\text{Pic}_c^R B = \text{Pic}^R B \cap \text{Pic}_c B$.

LEMMA 2.2. *Let the notations and assumptions be as above. Then there exists a natural isomorphism $\text{Pic}_c B \cong \mathbf{C}^*$ inducing $\text{Pic}_c^R B \cong \mathbf{R}^*$. Moreover, for any line bundle F in $\text{Pic}_c^R B$, σ induces a natural anti-holomorphic involution on the projective space $\mathbf{P}H^0(B, F) := (H^0(B, F) - 0)/\mathbf{C}^*$ which is induced by a complex conjugation on the vector space $H^0(B, F)$.*

Proof. Under our assumption on σ we see immediately that σ induces the identity of π_1 and hence $\text{Pic}_0^R B \cong \mathbf{R}^*$ by Lemma 2.1. The natural isomorphism $v : \text{Pic}_0 B \rightarrow \text{Pic}_c B$ induced by the multiplication by $K_T^{-1}|B$ is σ -equivariant since K_T^{-1} is σ -invariant. Indeed σ on T lifts naturally to an anti-holomorphic involution on K_T^{-1} . The first assertion follows. Moreover, combined with the second assertion of Lemma 2.1 it follows that any F as in the lemma admits a lift $\tilde{\sigma}$ of σ which is an involution, F being a product of $K_T^{-1}|B$ and an element of $\text{Pic}_0^R B$. From this the second assertion follows. \square

2.2 Let S be a projective nonsingular rational surface and $C = \bigcup_{1 \leq j \leq 2k} C_j$ a cycle of nonsingular rational curves embedded in S . (The numbering is in such a way that $C_i \cap C_{i+1} = \{p_i\}$, $1 \leq i \leq 2k$, and that there exist no other intersections, where $C_{2k+1} = C_1$.) Suppose that S admits a fixed-point-free anti-holomorphic involution σ which interchanges C_i and C_{i+k} for all i with $1 \leq i \leq k$. We call such a σ a *real structure* of S , or more precisely of (S, C) . If $u : S' \rightarrow S$ is the blowing-up of S with center p_i and p_{i+k} for some i , and $C' = u^{-1}(C)$ with reduced structure, σ lifts to a real structure of (S', C') in the sense defined above. We call such a blowing-up *admissible* and call *admissible* any bimeromorphic morphism which is a finite succession of admissible blowing-ups.

By a *toric surface* we shall mean here a projective nonsingular rational surface S on which the algebraic torus $\mathbf{C}^* \times \mathbf{C}^*$ acts algebraically with an open orbit. The complement C of the open orbit U consists of a cycle of nonsingular rational curves and is a member of the anti-canonical system $|K_S^{-1}|$. The complement C is called the *anti-canonical cycle* of S . If (S, C) admits a real structure, we have $C_i^2 = C_{i+k}^2$ for all i , $1 \leq i \leq k$, that is, S is a *symmetric* toric surface in the sense of [Fuj00, (2.5)].

Let $\mathbf{P} = \mathbf{C}(z) \cup \{\infty\}$ be the complex projective line. Let $S_0 := \mathbf{P} \times \mathbf{P}$ and $p_i : S_0 \rightarrow \mathbf{P}$ be the i th projections, $i = 1, 2$. Set $\bar{C}_1 = p_1^{-1}(0)$, $\bar{C}_2 = p_2^{-1}(0)$, $\bar{C}_3 = p_1^{-1}(\infty)$, and $\bar{C}_4 = p_2^{-1}(\infty)$. On \mathbf{P} we have anti-holomorphic involutions $\sigma_1 : z \rightarrow 1/\bar{z}$ and $\sigma_2 : z \rightarrow -1/\bar{z}$. We consider S_0 naturally as a toric surface with open orbit $S_0 - \bar{C}$ and with real structure σ on (S_0, \bar{C}) defined by $\sigma_1 \times \sigma_2$, where $\bar{C} = \bigcup_{j=1}^4 \bar{C}_j$. If $f : S \rightarrow S_0$ is any admissible bimeromorphic morphism, S is again a toric surface with naturally induced real structure.

3. Rational surfaces with anti-Kodaira dimension one

Let S be a projective nonsingular surface and set $A = \{m \geq 0 : |K_S^{-m}| \neq \emptyset\}$. When A is nonempty, the *anti-Kodaira dimension* $\kappa^{-1}(S)$ of S is defined to be

$$\kappa^{-1}(S) := \max_{m \in A} \dim f_m(S),$$

where f_m is the rational map defined by the anti-canonical system $|K_S^{-m}|$ and $f_m(S)$ denotes the rational image of S by f_m . If A is empty, we set $\kappa^{-1}(S) = -\infty$. The purpose of this section is to

construct a family of rational surfaces with anti-Kodaira dimension one by deforming certain real symmetric toric surfaces.

Suppose that S is a projective nonsingular rational surface with $|K_S^{-1}| \neq \emptyset$. Let C be a connected curve in $|K_S^{-1}|$. Denote by $N := [C]|_C$ the normal bundle of C in S so that $N \cong K_S^{-1}|_C$. We assume that N is a torsion line bundle on C of order $n \geq 1$.

LEMMA 3.1. *Let the notations and assumptions be as above. Then we have*

$$h^0(K_S^{-l}) = 1, \quad 1 \leq l < n, \quad \text{and} \quad h^0(K_S^{-n}) = 2. \tag{1}$$

$|K_S^{-n}|$ induces a structure $g : S \rightarrow \mathbf{P}$ of an elliptic surface on S . When $n > 1$, C is the unique member of $|K_S^{-1}|$ and it is the support of a unique multiple fiber of g , whose multiplicity is n .

Proof. By using the Riemann–Roch theorem we have

$$H^0(C, N^k) = H^1(C, N^k) = 0 \quad \text{for } 1 \leq k \leq n - 1. \tag{2}$$

Let I be the ideal sheaf of C in S . For any integer $l > 0$ we denote by $C_{(l)}$ the subspace of S defined by the ideal sheaf I^l . Then we have the short exact sequence

$$0 \rightarrow O_S \rightarrow K_S^{-l} \rightarrow K_S^{-l}|_{C_{(l)}} \rightarrow 0,$$

and the associated long exact sequence

$$0 \rightarrow H^0(S, O_S) \rightarrow H^0(S, K_S^{-l}) \rightarrow H^0(C_{(l)}, K_S^{-l}|_{C_{(l)}}) \rightarrow 0. \tag{3}$$

We compute $H^0(C_{(l)}, K_S^{-l}|_{C_{(l)}})$ for $l \leq n$. Noting that $N^{l-k} \cong I^k/I^{k+1} \otimes K_S^{-l}$ we have

$$0 \rightarrow N^{l-k} \rightarrow K_S^{-l}|_{C_{(k+1)}} \rightarrow K_S^{-l}|_{C_{(k)}} \rightarrow 0.$$

Using (2) we then get

$$H^0(C, K_S^{-l}|_C) \cong \dots \cong H^0(C_{(l)}, K_S^{-l}|_{C_{(l)}}) \quad \text{for any } l \leq n, \tag{4}$$

while $h^0(C, K_S^{-l}|_C) = 1$ if $l = n$, and $= 0$ otherwise. Thus by (3) we have $h^0(S, K_S^{-n}) = 2$ and $h^0(S, K_S^{-l}) = 1$ if $1 \leq l < n$. Therefore we get the first assertion. Also, (3) and (4) for $l = n$ show that $|K_S^{-n}|$ is base-point-free and $C_{(n)}$ is a fiber of the associated morphism $g : S \rightarrow \mathbf{P}$ which induces an elliptic surface structure on S . When $n > 1$, $C_{(n)}$ is a multiple fiber with multiplicity n . Finally, on a rational surface there exists at most one multiple fiber (cf. [Dol81, p. 133, Corollary 2]). \square

Consider the toric surface $S_0 := \mathbf{P} \times \mathbf{P}$ with its real structure σ given as in § 2.2. Denote by $\alpha : T \rightarrow S_0$ the blowing-up of the four nodes of \bar{C} , and by C_i , $1 \leq i \leq 4$, the proper transforms of \bar{C}_i in T . We further define E_i to be the inverse images of the nodes $\bar{C}_i \cap \bar{C}_{i+1}$, where $\bar{C}_5 = \bar{C}_1$. Then we set $q_i = E_i \cap C_i$, and $r_i = E_i \cap C_{i+1}$. The real structure σ on S_0 lifts to one on T , which is again denoted by σ . For $i = 1, 2$, σ interchanges members of each of the pairs (C_i, C_{i+2}) , (E_i, E_{i+2}) , (q_i, q_{i+2}) and (r_i, r_{i+2}) respectively. Let $B := \alpha^{-1}(\bar{C})$ with reduced structure, which is the union of C_i and E_j , $1 \leq i, j \leq 4$. Then T is again a toric surface with anti-canonical cycle B .

Now we take an admissible bimeromorphic morphism $u : T' \rightarrow T$, where we include the case where $T' = T$ and u is the identity. Again T' is a toric surface with anti-canonical cycle B' given by the reduced inverse image of B and with induced real structure (still denoted by σ).

Let E'_i be the proper transform of E_i in T' . Then u induces an isomorphism $E'_i \cong E_i$. In general we denote a point of E'_i corresponding to a point p of E_i by p' ; for example, we may speak of the points q'_i and r'_i of E'_i . We consider the product

$$E := E_1 \times E_2 \times E_3 \times E_4$$

and the holomorphic family $\{S_a\}_{a \in E}$ of nonsingular rational surfaces S_a parametrized by $a = (a_1, a_2, a_3, a_4) \in E$, $a_i \in E_i$, where S_a is obtained from T' by blowing up the four points a'_1, a'_2, a'_3 and a'_4 of T' . In general S_a is not a toric surface any more.

Suppose that $a_i \neq q_i, r_i$ for any i . Then we may also view the surface S_a as follows. Let $v_a : T_a \rightarrow T$ be the blowing-up of $a_i, 1 \leq i \leq 4$. Since $a_i \neq q_i, r_i$, there exists an admissible birational morphism $u_a : S_a \rightarrow T_a$ such that $v_a u_a = u w_a$, i.e.

$$\begin{array}{ccc}
 S_a & \xrightarrow{w_a} & T' \\
 u_a \downarrow & & \downarrow u \\
 T_a & \xrightarrow{v_a} & T
 \end{array} \tag{5}$$

where $w_a : S_a \rightarrow T'$ is the natural map.

Now we show the following lemma.

LEMMA 3.2. *Suppose that $a_i \neq q_i, r_i$ for any i as above. Let F_a be the line bundle on B defined by the Cartier divisor $a_1 + a_2 + a_3 + a_4$ on B . Suppose further that*

$$L_a := F_a^{-1} \otimes (K_T^{-1}|B) \in \text{Pic}_c B \tag{6}$$

is a torsion line bundle on B . Then we have $h^0(K_{S_a}^{-1}) \geq 1$ and $\kappa^{-1}(S_a) = 1$. The case $h^0(K_{S_a}^{-1}) > 1$ occurs if and only if $u : T' \rightarrow T$ is the identity and L_a is trivial. Moreover, in this case $h^0(K_{S_a}^{-1}) = 2$.

Proof. We first study the structure of T_a . Let n be the order of L_a on B . Let B_a be the proper transform of B in T_a . Then since $a_i \neq q_i, r_i$, $v_a|B_a : B_a \rightarrow B$ is isomorphic, and B_a is again an anti-canonical cycle of T_a . In particular as a line bundle we have

$$K_{T_a}^{-1}|B_a \cong [B_a]|B_a, \tag{7}$$

where K_{T_a} is the canonical bundle of T_a . On the other hand, if we denote the exceptional curve of v_a by $D_a := \bigcup_{1 \leq i \leq 4} v_a^{-1}(a_i)$ we have $[D_a]|B_a \cong (v_a|B_a)^* F_a$. Therefore we get

$$K_{T_a}^{-1}|B_a \cong (v_a^* K_T^{-1} \otimes [D_a]^{-1})|B_a \cong (v_a|B_a)^*(K_T^{-1}|B \otimes F_a^{-1}) = (v_a|B_a)^* L_a. \tag{8}$$

By (7), (8) and our assumption, $[B_a]|B_a$ is thus a torsion line bundle of order n . Then by Lemma 3.1, $|K_{T_a}^{-n}|$ defines a holomorphic map $f_a : T_a \rightarrow \mathbf{P}$ making T_a an elliptic surface over \mathbf{P} such that $B_{a(n)}$ is the unique multiple fiber of f_a .

Using this we next show that $\kappa^{-1}(S_a) = 1$. Since the morphism $u_a : S_a \rightarrow T_a$ is admissible, the inverse image $C_a := u_a^{-1}(B_a)$ of B_a in S_a with reduced structure is a member of $|K_{S_a}^{-1}|$ and C_a is the support of a fiber of the induced elliptic surface structure $f_a u_a : S_a \rightarrow \mathbf{P}$. In particular $h^0(K_{S_a}^{-1}) \geq 1$, and we have the equalities

$$\kappa^{-1}(S_a) = \kappa([C_a]; S_a) = \kappa([Q]; S_a) = 1,$$

where $\kappa(L; S_a)$ denotes the Iitaka dimension of a line bundle L on S_a in general [Uen75] and Q denotes a general fiber of $f_a u_a$.

Finally, when u_a is not isomorphic, the member C_a of $|K_{S_a}^{-1}|$ is reduced and is a support of a fiber, but does not coincide with the fiber. It follows that C_a cannot move and hence $\dim h^0(K_{S_a}^{-1}) = 1$. Suppose next that u_a is isomorphic so that $S_a \cong T_a$ and $T' \cong T$. In this case if $\dim |K_{T_a}^{-1}| > 0$ and B'_a is any member other than B_a , we must have $B_a \cap B'_a = 0$ since $K_{T_a}^{-1}|B_a$ is torsion. This implies that $K_{T_a}^{-1}|B_a$ is trivial, or L_a is trivial on B by (8). Conversely, if this condition is satisfied, we have $h^0(K_{T_a}^{-1}|B_a) = 1$, so that by the sequence (3) applied to $S = T_a$ and $l = 1$ we have $h^0(K_{T_a}^{-1}) = 2$. \square

Let $U \cong \mathbf{C}^{*4}$ be the Zariski open subset of E consisting of those points $a = (a_1, a_2, a_3, a_4)$ for which $a_i \neq q_i, r_i$ for any i . Let $\text{Pic}_c B$ be the connected component of $\text{Pic } B$ containing $K_T^{-1}|B$ as

defined in § 2.1. The real structure σ of T induces anti-holomorphic involutions of $E, U, \text{Pic}_c B$, etc., all denoted still by σ and referred to as real structures. Note that the induced real structure on E interchanges E_1 and E_3 , and E_2 and E_4 . A point which is fixed by σ is called a *real point*. $\text{Pic}_c^R B$ is thus the set of real points on $\text{Pic}_c B$. Now we consider the special real point $a^\# := (q_1, r_2, q_3, r_4)$ of E .

LEMMA 3.3. *For any line bundle F in $\text{Pic}_c B$ and for any neighborhood V of $a^\#$ in E there exists a point $a = (a_1, a_2, a_3, a_4)$ of $U \cap V$ such that the associated line bundle F_a on B is isomorphic to F . If F belongs to $\text{Pic}_c^R B$, we may take a to be a real point.*

Proof. For any element $F \in \text{Pic}_c B$ we have $\deg F|E_i = 1$ and $\deg F|C_j = 0$ for any i and j . By Serre duality and by our assumption we have $h^1(B, F) = h^0(B, F^{-1}) = 0$ in view of the triviality of $K_B := (K_T \otimes [B])|B$. Then by the Riemann–Roch theorem we get $h^0(B, F) = 4$. In fact it is readily shown that the restriction map $H^0(B, F) \rightarrow \bigoplus_j H^0(C_j, F|C_j) \cong \mathcal{C}^4$ is isomorphic. Let $p_j : H^0(B, F) \rightarrow H^0(C_j, F|C_j) \cong \mathcal{C}$ be the induced projection, and $K_j(F)$ its kernel. Then we obtain a \mathbf{P}^3 -bundle $h : W \rightarrow \text{Pic}_c B$ whose fiber over $F \in \text{Pic}_c B$ is the projectified vector space $\mathbf{P}H^0(B, F) := (H^0(B, F) - \{0\})/\mathcal{C}^*$. Moreover we obtain its \mathbf{P}^2 -subbundles $W_j \rightarrow \text{Pic}_c B, 1 \leq j \leq 4$, with fibers $\mathbf{P}K_j(F) \subseteq \mathbf{P}H^0(B, F)$.

Let $U_0 := W - \bigcup_j W_j$. Note that a point of W belongs to $\bigcup_j W_j$ if and only if a representative of it in $H^0(B, F)$ vanishes at least one of the points in $a^\#$; in fact since $\deg F|C_j = 0$, if a section of F vanishes at a point of C_j it vanishes identically on C_j . Then we have a natural biholomorphic map $g : U_0 \rightarrow U$ of U_0 onto U , where a point of U_0 represented by a section s of F , which never vanishes on any of the points in $a^\#$ by the above remark, corresponds to the point (a_1, a_2, a_3, a_4) of U , a_i being the unique zero of s on $E_i - \{q_i, r_i\}$.

Now for any neighborhood V of $a^\#$ in E if we take a sufficiently small neighborhood N of $W_1 \cap W_3$ in W we have $g(N \cap U_0) \subseteq V$ since points of $W_1 \cap W_3$ are represented by sections which vanish at all points in $a^\#$. Any point a of the set $g(N \cap U_0 \cap h^{-1}(F))$, which is clearly nonempty, then meets the requirement of the lemma.

Suppose finally that F belongs to $\text{Pic}_c^R B$. Then E and W have natural real structures of which U and U_0 are σ -invariant open subsets respectively. Moreover, g and h are compatible with real structures. By Lemma 2.2 the real part $h^{-1}(F)_R$ of $h^{-1}(F)$ is diffeomorphic to a three-dimensional real projective space and it has nonempty intersection with the σ -invariant \mathbf{P} -bundle $W_1 \cap W_3$ over $\text{Pic}_c B$. Thus $N \cap h^{-1}(F)_R$ is never empty, and we can take any point a of $g(N \cap h^{-1}(F)_R)$ as a desired real point. □

COROLLARY 3.4. *Let n be a positive integer. Then for any neighborhood V of $a^\#$ in E there exists a point a of $U \cap V$ such that the corresponding line bundle L_a (cf. (6)) is a torsion line bundle on B of order n . If $n = 1$ or 2 , we can take a to be a real point in $U \cap V$.*

Proof. For the first assertion we have only to take F in the lemma in such a way that $F^{-1} \otimes K_T^{-1}|B$ is a torsion line bundle of order n on B and then apply the above lemma to F . When $n = 1$ or 2 , $F^{-1} \otimes K_T^{-1}|B$, and hence F also, is real and the assertion again follows from the above lemma. □

For later reference we shall summarize what we have obtained in this section in terms of deformation of surfaces. Let S be the toric surface with real structure σ obtained from T' by blowing up the four points q'_1, r'_2, q'_3 and r'_4 , i.e. $S = S_{a^\#}$ for $a^\# = (q_1, r_2, q_3, r_4)$. For a neighborhood V of $a^\#$ in E we consider the surfaces $S_a, a \in V$, as small deformations of S . The surface S_a contains a cycle C_a of nonsingular rational curves and the point a defines a holomorphic line bundle L_a on the cycle of nonsingular rational curves B on T by the formula (6).

PROPOSITION 3.5. *Let n be any positive integer. Then for any neighborhood V of $a^\#$ as above, we can find a point $a \in U \cap V$ such that L_a is a torsion line bundle of order n and $\kappa^{-1}(S_a) = 1$. Moreover, if $n = 1$ or 2 , we can take S_a to be a real surface with real structure σ_a so that (S_a, σ_a) is a deformation of (S, σ) with real smooth parameters.*

Note that the last statement follows from the proof of Lemma 3.3.

4. Algebraic dimension of Z and anti-Kodaira dimension of S

Let Z be the twistor space associated to a self-dual structure of positive type on $m\mathbf{P}^2, m > 0$. In this section we shall show that the algebraic dimension $a(Z)$ of Z equals two, provided that Z contains a special rational surface as considered in Lemma 3.2 as a smooth member of the system $|K^{-1/2}|$, where $K := K_Z$ is the canonical bundle of Z and $K^{1/2}$ is its canonical square root.

Suppose now that $|K^{-1/2}|$ contains a smooth member S , which is necessarily a rational surface (cf. [PP94a]). By the Hitchin vanishing theorem $H^2(Z, O_Z) = 0$ (see [Hit80]). Then we get $H^1(Z, O_Z) = 0$ since $\chi(Z, O_Z) = c_1c_2/24 = 1/2(\chi - \tau) = 1$ (cf. [Hit81]), where χ and τ are respectively the Euler characteristic and the signature of $m\mathbf{P}$. Then in view of the relation $K^{-1/2}|_S = K_S^{-1}$ we get the exact sequence of sheaves

$$0 \rightarrow O_Z \rightarrow K^{-1/2} \rightarrow K_S^{-1} \rightarrow 0 \tag{9}$$

and the associated exact sequence of cohomology

$$0 \rightarrow H^0(Z, O_Z) \rightarrow H^0(Z, K^{-1/2}) \xrightarrow{b} H^0(S, K_S^{-1}) \rightarrow 0. \tag{10}$$

In particular we have $h^0(K^{-1/2}) = 1 + h^0(K_S^{-1})$. Suppose that $h^0(K_S^{-1}) > 0$ so that $\dim|K^{-1/2}| \geq 1$. Let P be a linear subpencil of $|K^{-1/2}|$ whose general member is smooth. Then there exists a countable or finite subset N of P such that for any $y \in P - N$ the corresponding surface S_y on Z is nonsingular and its anti-Kodaira dimension $\kappa^{-1}(S_y) = d$ for some fixed integer $d = d(P), 0 \leq d \leq 2$. We shall relate this number d with the algebraic dimension $a(Z)$ of Z .

PROPOSITION 4.1. *Let P be a linear pencil with $d = d(P)$ as above. Then we have the equality $a(Z) = d + 1$. In particular $d = d(P)$ is independent of the choice of such a P .*

Proof. Let S be any general smooth member of P . Then we have the inequality

$$a(Z) \leq \kappa^{-1}(S) + 1 = d + 1$$

by [Cam91, Proposition 1.5]. It thus suffices to show the reverse inequality $a(Z) \geq d + 1$. By eliminating the indeterminacy of the pencil P and taking a resolution of the singularities we obtain the diagram

$$\begin{array}{ccc} \hat{Z} & \xrightarrow{u} & Z \\ f \downarrow & & \\ & & P \end{array}$$

of compact complex manifolds, where u is a bimeromorphic morphism; moreover, the proper transform \hat{S} of a general member S of P in \hat{Z} is a fiber of f and is mapped isomorphically onto S by u . Then by [Uen75, Proposition 12.2] we get that

$$a(\hat{Z}) \geq d + \dim P = d + 1 = \kappa^{-1}(S) + 1. \quad \square$$

Suppose further that S is isomorphic to a nonsingular rational surface S_a for some $a = (a_1, \dots, a_4) \in U$ defined before Lemma 3.2. We identify S with S_a in what follows. Surface S contains a cycle C_a of nonsingular rational curves, which is a member of the anti-canonical system $|K_S^{-1}|$.

If $|K^{-1/2} - C_a|$ denotes the linear subsystem of $|K^{-1/2}|$ consisting of elements which contain C_a , we have $\dim |K^{-1/2} - C_a| = 1$ by (10).

PROPOSITION 4.2. *Suppose that the line bundle L_a on B in Lemma 3.2 is a torsion line bundle. Then we have $a(Z) = 2$.*

In view of Proposition 4.1, the proposition is a consequence of the following lemma.

LEMMA 4.3. *Let $P = |K^{-1/2} - C_a|$, which is a subpencil of $|K^{-1/2}|$ containing S . Then under the assumption of the previous proposition we have $d(P) = \kappa^{-1}(S) = 1$.*

Proof. Let a point $o \in P$ correspond to S . Then by the definition of $d(P)$ it suffices to show that there exists a neighborhood U of o in P such that for any point $t \in U$ the corresponding surface S_t is nonsingular and with $\kappa^{-1}(S_t) = 1$.

For this purpose first we note that $C_a = C_1 + \dots + C_{2k}, k = m + 2$, is naturally contained in every S_t and recall a sequence of bimeromorphic morphisms $w_a : S = S_a \rightarrow T', u : T' \rightarrow T$ and $\alpha : T \rightarrow S_0$ of nonsingular surfaces, where w_a contracts four (-1) -curves $D_{ai}, 1 \leq i \leq 4$, in S_a to the smooth points a'_i of $B' \subseteq T'$ and u and α are admissible bimeromorphic morphisms. More precisely, we have decompositions $u = \mu_{m-2} \cdots \mu_1$ and $\alpha = \mu_{m-1} \mu_m$, where $\mu_i : T_{i-1} \rightarrow T_i$ are admissible blowing-downs with $T_0 = T'$ and $T_m = S_0$, and for each i there exists a unique index $\kappa_i, 1 \leq \kappa_i \leq k$ (with $\kappa_i \neq \kappa_j$ for $i \neq j$) such that if we set $\nu_i := \mu_i \cdots \mu_1 : T' \rightarrow T_i, 1 \leq i \leq m$, and $\nu_0 := id_{T'}$, then $\nu_{i-1}(C_{\kappa_i})$ and $\nu_{i-1}(C_{\kappa_i+k})$ are (-1) -curves on T_{i-1} and μ_i is precisely the contraction of these two (-1) -curves.

We also recall a general fact: Let $\{X_t\}_{t \in A}$ be a holomorphic family of nonsingular complex surfaces parametrized by a domain A in \mathbf{C} . Let C be a (-1) -curve on X_o for some $o \in A$. Then there exist a neighborhood U of o in A and a unique holomorphic family $\{C_t\}$ of (-1) -curves C_t in X_t such that $C_o = C$ (cf. [Kod63]). Moreover, there exists a holomorphic family of bimeromorphic morphisms $X_t \rightarrow Y_t, t \in U$, which blow down C_t to nonsingular points p_t of Y_t (cf. [FN72]).

Applying this result to the family $\{S_t\}_{t \in P}$ in a neighborhood of o and the (-1) -curves $D_{ai}, 1 \leq i \leq 4$, in S we have the following: There exist a neighborhood U of o in P , holomorphic families of (-1) -curves $\{D_{ai,t}\}_{t \in U}$ in S_t with $D_{ai,o} = D_{ai}, 1 \leq i \leq 4$, and a holomorphic family of bimeromorphic morphisms $w_t : S_t \rightarrow T'_t$ contracting $D_{ai,t}$ to points where $T'_o = T'$ and $w_o = w_a$. Here, if we take U sufficiently small, we can assume that each $D_{ai,t}$ intersects transversally at a single point with a unique irreducible component of C_a which is independent of t . Thus C_a is mapped isomorphically onto a curve in T'_t denoted again by C_a independently of t .

The self-intersection number of each irreducible component of C_a in T'_t is independent of t ; in particular C_{κ_1} and C_{κ_1+k} are considered as (-1) -curves in T'_t for all t . Then μ_1 extends to a holomorphic family of admissible blowing-downs $\mu_{1t} : T_{0t} := T'_t \rightarrow T_{1t}$ which are indeed the blowing-downs of these (-1) -curves in T'_t with $\mu_{1o} = \mu_1$ and $T_{1o} = T_1$. Clearly the restriction $\mu_{1t}|_{C_a}$ is independent of t and hence so is the image $C_{at}^{(1)} := \mu_{1t}(C_a)$ in T_{1t} up to isomorphisms together with the self-intersection numbers of its irreducible components.

Then $\mu_{1t}(C_{\kappa_2})$ and $\mu_{1t}(C_{\kappa_1+k})$ are (-1) -curves in T_{1t} and we have again a holomorphic family of admissible blowing-downs $\mu_{2t} : T_{1t} \rightarrow T_{2t}$ contracting the two curves with $\mu_{2o} = \mu_2$ and $T_{2o} = T_2$. Continuing in this way we finally obtain holomorphic families of admissible blowing-downs $\mu_{it} : T_{i-1,t} \rightarrow T_{it}$ with $\mu_{io} = \mu_i$ and $T_{io} = T_i, 1 \leq i \leq m$, such that the image $C_{at}^{(i)} := \nu_{it}(C_a)$ is independent of t up to canonical isomorphisms, where $\nu_{it} = \mu_{it} \cdots \mu_{1t}$. In particular the pair $(T_{mt}, C_{at}^{(m)})$ is isomorphic to the pair $(T_{mo}, \nu_{mo}(C_a)) = (S_0, \nu_m(C_a)) \cong (\mathbf{P} \times \mathbf{P}, \bar{C})$ since the last pair is rigid under small (log-)deformations.

Then identifying $(T_{mt}, C_a^{(m)})$ with $(\mathbf{P} \times \mathbf{P}, \bar{C})$ if we follow the above process of successive blowing-downs in the reverse way as a succession of admissible blowing-ups, we conclude that the blowing-up

occurs in each step on the same pairs of points independently of t , and in particular (T'_t, C_a) are all isomorphic to (T', B) independently of t (after possible restriction of U).

Thus taking and fixing isomorphisms of these spaces we shall suppress the suffix t ; e.g. we write $T'_t = T', T_t = T, S_{0t} = S_0, u_t := \mu_{m-2,t} \cdots \mu_{1t} = u$, and $\alpha_t := \mu_{mt}\mu_{m-1,t} = \alpha$. Accordingly, we consider w_t as a bimeromorphic morphism $(S_t, C_a) \rightarrow (T', B')$, which is the blowing-down of four (-1) -curves $D_{it}, 1 \leq i \leq 4$, in S_t that are deformations of $D_{i0} = D_{ai}$. Thus if $a'_{it} = w_t(D_{it})$ are the points of T' which form the center of the blowing-up w_t , and $[D_t]$ is the line bundle on S_t defined by the divisor $D_t := D_{1t} + D_{2t} + D_{3t} + D_{4t}$, we get that $[D_t]|_{C_a}$ is the pull-back of the line bundle \tilde{F}_{at} on B' defined by the Cartier divisor $a'_{1t} + a'_{2t} + a'_{3t} + a'_{4t}$ on B' with respect to the isomorphism $w_t|_{C_a} : C_a \xrightarrow{\sim} B'$, which is independent of t and coincides with $w_B := w|_B$. Note that if we restrict U smaller if necessary, a'_{it} belongs to $E'_i - \{q'_i, r'_i\}$.

Now since $K_{S_t}^{-1}|_{C_a} \cong K^{-1/2}|_{C_a}$ by the adjunction formula, $K_{S_t}^{-1}|_{C_a}$ is independent of t . Furthermore we have

$$(K_{S_t}^{-1} \otimes [D_t])|_{C_a} \cong (w_t^* K_{T'}^{-1})|_{C_a} = (w_t|_{C_a})^*(K_{T'}^{-1}|_{B'}) = w_B^*(K_{T'}^{-1}|_{B'}).$$

Hence we get

$$(w_t|_{C_a})^* \tilde{F}_{at} \cong (K_{S_t}|_{C_a}) \otimes w_B^*(K_{T'}^{-1}|_{B'}).$$

By what we have shown above, the right-hand side is independent of t , and hence the same is true for the left-hand side. Moreover, since $(w_t|_{C_a})_* O_{C_a}^* \cong O_{B'}^*$, $w_t^* : \text{Pic } B' \rightarrow \text{Pic } C_a$ is injective. This implies that \tilde{F}_{at} itself is independent of t as an element of $\text{Pic } B'$. Furthermore, since $u|_{B'} : B' \rightarrow B$ induces isomorphisms of E'_i and E_i , we have $\tilde{F}_{at} \cong u^* F_{at}$, where F_{at} is the line bundle on B defined by the Cartier divisor $a_{1t} + a_{2t} + a_{3t} + a_{4t}$ on B with $a_{it} = u(a'_{it})$.

Thus if we set $a_t = (a_{1t}, \dots, a_{4t})$, we can identify S_t with S_{a_t} and F_{at} with F_{a_t} in the notation of Lemma 3.2, and therefore $L_{a_t} := F_{a_t}^{-1} \otimes (K_T^{-1}|_B)$ is also independent of t as a line bundle on B . In particular it is a torsion line bundle on B as well as L_a . Hence, by Lemma 3.2, $\kappa^{-1}(S_t) = \kappa^{-1}(S_{a_t}) = 1$ for all $t \in U$ as was desired. \square

5. Deformations of Moishezon twistor spaces

Let Z be a Moishezon twistor space associated to a compact self-dual manifold $(M, [g])$ with $M = m\mathbf{P}^2$, $m > 0$, which is necessarily of positive type [Poo88]. Suppose that Z contains a smooth member S of $|K^{-1/2}|$, which is a rational surface (cf. [PP94a]). In this section we show the vanishing of the obstructions related to deformations of Z and S . Similar but somewhat weaker results can be found in the articles by Campana [Cam91, Cam94], LeBrun [LeB92], and Pedersen and Poon [PP94b]. The proof of the following theorem is given after Lemma 5.5

THEOREM 5.1.

- 1) For $i = 2, 3$ we have

$$H^i(Z, \Theta) = H^i(Z, \Theta(-\log S)) = H^i(Z, \Theta(-S)) = H^i(S, \Theta) = 0.$$

- 2) The natural map $H^1(Z, \Theta(-\log S)) \rightarrow H^1(S, \Theta)$ is surjective.
- 3) We have

$$h^1(Z, \Theta) - h^0(Z, \Theta) = 7m - 15, \quad h^1(\Theta_Z(-\log S)) - h^0(\Theta_Z(-\log S)) = 5m - 6.$$

Here Θ denotes the sheaf of germs of holomorphic vector fields on Z , and $\Theta(-\log S)$ (respectively $\Theta(-S)$) the subsheaf of Θ defined by those vector fields which are tangent to (respectively vanish on) S . Note that if $H^2(Z, \Theta(-\log S))$ vanishes, then the local deformations of the pair (Z, S)

are unobstructed and the tangent space of the corresponding Kuranishi space is naturally identified with $H^1(Z, \Theta(-\log S))$. In particular we obtain the following corollary.

COROLLARY 5.2. *The Kuranishi spaces T_Z, T_S and $T_{(Z,S)}$ for the local deformations of Z, S and the pair (Z, S) respectively are all smooth. The versal holomorphic maps $T_{(Z,S)} \rightarrow T_S$ are submersive at the base point.*

Let $\Omega = \Omega_Z^1$ be the sheaf of holomorphic 1-forms on Z . Applying $\otimes K^{1/2}$ to (9) we obtain a short exact sequence

$$0 \rightarrow K \rightarrow K^{1/2} \rightarrow K_S \rightarrow 0.$$

Applying $\otimes \Omega$ to (9) and to this sequence we further get

$$0 \rightarrow \Omega K^{1/2} \rightarrow \Omega \rightarrow \Omega|_S \rightarrow 0$$

and

$$0 \rightarrow \Omega K \rightarrow \Omega K^{1/2} \rightarrow \Omega|_S K_S \rightarrow 0,$$

respectively. (Here and in what follows the juxtaposition denotes the tensor product.) Consider the associated long cohomology exact sequences

$$\rightarrow H^0(S, \Omega|_S) \rightarrow H^1(Z, \Omega K^{1/2}) \rightarrow H^1(Z, \Omega) \xrightarrow{a} H^1(S, \Omega|_S) \rightarrow \tag{11}$$

and

$$\rightarrow H^0(S, \Omega|_S K_S) \rightarrow H^1(Z, \Omega K) \rightarrow H^1(Z, \Omega K^{1/2}) \rightarrow H^1(S, \Omega|_S K_S) \rightarrow . \tag{12}$$

On the other hand, by applying $\otimes K_S$ to the standard exact sequence

$$0 \rightarrow N^* \rightarrow \Omega|_S \rightarrow \Omega_S \rightarrow 0,$$

we obtain

$$0 \rightarrow N^* K_S \rightarrow \Omega|_S K_S \rightarrow \Omega_S K_S \rightarrow 0,$$

where N is the normal bundle of S in Z and N^* its dual. From these we get the associated cohomology exact sequences

$$\begin{aligned} 0 \rightarrow H^0(S, N^*) \rightarrow H^0(S, \Omega|_S) \rightarrow H^0(S, \Omega_S) \\ \rightarrow H^1(S, N^*) \rightarrow H^1(S, \Omega|_S) \xrightarrow{b} H^1(S, \Omega_S) \rightarrow \end{aligned} \tag{13}$$

and

$$0 \rightarrow H^0(S, N^* K_S) \rightarrow H^0(S, \Omega|_S K_S) \rightarrow H^0(S, \Omega_S K_S) \rightarrow H^1(S, N^* K_S) \rightarrow . \tag{14}$$

Since S is rational, in view of $N^* = K_S$ and $N^* K_S = 2K_S$ we see that $H^0(S, N^*) = H^0(S, N^* K_S) = 0$, $H^1(S, N^*) = 0$ and $H^0(S, \Omega_S K_S) = 0$. Substituting these into (13) and (14) we get the next lemma.

LEMMA 5.3. *We have that $H^0(S, \Omega|_S) = 0$ and $H^0(S, \Omega|_S K_S) = 0$.*

Next we prove the following.

LEMMA 5.4. *The restriction map $H^1(Z, \Omega) \rightarrow H^1(S, \Omega_S)$ is injective.*

Proof. Since both Z and S are Moishezon with $h^{2,0} = 0$, it suffices to show the injectivity of the restriction map $r : H^2(Z, \mathbf{C}) \rightarrow H^2(S, \mathbf{C})$. Let $t : Z \rightarrow M$ be the C^∞ twistor fibration. We consider the commutative diagram

$$\begin{array}{ccc} H^2(Z, \mathbf{C}) \cong H^2(M, \mathbf{C}) \oplus \mathbf{C}c_1(Z) & & \\ \searrow r & \downarrow t_S^* & \downarrow r \\ & H^2(S, \mathbf{C}) \supseteq \mathbf{C}c_1(S) & \end{array}$$

where the horizontal isomorphism is via the Leray–Hirsch theorem (cf. [Hit81]), $t_S : S \rightarrow M$ is the restriction of t to S and c_1 denotes the first Chern class.

We know the injectivity of t_S^* ; indeed, if we describe S as a surface obtained from $\mathbf{P} \times \mathbf{P}$ by a succession of blowing-ups of pairwise σ -conjugate points, the Chern classes $c(i)$ of the exceptional curves $E_i + \bar{E}_i$ in S coming from the i th blowing-up forms a basis of $t_S^*H^2(M, \mathbf{C})$ (cf. [PP94a, § 3]). Moreover, $c_1(S)$ is not a linear combination of $c(i)$ in $H^2(S, \mathbf{C})$. Thus the injectivity follows from the above diagram. □

Applying these to the above sequences (11) and (12) we obtain our next lemma.

LEMMA 5.5. *We have that $H^i(Z, \Omega K) = H^i(Z, \Omega K^{1/2}) = 0$, $i = 0, 1$.*

Proof. The assertion for $i = 0$ follows from the fact that $K^{-1/2}$ is effective and $H^0(Z, \Omega) = 0$ (cf. [Hit81]). For $i = 1$ we note that the restriction map $H^1(Z, \Omega) \rightarrow H^1(S, \Omega_S)$ factors as

$$H^1(Z, \Omega) \xrightarrow{a} H^1(S, \Omega|_S) \xrightarrow{b} H^1(S, \Omega_S).$$

Since ba is injective by Lemma 5.4, a also is injective. Thus in view of (11) and Lemma 5.3 we get $H^1(Z, \Omega K^{1/2}) = 0$, which in turn yields the vanishing of $H^1(Z, \Omega K)$ in view of (12) and Lemma 5.3. □

Proof of Theorem 5.1. By Serre duality and Lemma 5.5 we get

$$H^i(Z, \Theta_Z) = H^i(Z, \Theta_Z(-S)) = 0 \quad \text{for } i = 2, 3.$$

Consider the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(Z, \Omega K(\log S)) \rightarrow H^0(Z, \Omega K^{1/2}) \rightarrow H^0(S, \Omega_S K_S) \rightarrow \\ \rightarrow H^1(Z, \Omega K(\log S)) \rightarrow H^1(Z, \Omega K^{1/2}) \rightarrow H^1(S, \Omega_S K_S) \rightarrow \end{aligned}$$

associated to the sheaf exact sequence

$$0 \rightarrow \Omega K(\log S) \rightarrow \Omega(S)K \rightarrow \Omega_S K_S \rightarrow 0,$$

where we have used the isomorphism $\Omega(S)K \cong \Omega(K^{1/2})$. By Lemma 5.5, $H^i(Z, \Omega K^{1/2})$ vanish for $i = 0, 1$ and $H^0(S, \Omega_S K_S) = 0$ since S is rational. Thus we get

$$H^0(Z, \Omega K(\log S)) = H^1(Z, \Omega K(\log S)) = 0.$$

By taking the Serre dual we get

$$H^2(Z, \Theta(-\log S)) = H^3(Z, \Theta(-\log S)) = 0.$$

Thus part 1 is proved. Further, from the exact sequence of cohomology

$$0 \rightarrow H^0(S, \Theta_S) \rightarrow H^1(Z, \Theta(-S)) \rightarrow H^1(Z, \Theta(-\log S)) \rightarrow H^1(S, \Theta_S) \rightarrow 0,$$

arising from the sheaf exact sequence

$$0 \rightarrow \Theta(-S) \rightarrow \Theta(-\log S) \rightarrow \Theta_S \rightarrow 0,$$

we get part 2 of the theorem.

In view of the Riemann–Roch theorem and the relations (cf. [Hit81])

$$c_1^3 = 16(2\chi - 3\tau), \quad c_1 c_2 = 12(\chi - \tau) \quad \text{and} \quad c_3 = 2\chi,$$

where c_i are Chern classes of Z , and χ and τ are the Euler number and the signature of M respectively, we obtain

$$\chi(Z, \Theta) = \frac{1}{2}\{4(12c_1^3 - 19c_1 c_2 + 12c_3)\} = \frac{15}{2}\chi - \frac{19}{2}\tau = -7m + 15,$$

since $\chi = m + 2$ and $\tau = m$. Hence by part 1 we have

$$h^1(Z, \Theta) - h^0(Z, \Theta) = 7m - 15.$$

Further, from the sheaf exact sequence

$$0 \rightarrow \Theta_Z(-\log S) \rightarrow \Theta_Z \rightarrow N_{S/Z} \rightarrow 0,$$

we get

$$\chi(\Theta_Z(-\log S)) = \chi(\Theta_Z) - \chi(S, K_S^{-1}) = -7m + 15 - (c_1^2(S) + 1) = -5m + 6.$$

Hence by part 1 we have

$$h^1(\Theta_Z(-\log S)) - h^0(\Theta_Z(-\log S)) = 5m - 6.$$

Thus part 3 is proved. This completes the proof of Theorem 5.1. □

Let $K := S^1 \times S^1$ be the real two-torus and $G := \mathbf{C}^* \times \mathbf{C}^*$ its complexification. Let m be a positive integer and fix a smooth effective action of K on $m\mathbf{P}^2$. In [Joy95] Joyce has constructed a family of K -invariant self-dual metrics on $m\mathbf{P}^2$. We call a twistor space corresponding to any such invariant self-dual metric a *Joyce twistor space* (associated to the given K -action on $m\mathbf{P}^2$). It admits a naturally induced biholomorphic G -action. The structure of a Joyce twistor space has been studied in detail in [Fuj00]. In particular it is Moishezon.

The class of Joyce twistor spaces contains as a special case the degenerate case of LeBrun’s twistor spaces [LeB91], which we call of *LeBrun type* (cf. [Fuj00, Proposition 6.14] for the details). Now we specialize the above consideration to the case of Joyce twistor spaces and determine the dimensions of the individual cohomology groups in Theorem 5.1.

PROPOSITION 5.6. *In Theorem 5.1 suppose that Z is a Joyce twistor space and S is G -invariant (which is automatic unless Z is of LeBrun type). Suppose further that $m \geq 3$. Then we have*

- 1) $h^0(\Theta_Z(-\log S)) = 2$ and $h^1(\Theta_Z(-\log S)) = 5m - 4$
- 2) $h^0(Z, \Theta_Z) = 2$ and $h^1(Z, \Theta) = 7m - 13$.

Proof. The assertion on h^1 follows from the assertion on h^0 by Theorem 5.1. So we shall compute h^0 . (For the details on the structure of Z used below we refer to [Fuj00, § 6].) Since Z admits a G -action which preserves S , it is clear that $2 \leq h^0(\Theta_Z(-\log S)) \leq h^0(\Theta_Z)$. Thus it suffices to show that $h^0(\Theta_Z) = 2$.

First we show this assuming that Z is not of LeBrun type. (In particular $m \geq 3$.) In this case $|K^{-1/2}|$ is a pencil and its base locus C is a cycle of nonsingular rational curves. Let $\mu : \hat{Z} \rightarrow Z$ be the blowing-up of Z with center C . Then \hat{Z} becomes a holomorphic fiber space $f : \hat{Z} \rightarrow P$ over a nonsingular rational curve P and $E := \mu^{-1}(C)$ is isomorphic to the product $C \times P$. Clearly, the biholomorphic automorphism group $\text{Aut } Z$ of Z preserves $|K^{-1/2}|$ and hence the action lifts to one on \hat{Z} preserving the fibers of f and E . Note that the induced action of $\text{Aut}_0 Z$ on $|K^{-1/2}|$ is trivial since for $m \geq 3$ it contains more than three singular fibers as pointed out below. Let $\text{Aut}(\hat{Z}, E)/P$ be the relative Lie group over P whose fiber over a point $y \in P$ is just the automorphism group of the pair (Z_y, E_y) . Then an irreducible component of $\text{Aut}(\hat{Z}, E)/P$ which contains the identity section and is mapped surjectively onto P is unique and is isomorphic to $G \times P$. Since only holomorphic sections of the projection $G \times P \rightarrow P$ are constant sections, the identity component $\text{Aut}_0 Z$ must coincide with G . It follows that $h^0(Z, \Theta_Z) = \dim G = 2$. Thus the non-LeBrun case is finished.

In the LeBrun case the linear subsystem $|K^{-1/2}|^G := \mathbf{P}(H^0(Z, K^{-1/2})^G)$ of $|K^{-1/2}|$ is a pencil. There exist $m + 2$ pairs (S_i^+, S_i^-) , $1 \leq i \leq m + 2$, of σ -conjugate divisors such that $S_i := S_i^+ + S_i^-$ belongs to $|K^{-1/2}|^G$. Moreover, one can check that S_i^\pm do not move in Z except for exactly two successive i , say $m + 1$ and $m + 2$, if $m \geq 3$ (cf. [Fuj00, Lemmas 6.13 and 2.10]). In this case

$S_i^\pm, i \neq m + 1, m + 2$, are preserved by the action of $\text{Aut}_0 Z$. Since S_1 and S_2 generate the pencil $|K^{-1/2}|^G$, $\text{Aut}_0 Z$ also preserves $|K^{-1/2}|^G$. The rest of the argument is then precisely the same as in the non-LeBrun case, starting from $|K^{-1/2}|^G$ instead of $|K^{-1/2}|$. \square

Remark. For general LeBrun twistor spaces we have $h^1(\Theta_Z(-\log S)) = 5m - 5$ [PP94b, Theorem 2.3] and $h^1(Z, \Theta) = 7m - 14$ [LeB92, p. 302, Corollary 1] for $m \geq 3$.

6. Proof of Main Theorem

Recall that a member of the system $|K^{-1/2}|$ is called a fundamental divisor. Let $S = S_{a\#}$ be the symmetric toric surface defined before Proposition 3.5. It has the natural real structure induced from that of S_0 (cf. § 2.2). Then the second Betti number b_2 of S is always even and ≥ 10 by our construction. We start with the following proposition.

PROPOSITION 6.1. *There exists a Joyce twistor space Z associated to a self-dual structure on $m\mathbf{P}^2$ with $2m + 2 = b_2$ such that it contains the surface S as a real smooth fundamental divisor, where the real structure on S induced from Z coincides with the original one. In particular $m \geq 4$.*

Proof. Set $k = b_2/2 + 1$. By our construction of S we have $b_2 \geq 10$ and hence $k \geq 6$. The toric surface S is determined by the associated fan which consists of primitive elements $\rho_i = (s_i, t_i)$ in $\mathbf{Z}^2, 1 \leq i \leq 2k$, arranged counterclockwise such that $(s_{i+k}, t_{i+k}) = -(s_i, t_i)$ for $1 \leq i \leq k$, where s_i and t_i are coprime integers. We then define $(m_i, n_i) := (-t_i, s_i)$ for $1 \leq i \leq k$. The smoothness of S implies for the determinant that

$$\begin{vmatrix} m_i & n_i \\ m_{i+1} & n_{i+1} \end{vmatrix} = \begin{vmatrix} s_i & t_i \\ s_{i+1} & t_{i+1} \end{vmatrix} = 1. \tag{15}$$

Hence the sequence $\{\pm(m_i, n_i)\}_{1 \leq i \leq k}$ is the Orlik–Raymond invariant of a unique smooth K -action on $m\mathbf{P}^2$, where $m = k - 2 \geq 4$ (cf. [Fuj00, (3.3) and (3.4)] and [Joy95, 3.1]). Take any of the Joyce twistor spaces Z associated to this action. Then by [Fuj00, Lemma 5.12] together with the results of [Fuj00, § 6] Z contains a real smooth G -invariant fundamental divisor, which is isomorphic to the original S and will be identified with S . The final assertion on the real structure on S then follows from the general result of Pedersen and Poon [PP94a, § 3] on the induced real structure of a real fundamental divisor. \square

Remark. If we take S_a in such a way that each successive blowing-up forming the morphism $T' \rightarrow T$ takes place always on the proper transform of E_1 and E_3 , then the resulting twistor space is of LeBrun type and *vice versa*.

For the proof of the Main Theorem we need some remarks about the real structure on the spaces considered in § 5. Let Z be a Joyce twistor space. Then the linear system $|K^{-1/2}|$ contains the real pencil $|K^{-1/2}|^G$ whose general member S is a smooth projective toric surface with respect to the induced G -action. Consider a triple $(Z, S, \{p, \bar{p}\})$ consisting of such a pair (Z, S) with S real and a pair (p, \bar{p}) of a point p and its conjugate $\bar{p} := \sigma(p)$, where p , and hence \bar{p} also, is assumed to be in the open orbit U on S . We then consider the local deformations $\{(Z_t, S_t, \{p_t, q_t\})\}_{t \in T}$ of the triple $(Z, S, \{p, \bar{p}\})$, where Z_t is a compact complex threefold, S_t a smooth divisor on Z_t , and p_t and q_t are two points of S_t . From Theorem 5.1, Corollary 5.2 and Proposition 5.6 we deduce easily that the Kuranishi family for such deformations of triples exists and its parameter space T is smooth of dimension $5m - 2$. Moreover, T admits a natural submersion u onto the smooth parameter space of the Kuranishi family of the deformation of the pairs $(S, \{p, \bar{p}\})$ defined similarly.

On the other hand, since p and \bar{p} belong to U , by part 1 of Proposition 5.6, the identity component of the automorphism group of the triple $(Z, S, \{p, \bar{p}\})$ (respectively the pair $(S, \{p, \bar{p}\})$) reduces to

the identity. It follows that the above Kuranishi families are both universal. Therefore as in [DF89, Lemma 2.1] these Kuranishi families admit natural real structures and u is compatible with them. In particular u induces a submersion between the real parts of the parameter spaces.

Proof of the Main Theorem. Fix an integer $m \geq 4$. Consider the toric surface S with real structure σ in Proposition 6.1. By that proposition we can find a Moishezon twistor space Z associated to a self-dual structure on $m\mathbf{P}^2$ which contains S as a real fundamental divisor. Taking $n = 1$ or 2 in Proposition 3.5 we find arbitrarily small deformations (S_a, σ_a) of (S, σ) as a surface with real structure such that $\kappa^{-1}(S_a) = 1$. For any point p of the open orbit U of S we can find real deformations $(S_a, \{p_a, \bar{p}_a\})$ of $(S, \{p, \bar{p}\})$ for a suitable choice of points p_a of S_a , where $\bar{p}_a = \sigma_a(p_a)$. By the description preceding the proof these deformations of the pairs can be extended to real deformations $(Z_a, S_a, \{p_a, \bar{p}_a\})$ of the triple $(Z, S, \{p, \bar{p}\})$.

It is then known by a standard argument that Z is a twistor space associated to some self-dual structure on $m\mathbf{P}^2$. Moreover, it contains S_a as a fundamental divisor as follows the relations $l_a \cdot S_a = l \cdot S = 2$ and $H^1(Z_a, O_{Z_a}) = 0$, where l_a and l are twistor lines on Z_a and Z respectively. By Proposition 3.5 the line bundle L_a in (6) is a torsion line bundle of order n . Thus, by Proposition 4.1, $a(Z_a) = 2$, in which case the corresponding self-dual metric is necessarily of positive type by [Pon91]. This completes the proof of the Main Theorem. \square

We conclude by making a remark on the case $m = 4$. In this case the surface S_a is an elliptic surface over \mathbf{P} with a unique multiple fiber C_a with multiplicity n , where C_a is a cycle of eight nonsingular rational curves. Suppose that S_a is realized as a real fundamental divisor of a twistor space of algebraic dimension two. Then we have seen that $n = 1$ or 2 and in fact that both cases are actually realized. On the other hand, in the original examples of Campana and Kreuzler [CK99] the corresponding fundamental divisor is an elliptic surface whose multiple fiber is a multiple of a nonsingular elliptic curve. Moreover, it is shown by Honda [Hon00] that the multiplicity n can take arbitrary positive values.

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Akira Fujiki fujiki@math.sci.osaka-u.ac.jp

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan