MATCHED PAIRS OF LIE ALGEBROIDS

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(Received 16 October, 1995)

Abstract. We extend to Lie algebroids the notion variously known as a double Lie algebra (Lu and Weinstein), matched pair of Lie algebras (Majid), or twilled extension of Lie algebras (Kosmann-Schwarzbach and Magri). It is proved that a matched pair of Lie groupoids induces a matched pair of Lie algebroids. Conversely, we show that under certain conditions a matched pair of Lie algebroids integrates to a matched pair of Lie groupoids. The importance of matched pairs of Lie algebroids has been recently demonstrated by Lu.

Introduction. The notion of a *Lie bialgebra*, due to Drinfel'd [**5**], has arisen in the study of Poisson Lie groups. A Poisson Lie group G, that is a Lie group G with a compatible Poisson bracket { }, induces a Lie algebra structure on the vector space dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of G, such that the direct sum $\mathcal{G} \oplus \mathcal{G}^*$ of vector spaces acquires a Lie algebra structure with $\mathcal{G}$ and $\mathcal{G}^*$ as Lie subalgebras. If G^* is the connected and simply connected Lie group whose Lie algebra is $\mathcal{G}^*$, then it is sometimes possible to equip the product manifold $G \times G^*$ with a Lie group structure whose Lie algebra is $\mathcal{G} \oplus \mathcal{G}^*$, in such a way that the triple $(G, G^*, G \times G^*)$ becomes a *matched pair of Lie groups* [**17**]. Abstracting the structures of $G \times G^*$ and $\mathcal{G} \oplus \mathcal{G}^*$ leads to the concepts of *matched pairs of Lie groups* and *matched pairs of Lie algebras.*

A matched pair of Lie groups is a triple (V, H, G) of three Lie groups V, H and G such that V and H are Lie subgroups of G, and the map $V \times H \to G$ defined by $(v, h) \mapsto vh$ is a diffeomorphism. Similarly a double Lie algebra is a triple $(\mathcal{V}, \mathcal{H}, \mathcal{G})$ of three Lie algebras $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{G}$ such that $\mathcal{G} = \mathcal{V} \oplus \mathcal{H}$ as vector spaces, and $\mathcal{V}$, $\mathcal{H}$ are Lie subalgebras of $\mathcal{G}$. Matched pairs of Lie groups and matched pairs of Lie algebras are also natural extensions of semi direct products of Lie groups and of their Lie algebras; this is the case where one action is trivial.

Matched pairs of Lie groups and matched pairs of Lie algebras have arisen in the work of several authors in connection with Poisson geometry and dressing transformations, notably Kosmann-Schwarzbach and Magri in [**7**], where matched pairs of Lie algebras are studied under the name of *twilled extensions of Lie algebras*, Majid in [**17**] (see also [**18**]), and Lu and Weinstein in [**11**]. In [**12**] Lu and Weinstein showed that the Lie bialgebra $(\mathcal{G}, \mathcal{G}^*)$ associated with any Poisson Lie group G may be integrated to a *double Lie groupoid* in the sense of [**14**].

Mackenzie [**14**] extended the concept of double Lie group to the context of Lie groupoids, showing that there is a double Lie groupoid structure associated with a double Lie group (this is not as obvious as the terminology makes it seem). It has been also shown [**14**] that the double groupoids which arise from double Lie groups can be characterized amongst general double groupoids by a simple condition called *vacancy.* Since the concept of a double Lie groupoid is more general than that of a vacant double groupoid the terminology *matched pairs of Lie groupoids* is more appropriate for the groupoid version of a double Lie group. Although the paper [**14**] studied the infinitesimal actions associated with a matched pair of Lie groupoids, it did not consider any notion of matched pair for Lie algebroids.

In this paper we introduce a natural notion of matched pair of Lie algebroids and show that a matched pair of Lie groupoids induces a matched pair of Lie algebroids. This process is significantly more involved than for matched pairs of Lie groups and algebras. The Lie algebroid representations which constitute the matched pair may be obtained either by linearizing the groupoid actions and then applying the Lie functor, or by linearizing the infinitesimal actions on the Lie algebroid level. The importance of matched pairs of Lie groupoids and Lie algebroids has recently been dramatically reaffirmed by new work of Lu [**10**], who shows that matched pairs underlie any Poisson action of a Poisson Lie group.

The article is organized as follows: in Section 1 we review matched pairs of Lie groupoids. In Sections 2 and 3 we study the induced algebroid actions of AV on H and AH on V, where (V, H) is a matched pair of Lie groupoids. These actions yield representations of the Lie algebroids AV and AH on each other by "twisted derivations". In Section 4 we define a matched pair of abstract Lie algebroids by a Lie algebroid structure on the vector bundle direct sum, exactly as for double Lie algebras. In Section 5 we prove that the Lie algebroids AV and AH of a matched pair V and H of Lie groupoids form a matched pair of Lie algebroids. As an example, we show that for a compact Poisson Lie group G, the Lie algebroid T^*G of 1 forms on G, and the Lie algebroid TG associated to the tangent bundle $TG \to G$ form a matched pair of Lie algebroids. In Section 6 we prove an integration result for matched pairs of Lie algebroids, modelled on a result by Lu and Weinstein ([**11**], Section 3.7). Some further integration results will be given in another paper.

All manifolds are C^∞, real, Hausdorff, and second countable. A Lie groupoid Ω over base B with the source map $\alpha:\Omega \to V$, and the target map $\beta:\Omega \to B$ is denoted by $\Omega \overset{\alpha,\beta}{\rightrightarrows} B$. The identity map of Ω is denoted by 1^Ω and its value at $b \in B$ by 1_b^Ω, or simply by 1_b. The notations V and H are always for two Lie groupoids with the same base B; we denote by $v_1, v_2, \ldots, v_n, \ldots$, the elements of V and by $h_1, h_2, \ldots, h_n, \ldots$ the elements of H. We use the maps α_V, β_V, α_H and β_H for the source and the target maps of V and H respectively, but we omit the subscripts V and H if there is no confusion in doing so. Finally, we denote by $C(M)$ the module of smooth real valued maps on a manifold M.

1. Matched pairs of Lie groupoids. In this section, we review briefly matched pairs of Lie groupoids. More details may be found in [**14**] (see also [**19**]).

Definition 1.1. [**14**]. Let V and H be two Lie groupoids over the same base B. The groupoids V and H form a *matched pair of Lie groupoids*, or (V, H) is a matched pair of Lie groupoids, if there exits a left action $(h, v) \mapsto {}^h v$ of H on V and a right action $(h, v) \mapsto h^v$ of V on H such that the following conditions are satisfied

(i) $\alpha_V({}^h v) = \beta_H(h^v)$ for all (h, v) with $\alpha_H(h) = \beta_V(v)$;

(ii) ${}^h(v_1 v_2) = ({}^h v_1)({}^{h^{v_1}} v_2)$ with $\alpha_V(v_1) = \beta_V(v_2)$ and $\alpha_H(h) = \beta_V(v_1)$;

(iii) $(h_1 h_2)^v = (h_1^{h_2 v}) . (h_2^v)$ for $v \in V$, $h_1, h_2 \in H$ with $\alpha_H(h_1) = \beta_H(h_2)$.

The following two properties are a consequence of the properties (ii) and (iii) in the above definition:

(iv) ${}^{h}(1_b^V) = 1_c^V$, for $h \in H$ with $\alpha_H(h) = b$ and $\beta_H(h) = c$;
(v) $(1_c^H)^v = 1_b^H$, for $v \in V$ with $\beta_V(v) = c$ and $\alpha_V(v) = b$.

For two Lie groupoids V and H with the same base B, we denote by $V * H$ the submanifold $\{(v,h) \in V \times H \mid \alpha_V(v) = \beta_H(h)\}$ of $V \times H$. If V and H form a matched pair of Lie groupoids, then we denote by $\Phi:(h,v) \mapsto \Phi_h(v)$ and by $\Psi:(h,v) \mapsto \Psi_v(h)$ the left action of H on V and the right action of V on H, respectively.

PROPOSITION 1.2 [**14**] (p. 199, Theorem 2.10). *(V,H) is a matched pair of Lie groupoids if and only if the manfold $G = V * H$ has a Lie groupoid structure on base B, such that*

(i) *the maps $v \mapsto (v, 1^H_{\alpha(v)})$ and $h \mapsto (1^V_{\beta(h)}, h)$ are Lie groupoid morphisms from V and H to G, respectively. We denote by V' and by H' the images of V and H under such morphisms.*

(ii) *The multiplication $m: V' * H' \to g$ is a diffeomorphism, where $V' * H'$ is the set of all composable pairs $(v',h') \in V' \times H'$.*

We recall [**13**] that if a Lie group G acts on a manifold M, one can construct a Lie groupoid structure on the product manifold $G \times M$ with base M, called the *action groupoid*. This structure is described as follows: α is the projection onto the second factor of $G \times M$ and β is the action $G \times M \to M$ itself; the identity map is $m \mapsto (1,m)$; the inverse of (g,m) is (g^{-1}, gm) and the partial multiplication is $(g,m)(g',m') = (gg',m')$, for $m = g'm'$. We recall also that for any manifold B, there is a Lie groupoid structure on the product manifold $B \times B$, with base B, called *the coarse groupoid structure*. The source and the target maps are the projections of $B \times B$ onto the second and the first factor, respectively, and the multiplication is defined by $(b_1,b_2)(b_3,b_4) = (b_1,b_4)$, for $b_2 = b_3$. Now the next proposition gives a class of examples of a matched pair of Lie groupoids; see [**19**] for others.

PROPOSITION 1.3. *Let H be a Lie group, and $\Phi: H \times M \to M$, $(h,m) \mapsto hm$, an action of H on a manifold M. Then the action groupoid $H \ltimes M$, and the coarse groupoid $M \times M$ form a matched pair of Lie groupoids over base M.*

Proof. If $\tilde{H} = H \ltimes M$ and $\tilde{V} = M \times M$, the pullback

$$\tilde{H} * \tilde{V} = \{(\tilde{h}, \tilde{v}) \in \tilde{H} \times \tilde{V} \mid \alpha_{\tilde{H}}(\tilde{h}) = \beta_{\tilde{V}}(\tilde{v})\}$$

is just the trivial Lie groupoid $M \times H \times M$. The maps $(h,m) \mapsto (hm,h,m)$ and $(x,y) \mapsto (x,1,y)$ are groupoid morphisms, and represent $\tilde{H}$ and $\tilde{V}$ embedded as wide subgroupoids in $M \times H \times M$. The multiplication of an element of $\tilde{H}$ by an element of $\tilde{V}$ is

$$(hm,h,m)(m,1,x) = (hm,h,x).$$

For $(y,h,m) \in M \times H \times M$ we have $(y,h,h^{-1}y)(h^{-1}y,1,x) = (y,h,x)$, in a unique way, with $(y,h,h^{-1}y) \in \tilde{H}$, and $(h^{-1}y,1,x) \in \tilde{V}$. It follows from the Proposition 1.2, that $(\tilde{H},\tilde{V})$ is a matched pair of Lie groupoids. □

If G is a Lie groupoid on base B with m as multiplication, the tangent bundle TG equipped with Tm as multiplication is then a Lie groupoid over TB. Assume now that

(V, H) is a matched pair of Lie groupoids, and let m be the multiplication in $V * H$. Since $T(V * H) = TV * TH$, there is a Lie groupoid structure on $TV * TH$, such that the maps: $X \mapsto (X, T(1)T\alpha X)$ and $Y \mapsto (T(1)T\beta Y, Y)$ represent TV and TH as embedded wide subgroupoids in $T(V * H)$. The proof of the following proposition follows now from the Proposition 1.2.

PROPOSITION 1.4. *If (V, H) is a matched pair of Lie groupoids then (TV, TH) is also a matched pair of Lie groupoids.*

2. Lie algebroids. Here we briefly recall the concept of Lie algebroid. *A Lie algebroid on base B* is a vector bundle $q: A \to B$ together with a map $a: A \to TB$ of vector bundles over B, called the *anchor* of A, and an $\mathbb{R}$-bilinear, antisymmetric bracket of sections, $[\ ,\ ]: \Gamma A \times \Gamma A \to \Gamma A$, which obey the Jacobi identity, and satisfies the relations (i) $a[X, Y] = [a(X), a(Y)]$ and (ii) $[X, fY] = f[X, Y] + (a(X)f)Y)$, for $X, Y \in \Gamma A$, $f \in C(B)$. Here $a(X)(f)$ is the Lie derivative of f with respect to the vector field $a(X)$.

Let A be a Lie algebroid on B and let $f: M \mapsto B$ be a smooth map. Then *an action* of A on M is an $\mathbb{R}$-linear map $X \mapsto X^\dagger$, $\Gamma A \to \Gamma A$ such that (i) $[X, Y]^\dagger = [X^\dagger, Y^\dagger]$, for $X, Y \in \Gamma A$; (ii) $(uX)^\dagger = (u \circ f)X^\dagger$ for $X \in \Gamma A$, $u \in C(B)$; (iii) $Tf(X^\dagger(m)) = a(X)(f)(m)$ for $X \in \Gamma A$, $m \in M$.

The construction of the Lie algebroid of a Lie groupoid follows closely the construction of a Lie algebra of a Lie group; we refer to [13] for full account. Let $G \rightrightarrows B$ be a Lie groupoid, and let $T^\alpha B = \mathrm{Ker}(T\alpha)$ be the vertical bundle along the fibers of α. Let $AG \to B$ be the vector bundle pullback of the vector bundle $T^\alpha G$ across the identity map $1: B \to G$. Notice that a section $X \in \Gamma AG$ is characterized by $X(b) \in T_{1_b}G_b$, $\forall b \in B$, where $G_b = \alpha^{-1}(b)$. Now take $X \in \Gamma AG$ and denote by $\vec{X}$ the right invariant vector field on G, defined by $\vec{X}(g) = TR_g X(\beta(g))$; the correspondence $X \mapsto \vec{X}$ from ΓAG to the module of right invariant vector fields is a bijection; we equip ΓAG with the Lie algebra structure obtained by transferring the Lie algebra structure of the module of right invariant vector fields on G to ΓAG, via the bijection $X \mapsto \vec{X}$. Namely, if $X, Y \in \Gamma AG$, we define $[X, Y] = [\vec{X}, \vec{Y}] \circ 1$, and $a: AG \to TB$, by $a(X_b) = T\beta X_b$. The vector bundle AG constructed above is called *the Lie algebroid of G*; we will denote it by $q_G: AG \to B$, and its anchor map by $a_G: AG \to TB$. If there is no confusion we will omit the subscript. The pullback of $\vec{X}$ by the inversion map i of G, is denoted by $\overleftarrow{X}$ and is a left invariant vector field on G. Lastly, if (x_t) is the one parameter group of local diffeomorphisms which generates a right invariant vector field $\vec{X}$ on G, then $x_t(v) = x_t(1_{\beta(v)})v$, $\forall v \in G$, and we write $x_t(v) = \exp tX(\beta(v))v$, where $\exp tX(b) = x_t(1_b)$.

Now assume that (V, H) is a matched pair of Lie groupoids. The left action Φ of H on $\beta: V \to B$ induces an action of the Lie algebroid AH on the groupoid V. This action associates to $Y \in \Gamma AH$ the vector field $Y^\dagger$ on V, called the *fundamental vector field generated by* Y, defined by:

$$Y^\dagger(v) = T(h \mapsto \Phi_h(v))Y(\beta(v)), \forall v \in V,$$

and a left action of H on $q: AV \to B$, defined by:

$$\chi_h(X) = TiT\Phi_h(TiX),$$

for all $(h, X) \in H \times AV$ with $\alpha_H(h) = q_V(X)$.

Similarly, the right action Ψ of V on $\alpha: H \to B$, induces an infinitesimal action of the

Lie algebroid AV on H. This action associates to $X \in \Gamma AV$ the vector field $X^\dagger$ on H defined by:

$$X^\dagger(h) = T(v \mapsto \Psi_v(h))TiX(\alpha(h)), \forall h \in H,$$

and a right action of the Lie groupoid V on $q: AH \to B$, given by the formula:

$$\chi'_v(Y) = T\Psi_v Y,$$

for all $(Y, v) \in AH \times V$ such that $q_H(Y) = \beta_V(v)$. Finally, we will denote by m and m' the multiplications in the groupoids V and H respectively.

Lastly, we have

Proposition 2.1 [**14**] (p. 224, Theorem 4.9). *If (V, H) is a matched pair of Lie groupoids, then for $Y \in \Gamma AH$, $X \in \Gamma AV$, v and w in V, h and l in H, such that $\alpha(v) = \beta(w)$ and $\alpha(h) = \beta(l)$, the following relations are satisfied*:

$$Y^\dagger(vw) = Tm(Y^\dagger(v), (\chi'_v(Y(\beta(v)))^\dagger(w)), \tag{1}$$

and

$$X^\dagger(hl) = Tm'((\chi_l(X(\alpha(l)))^\dagger(h), X^\dagger(l)). \tag{2}$$

Notice that from 1.1 the vector fields $Y^\dagger$ and $X^\dagger$ satisfy the relations

$$Y^\dagger(1_b^H) = T(1^H)a_H(Y)(b), \tag{3}$$

and

$$X^\dagger(1_b^V) = T(1^V)a_V(X)(b). \tag{4}$$

3. The representations induced by a matched pair. For a matched pair (V, H) of Lie groupoids, there are induced representations of the Lie algebroids AV and AH on each other by "twisted observations", exactly as in the Lie algebra case. We recall now from [**6**] the definition of a representation of a Lie algebroid on a vector bundle. Let $\mathscr{A}$ be a Lie algebroid over B with anchor map $a: \mathscr{A} \to TB$. Let E be a vector bundle over B. A *representation* of $\mathscr{A}$ on E is a $\mathbb{R}$ bilinear map

$$\rho: \Gamma\mathscr{A} \times \Gamma E \to \Gamma E: (X, \mu) \mapsto \rho_X(\mu),$$

such that for all X, $Y \in \Gamma\mathscr{A}$, $\mu \in \Gamma E$ and $f \in C(B)$,

(i) $\rho_{fX}(\mu) = f\rho_X(\mu)$;

(ii) $\rho_X(f\mu) = f\rho_X(\mu) + (a(X)f)\mu$;

(iii) $\rho_{[X,Y]}(\mu) = \rho_X(\rho_Y(\mu)) - \rho_Y(\rho_X(\mu))$.

For a matched pair of Lie groupoids (V, H) and $Y \in \Gamma AH$, let $\rho_Y: \Gamma AV \to \Gamma AV$ be the map defined by:

$$\rho_Y(X)(b) = [i, Y^\dagger, \vec{X}](1_b), \qquad \forall b \in B. \tag{5}$$

Since $Y^\dagger$ is β projectable, $i_* Y^\dagger$ is α projectable and then $\rho_Y(X) \in \Gamma AV$. Similarly, ρ'_X for $X \in \Gamma AV$ is the map $\Gamma AH \to \Gamma AH$ defined by:

$$\rho'_X(Y)(b) = [X^\dagger, \vec{Y}](1_b), \qquad \forall b \in B. \tag{6}$$

Since the vector field $X^\dagger$ is α projectable the map ρ'_X takes its value in ΓAH.

Proposition 3.1. *The map $Y \mapsto \rho_Y$ is a representation of the Lie algebroid AH on the*

https://doi.org/10.1017/S0017089500032055 Published online by Cambridge University Press

vector bundle AV, and the map $X \mapsto \rho'_X$ is a representation of the Lie algebroid AV on the vector bundle AH.

LEMMA 3.2. *If W is a vector field on V tangent to the α fibres, that is $T\alpha W(v) = 0$, for any $v \in V$, then:*

$$[i_*Y^\dagger, \overrightarrow{W \circ 1}](1_b) = [i_*Y^\dagger, W](1_b),$$

for any $b \in B$, where $\overrightarrow{W \circ 1}$ is the right invariant vector field on V associated to the section $W \circ 1 \in \Gamma AV$.

Proof of the lemma. The vector field $X = W - \overrightarrow{W \circ 1}$ vanishes identically on 1_B and the relation $i_*Y^\dagger(1_b) = T(1)T\beta Y(1_b)$ shows that $i_*Y^\dagger$ restricts to the base 1_B as a vector field. It follows that $[i_*Y^\dagger, X]$ vanishes identically on 1_B. □

Proof of the proposition 3.1. For $f \in C(B)$, we have

$$\begin{aligned}\rho_Y(fX)(b) &= [i_*Y^\dagger, \overrightarrow{fX}](1_b)\\ &= [i_*Y^\dagger, (f \circ \beta)\vec{X}](1_b)\\ &= (f \circ \beta)[i_*Y^\dagger, \vec{X}](1_b) + (i_*Y^\dagger(f \circ \beta))_{1_b}\vec{X}(1_b).\end{aligned}$$

Since $\beta(1_b) = b$ and $\vec{X}(1_b) = X(b)$, for all $b \in B$, we have

$$\rho_Y(fX)(b) = f(\rho_Y(X)(b)) + (i_*Y^\dagger(1_b)(f \circ \beta))X(b).$$

But

$$i_*Y^\dagger(1_b)(f \circ \beta) = (df \circ T\alpha)Y^\dagger(1_b) = (a_H(Y)f)(b),$$

then

$$\begin{aligned}\rho_Y(f(X)(b) &= f\rho_Y(X)(b) + df(a_H(Y)(b))X(b)\\ &= f\rho_Y(X)(b) + a_H(Y)f(b)X(b).\end{aligned}$$

It follows that

$$\rho_Y(fX) = f\rho_Y(X) + a_H(Y)(f)X.$$

For $f \in C(B)$, we have $(fY)^\dagger = (f \circ \beta)Y^\dagger$, by the very definition of $Y^\dagger$, therefore

$$i_*(fY)^\dagger = i_*(f \circ \beta)Y^\dagger = (f \circ \alpha)i_*Y^\dagger.$$

It follows, by noticing that

$$\vec{X}(f \circ \alpha) = df \circ T\alpha\vec{X} = 0,$$

that

$$\rho_{fY}(X)b = f(b)\rho_Y(X)(b).$$

The assertion

$$[\rho_Y, \rho_{Y'}] = \rho_{[Y, Y']}, \forall Y, Y' \in \Gamma AH$$

follows from the Jacobi identity and from the relation:

$$[i_*Y^\dagger, [i_*Y^\dagger, \vec{X}]](1_b) = [i_*Y^\dagger, \overrightarrow{(\rho_{Y'}(X))}](1_b), \qquad \forall b \in B,$$

which is a consequence of the Lemma 3.2.

The second part of the proof concerning ρ'_X is proved in the same way. □

The following proposition shows that the representations ρ and ρ' may also be obtained by linearizing the actions Φ and Ψ, as for the Lie group case.

PROPOSITION 3.3. *For all $Y \in \Gamma AH$, all $X \in \Gamma AV$, and all $b \in B$, we have the relations*

(i) $\rho'_X(Y)(b) = \left.\dfrac{d}{dt}\right|_{t=0} T\Psi_{\exp tX(b)} Y(\beta \circ \exp tX(b))$,

(ii) $Ti\rho_Y(X)(b) = \left.\dfrac{d}{dt}\right|_{t=0} T\Psi_{i \exp tY(b)} TiX(\beta \circ \exp tY(b))$.

Proof. We prove the first assertion. The second one can be proved by the same argument. Let $F_t(h) = \Psi_{i \exp X(\alpha h)}(h)$ be the (local) flow of $X^\dagger$. We have

$$[X^\dagger, \vec{Y}](1_b^H) = \left.\frac{d}{dt}\right|_{t=0} TF_t^{-1} Y(\beta \circ (\exp tX)(b));$$

but

$$\begin{aligned} TF_t^{-1} Y(\beta \circ (\exp tX(b))) &= \left.\frac{d}{ds}\right|_{s=0} F_t^{-1} (\exp sY(\beta \circ (\exp tX)(b))) \\ &= \left.\frac{d}{ds}\right|_{s=0} \Psi_{\exp tX(b)} (\exp sY(\beta \circ \exp tX)(b)) \\ &= T\Psi_{\exp tX(b)}(Y(\beta \circ \exp tX)(b)), \end{aligned}$$

by using the relation $\exp tX(b) = i(\exp - tX(\beta \circ \exp tX(b)))$; see ([**13**], Chap. 2, Sect. 5), and that proves the proposition. □

We need now the following lemma.

LEMMA 3.4. *For all $X \in \Gamma AV$, for all $Y \in \Gamma AH$, and for all $b \in B$, the following properties hold*:

(i) $(X + TiX(b) = T(1)(T\alpha(X) + T\beta(X))(b)$;

(ii) $[Y^\dagger, i_*\vec{X}](1_b) = -[Y^\dagger, \vec{X}](1_b) + T(1)[a_H(Y), a_V(X)]$;

(iii) $\rho_Y(X)b = [Y^\dagger, \vec{X}](1_b) - T(1)T\alpha[Y^\dagger, \vec{X}](1_b)$,

(iv) $\rho'_X(Y)b = [i_*Y^\dagger, \vec{Y}](1_b) - T(1)T\alpha[i_*Y^\dagger, \vec{Y}](1_b)$.

Proof. (i) We have $Tm(X(b), TiX(b)) = T(1)T\beta X(b)$, since TV is a Lie groupoid with base TB and multiplication Tm. But

$$\begin{aligned} Tm(X(b), TiX(b)) &= Tm(X(b) - T(1)T\alpha X(b), 0) + Tm(T(1)T\alpha X(b), TiX(b)) \\ &= X(b) - T(1)T\alpha X(b) + TiX(b), \end{aligned}$$

thus the result.

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(ii) Let $F_t(v) = \Phi_{\exp Y(\beta(v))}(v)$ be the (local) flow of $Y^\dagger$, then

$$[Y^\dagger, i_*\vec{X}](1_b) = \frac{d}{dt}\bigg|_{t=0} TF_t^{-1}Ti\vec{X}(1_{\beta(\exp tY(b))})$$
$$= \frac{d}{dt}\bigg|_{t=0} - TF_t^{-1}\vec{X}(1_{\beta(\exp tY(b))}) + \frac{d}{dt}\bigg|_{t=0} TF_t^{-1}T(1^V)T\beta\vec{X}(1_{\beta\circ(\exp tY(b))})$$
$$= -[Y^\dagger, \vec{X}](1_b) + T(1^V)[a_H(Y), a_V(X)](b),$$

by using the formula $TiX + X = T(1^V)A(X)$, from the first part of the lemma.

(iii) Notice that the two relations $T\beta Y^\dagger(v) = a_H(Y)(\beta v)$ and $T\beta\vec{X}(v) = a_V X(\beta v)$ imply

$$T\beta[Y^\dagger, \vec{X}](1_b) = [a_H(Y), a_V(X)](b), \qquad \forall b \in B.$$

We have

$$\begin{aligned}\rho_Y(X)(b) &= [i_*Y^\dagger, \vec{X}](1_b)\\ &= Ti[Y^\dagger, i_*\vec{X}](1_b)\\ &= -Ti[Y^\dagger, \vec{X}](1_b) + T(1^V)[a_H(Y), a_V(X)](b) \text{ (by (1))},\\ &= [y^\dagger, \vec{X}](1_b) - T(1^V)(T\alpha[Y^\dagger, \vec{X}](1_b) + T\beta[Y^\dagger, \vec{X}])(1_b)\\ &\quad + T(1^V)[a_H(Y), a_V(X)](b) \text{ (by (1) of 3.4)},\\ &= [Y^\dagger, \vec{X}](1_b) - T(1)T\alpha[Y^\dagger, \vec{X}](1_b).\end{aligned}$$

(iv) The statement for $\rho'_X(Y)$ is proved by the same method. □

PROPOSITION 3.5. *If (V, H) is a matched pair of Lie groupoids then the representations ρ and ρ' satisfy the following relations:*

$$\overrightarrow{\rho_Y(X)}(v) = [Y^\dagger, \vec{X}](v) + (\rho'_X(Y))^\dagger(v), \tag{7}$$

for all v in V, and

$$\overrightarrow{\rho'_X(Y)}(h) = [i_*X^\dagger, \vec{Y}](h) + i_*(\rho_Y(X))^\dagger(h), \tag{8}$$

for all $h \in H$.

Proof. Let $\vec{X}$ be a right invariant vector field on V with $X \in \Gamma AV$, and let f_t be its (local) flow. We recall that

$$f_t(vw) = f_t(v)w, \qquad \forall u, w \in V, \text{ such that } \alpha(v) = \beta(w),$$

with $f_t(v) = \exp tX(\beta(v))v$; it follows that if Z and W are two vector fields in V, with $T\alpha Z_v = T\beta W_w$, then the following relation holds

$$Tf_tTm(Z_v, W_w) = Tm(Tf_tZ_v, W_w).$$

Now if in the relation (1) we replace v by $f_t(1_b)$, with $b = \beta(w)$, and then if we apply this operator Tf_t^{-1} to both sides of this relation by Tf_t^{-1}, we get

$$(f_t)^*Y^\dagger(w) = Tm((f_t)^*Y^\dagger(1_b), \qquad T(h \to \Phi_h(v))T\Psi_{f_t(1_b)}Y(\beta f_t(1_b))).$$

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Differentiating the above relation with respect to t at $t=0$, we get

$$-[Y^\dagger, \vec{X}](w) = Tm(-[Y^\dagger, \vec{X}](1_b), \rho'_X(Y)^\dagger(w)),$$

by the Proposition 3.3. Now the above relation may be written as follows

$$\begin{aligned}[Y^\dagger, \vec{X}](w) &= Tm([Y^\dagger, \vec{X}](1_b), -\rho'_X(Y)^\dagger(w)) \\ &= Tm([Y^\dagger, \vec{X}](1_b) - T(1)T\alpha[Y^\dagger, \vec{X}](1_b), 0) \\ &\quad + Tm(T(1)T\alpha[Y^\dagger, \vec{X}](1_b), -\rho'_X(Y)^\dagger(w)) \\ &= \overrightarrow{\rho_Y(X)}(w) - (\rho'_X(Y))^\dagger(w),\end{aligned}$$

by the Lemma 3.4.

From the relation (2) we deduce similarly

$$i_*X^\dagger(hl) = Tm'(i_*X^\dagger(h), TiT(v \to \Psi_v(il)T\Phi_{ih}TiX(\beta h)),$$

and if $\vec{Y}$ is any right invariant vector field on H, then we have by the same argument

$$\begin{aligned}[i_*X^\dagger, \vec{Y}](h) &= Tm'([i_*X^\dagger, \vec{Y}](1_a), -i_*\rho_Y(X)^\dagger(h)) \\ &= \overrightarrow{\rho'_X(Y)}(h) - i_*(\rho_Y(X))^\dagger(h),\end{aligned}$$

where $a = \beta(h)$. □

THEOREM 3.6. *If (V, H) is a matched pair of Lie groupoids and if ρ and ρ' are the induced representations defined by (5) and (6), then for all $Y, W \in \Gamma AH$, and all $X, Z \in \Gamma AV$, the following relations are satisfied*

(i) $\rho_Y[X, Z] = [\rho_Y(X), Z] + [X, \rho_Y(Z)] - \rho_{\rho'_X(Y)}(Z) + \rho_{\rho'_Z(Y)}(X)$,

(ii) $\rho'_X[Y, W] = [\rho'_X(Y), W] + [Y, \rho'_X(W)] - \rho'_{\rho_Y(X)}(W) + \rho'_{\rho_W(X)}(Y)$.

(iii) $a_V(\rho_Y(X)) - a_H(\rho'_X(Y)) = [a_H(Y), a_V(X)]$.

Proof. To avoid overloading notations, we shall denote here by the same letter a section of the Lie algebroid AV and the corresponding invariant vector field on V.

(i) and (ii). From the relation (7) we deduce that

(a) $\rho_Y[X, Z] = [Y^\dagger, [X, Z]] + (\rho'_{[X,Z]}(Y))^\dagger$;

(b) $[\rho_Y(X), Z] = [[Y^\dagger, X], Z] + [(\rho'_X(Y))^\dagger, Z]$;

(c) $[X, \rho_Y(Z)] = [X, [Y^\dagger, Z]] + [X, (\rho'_Z(Y))^\dagger]$;

(d) $\rho_{\rho'_Z(Y)}(X) = [(\rho'_Z(Y))^\dagger, X] + (\rho'_X(\rho'_Z(Y)))^\dagger$;

(e) $\rho_{\rho'_X(Y)}(Z) = [(\rho'_X(Y))^\dagger, Z] + (\rho'_Z(\rho'_X(Y)))^\dagger$.

If we add the equations (b), (c) and (d) and then subtract (e), we get the equation (a), by using the Jacobi identity and the relation $\rho'_{[X,Z]} = \rho'_X\rho'_Z - \rho'_Z\rho'_Z$. We prove the similar assertion for ρ by the same method.

(iii) The relation (7) gives

$$\alpha_V\rho_Y(X)b = T\beta[Y^\dagger, \vec{X}](1_b) + T\beta(\rho'_X(Y))^\dagger(1^b);$$

but $T\beta[Y^\dagger, \vec{X}](1_b) = [a_H(Y), a_V(X)](b)$, $\forall b \in B$. As the vector field $Y^\dagger$ on V is β projectable to the vector field $a_H(Y)$ on B, and $(\rho'_X(Y))^\dagger(1^V_b) = T(1^V)(a_H\rho'_X(Y)(b))$ by (3), the result follows. □

In the next section we will see that the equations proved in the Theorem 3.6 may be used to define an abstract notion of matched pairs of Lie algebroids.

We will prove later that under some topological assumptions, a matched pair of Lie algebroids is integrable to a matched pair of Lie groupoids, provided that the Lie algebroids arise from Lie groupoids. The last assumption is crucial since not all Lie algebroids are integrable.

4. Matched pairs of Lie algebroids. The following definition is an extension to Lie algebroids of the definition of a double Lie algebra in **[11]** or a matched pair of Lie algebras in **[17]**, or of a twilled extension of Lie algebras in **[7]**.

DEFINITION 4.1. Two Lie algebroids $\mathcal{A}$ and $\mathcal{B}$, with the same base B form *a matched pair of Lie algebroids* if the direct sum $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$, of vector bundles has a Lie algebroid structure on base B, such that $\mathcal{A}$ and $\mathcal{B}$ are Lie subalgebroids of $\mathcal{C}$.

The following theorem shows that a matched pair of Lie algebroids $(\mathcal{A}, \mathcal{B})$ induces a representation of $\mathcal{A}$ on $\mathcal{B}$ and a representation of $\mathcal{B}$ on $\mathcal{A}$, and these two representations are by "twisted derivations", as in the Lie algebra case.

THEOREM 4.2. *If the Lie algebroids $\mathcal{A}$ and $\mathcal{B}$ form a matched pair of Lie algebroids, there exists a representation*

$$\rho : \Gamma\mathcal{B} \times \Gamma\mathcal{A} \to \Gamma\mathcal{A},$$

of $\mathcal{B}$ on $\mathcal{A}$ and a representation

$$\rho' : \Gamma\mathcal{A} \times \Gamma\mathcal{B} \to \Gamma\mathcal{B}$$

of $\mathcal{A}$ on $\mathcal{B}$. These two representations satisfy the relations

(i) $\rho_Y[X, Z] = [\rho_Y(X), Z] + [X, \rho_Y(Z)] - \rho_{\rho'_X(Y)}(Z) + \rho_{\rho'_Z(Y)}(X)$,

(ii) $\rho'_X[Y, W] = [\rho'_X(Y), W] + [Y, \rho'_X(W)] - \rho'_{\rho_Y(X)}(W) + \rho'_{\rho_W(X)}(Y)$,

(iii) $a(\rho_Y(X)) - a'(\rho'_X(Y)) = [a'(Y), a(X)]$.

for all $X, Z \in \Gamma\mathcal{A}$, and $Y, W \in \Gamma\mathcal{B}$.

Proof. Let Π_1 and Π_2 be the projections of $\mathcal{A} \oplus \mathcal{B}$ onto the first and the second factor, respectively. We define ρ, and ρ' by:

$$\rho_Y(X) = -\Pi_1[X, Y],$$

and

$$\rho'_X(Y) = \Pi_2[X, Y],$$

for all $X \in \Gamma\mathcal{A}$ and all $Y \in \Gamma\mathcal{B}$. It follows that:

$$-\rho_Y(X) + \rho'_X(Y) = [X, Y], \tag{9}$$

and therefore, the relation (iii) of 4.2 follows by applying the enchor map to the left and right hand side of (9). We have also

$$\rho_Y(fX) = f\rho_Y(X) + a_{\mathcal{A}}(X)fY,$$

and

$$\rho_{fY}(X) = f\rho_Y(X),$$

for all $f \in C(B)$, by the properties of a Lie bracket. The similar statements for $\rho'_{fX}(Y)$ and $\rho'_X(fY)$ hold, by the same argument.

Now for $Y' \in \Gamma\mathscr{B}$, we have

$$
\begin{aligned}
-\rho_{[Y,Y']}(X) + \rho'_X[Y,Y'] &= [X,[Y,Y]] \\
&= [[X,Y],Y'] + [Y,[X,Y']] \\
&= [-\rho_Y(X) + \rho'_X(Y), Y'] + [Y, -\rho_{Y'}(X) + \rho'_X(Y')] \\
&= -[\rho_Y(X), Y'] + [\rho'_X(Y), Y'] - [Y, \rho_{Y'}(X)] + [Y, \rho'_X(Y')] \\
&= +\rho_{Y'}\rho_Y(X) - \rho'_{\rho_Y(X)}(Y') + [\rho'_X(Y), Y'] + [Y, \rho'_X(Y')] \\
&\quad - \rho_Y\rho_{Y'}(X) + \rho'_{\rho_{Y'}(X)}(Y);
\end{aligned}
$$

the second equality is by using the Jacobi identity. The relation (ii) of 4.2 and the relation

$$[\rho_Y, \rho_{Y'}] = \rho_Y\rho_{Y'} - \rho_{Y'}\rho_Y$$

follow from the last equation. The similar results for ρ' are proved by the same method.

The next theorem is a converse of the Theorem 4.2.

THEOREM 4.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be two Lie algebroids with the same base B, with anchor maps a and a', respectively. If there exist representations ρ and ρ' of $\mathscr{B}$ and $\mathscr{A}$ on each other, such that the relations* (i) *and* (ii) *and* (iii) *in the Theorem* 4.2 *hold, then there is a unique matched pair structure on $\mathscr{A}$ and $\mathscr{B}$ which induces the representations ρ and ρ'.*

Proof. Let $a: \mathscr{A} \oplus \mathscr{B} \to TB$ be defined by $a(X \oplus Y) = a(X) + a'(Y)$. We define on $\mathscr{A} \oplus \mathscr{B}$ bracket by setting

$$[X \oplus 0, 0 \oplus Y] = [X, Y] = -\rho_Y(X) + \rho'_X(Y). \tag{10}$$

We have then

$$
\begin{aligned}
[X + Y, X' + Y'] = [X, X'] &+ \rho_Y(X') - \rho_{Y'}(X) \\
&+ [Y, Y'] + \rho'_X(Y') - \rho'_{X'}(Y).
\end{aligned}
$$

Equipped with the above brackets and with the map a is an enchor map the vector bundle $\mathscr{A} \oplus \mathscr{B}$ is a Lie algebroid; the verification is straightforward.

Since the injections of $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{A} \oplus \mathscr{B}$ are Lie algebroid morphisms $(\mathscr{A}, \mathscr{B})$ is a matched pair of Lie algebroids with ρ and ρ' as induced representations. The uniqueness of the Lie algebroid structure on $\mathscr{A} \oplus \mathscr{B}$ with ρ and ρ' as induced representations follows from the relation (10). □

For a matched pair $(\mathscr{A}, \mathscr{B})$ of Lie algebroids, we denote by $\mathscr{A} \bowtie \mathscr{B}$ the Lie algebroid $\mathscr{A} \oplus \mathscr{B}$, following the convention in use for the Lie algebra case [[**17**], [**11**]].

5. Matched pairs of Lie algebroids induced by matched pairs of Lie groupoids. Matched pairs of Lie algebroids arise from matched pairs of Lie groupoids in the same way that matched pairs of Lie algebras arise from matched pairs of Lie groups, by applying the Lie functor to the Lie groupoids.

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In this section V and H are always two Lie groupoids over the same base B, and the manifold G is the pullback $V * H$, defined in Section 1. The maps α_G and β_G are defined by $\alpha_G(v,h)=\alpha_H(h)$, and $\beta_G(v,h)=\beta_V(v)$. If we denote by K the submanifold $\{(X_{1^V_b}, Y_{1^H_b}) \in T_{(1^V_b, 1^H_b)}G \mid T\alpha_H Y_{1^H_b}=0\}$ of $T(V \times H)$, then K may be regarded as a vector bundle on B and the module ΓK of sections of K may be identified with the submodule

$$\{(X + T(1^V)T\beta Y, Y) \mid X \in \Gamma AB, Y \in \Gamma AH\}$$

of $\Gamma_{|B}T(V \times H)$.

Proposition 5.1. *The Lie algebroids AV and AH form a matched pair of Lie algebroids if and only if K has a Lie algebroid structure with base B, such that the maps $X \mapsto (X,0)$, and $Y \mapsto (T(1^V)T\beta Y, Y)$ are Lie algebroid morphisms, from AV and AH respectively, to K.*

Proof. Let F be the map $AV \oplus AH \to K$, defined by

$$F(X \oplus Y) = (X + T(1^V)T\beta Y, Y), \forall X \oplus Y \in AV \oplus AH,$$

then F is clearly a vector bundle isomorphism. This isomorphism carries the Lie algebroid structure from $AV \oplus AH$ to K, and vice versa.

Assume that (AV, AH) is a matched pair of Lie algebroids, then K has a Lie algebroid structure isomorphic to $AV \oplus AH$, by the isomorphism F. Since the maps $i_V : X \mapsto (X,0)$ and $i_H : Y \mapsto (0,Y)$ are Lie algebroid morphisms from AV and AH to $AV \oplus AH$, the maps $i_V \circ F$ and $i_H \circ F$ are Lie algebroid morphisms, as well.

Conversely, assume that K has a Lie algebroid structure such that $j_V : X \mapsto (X,0)$ and $j_H : Y \mapsto T(1^V)T\beta Y, Y)$ are Lie algebroid morphisms. We have only to prove that AV and AH are Lie subalgebroids of $AV \oplus AH$; this follows from the relations $i_V = F^{-1} \circ j_V$, and $i_H = F^{-1} \circ j_H$. □

Theorem 5.2. *If (V,H) is a matched pair of Lie groupoids, then (AV, AH) is a matched pair of Lie algebroids, and for $X \in \Gamma AV$, $Y \in \Gamma AH$, the bracket $[X, Y]$ is given by*

$$[X, Y] = -\rho_Y(X) + \rho'_X(Y),$$

where the maps ρ and ρ' are the representations defined by the relations (5) *and* (6).

Proof. The theorem is a consequence of the Theorems 3.6 and 4.3. □

Example 5.3. Let (V,H) be a matched pair of Lie groupoids over the base B. Assume H is the trivial Lie groupoid $B \times B \rightrightarrows B$, and V is a symplectic groupoid [2]. Then,

$$AV \oplus AH = T^*B \oplus TB,$$

and (T^*B, TB) is a matched pair of Lie algebroids.

Example 5.4. If (V,H) is a matched pair of Lie groupoids, then (TAV, TAH) is a matched pair of Lie algebroids. Indeed, (TV, TH) is a matched pair of Lie groupoids

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(over TB), by 1.4 and the Lie algebroids ATV and ATH are isomorphic to the Lie algebroids TAV and TAH by ([15], Theorem 7.1).

EXAMPLE 5.5. Let $G \times M \to M$ be an action of a Lie group G on a manifold M and let $\mathcal{G}$ be the Lie algebra of G. The trivial vector bundle $\mathcal{G} \times M$ on M has a Lie algebroid structure on base M, denoted by $\mathcal{G} \ltimes M$ and called *the action Lie algebroid.* Furthermore $\mathcal{G} \ltimes M = A(G \ltimes M)$ [6], and now from the Proposition 1.3 the Lie algebroids TM and $\mathcal{G} \ltimes M$ form a matched pair of Lie algebroids.

Let (P, π) be a Poisson manifold with Poisson bivector field π. Let $\pi^{\#}$ be the associated bundle map $\pi^{\#}(p): T_p^*P \to T_pP$ defined by

$$(\alpha_p, \pi^{\#}(\beta_p)) = \pi(p)(\alpha_p, \beta_p).$$

Then, with $-\pi^{\#}$ as the anchor map, the cotangent bundle T^*P has a Lie algebroid structure over P, where the Lie bracket on the space $\Omega^1(P)$ of 1-forms on P is given by

$$\{\alpha, \beta\} = -D(\pi(\alpha, \beta)) - L_{\pi^{\#}\alpha}\beta + L_{\pi^{\#}\beta}\alpha,$$

where $L_{\pi^{\#}\alpha}\beta$ denotes the Lie derivative of the 1-form β in the direction of the vector field $\pi^{\#}\alpha$, see **[11]** (and the references given there).

For a compact and simply connected Poisson Lie group G, we prove that the Lie algebroid TG which arise from the tangent bundle $TG \to G$, and the Lie algebroid T^*G of one forms on G, form a matched pair of Lie algebroids.

THEOREM 5.6. *Let G be a compact and simply connected Poisson Lie group, with Poisson tensor π, then (TG, T^*G) is a matched pair of Lie algebroids.*

Proof. Let G^* be the dual Poisson Lie group of G, and let $\mathcal{G}$ and $\mathcal{G}^*$ be the Lie algebras of G, and G^*, respectively [5] (see also **[11]**).

For $\alpha \in \mathcal{G}^*$, let $\alpha_l^{\dagger} = \pi(\alpha_l)$, where α_l is the left invariant one form on G with $\alpha_l(e) = \alpha$. Since G is compact, the left infinitesimal dressing action of $\mathcal{G}^*$ on G, $\alpha \mapsto \alpha_l^{\dagger}$ is integrated to a global left action of G^* on G. If we identify T^*G with $G \times \mathcal{G}^*$, via right translations $\theta_g \in T_g^*G \mapsto (g, \theta_g \circ TR_g)$, then the Lie algebroid structure on T^*G, defined by π coincides with the action Lie algebroid $A(G^* \bowtie G)$, see **[9]**. It follows from the Proposition 1.3, that $(T^*G, A(G \times G))$ is a matched pair of Lie algebroids; since $A(G \times G) = TG$, the theorem is now proved. □

6. Integration of matched pairs of Lie algebroids. In the theorem of this section we prove that under some topological assumptions, a weakly integrable matched pair of Lie algebroids gives rise to a matched pair of Lie groupoids, generalizing the Theorem 3.7 in **[11]** (see also **[17]** for an integration result of a different type).

DEFINITION 6.1. A matched pair of Lie algebroids $(\mathcal{A}, \mathcal{B})$ is *weakly integrable* if there exist three Lie groupoids V, H and G, with the same base B, such that G is connected and simply connected, and $\mathcal{A} = AV$, $\mathcal{B} = AH$, and $AV \oplus AH = AG$.

Let M and B be two manifolds and $\pi: M \to B$ a surjective submersion. We call M a *π-(simply) connected* manifold if for all $b \in B$ the fibers $\pi^{-1}(b)$ are (simply) connected subspaces of M.

THEOREM 6.2. *Let $(\mathcal{A}, \mathcal{B})$ be a weakly integrable matched pair of Lie algebroids, on* *base B. Let V, H and G be the Lie groupoids with Lie algebroids $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{A} \bowtie \mathscr{B}$, respectively. Then, if V and H are compact, α-connected and simply connected, (V, H) is a matched pair of Lie algebroids.*

LEMMA 6.3. *Under the hypothesis of* 6.2 *the manifold*

$$V * H = \{(v, h) \in V \times H \mid \alpha_V(v) = \beta_H(h)\}$$

is connected.

Proof. The manifold B is connected, since B is the base of the connected and simply connected Lie groupoid G whose Lie algebroid is $AV \oplus AH$. Let $C = \{(1_b^V, 1_b^H) \mid b \in B\}$, and $C_b = \alpha_V^{-1}(b) \times \beta_H^{-1}(b)$. Then,

$$V * H = C \cap \left(\bigcup_{b \in b} C_b \right).$$

Since C and C_b are connected for all $b \in B$, and since $C \cap C_b$ is not empty, $V * H$ is connected. □

Proof of Theorem 6.2. Let G be the connected and simply connected Lie groupoids on base B with Lie algebroid $AV \oplus AH$. Since the α fibers of V and H are simply connected, the Lie algebroid morphisms $X \mapsto (X, 0)$ and $Y \mapsto (0, Y)$ from AV and AH to AG can be integrated to get V and H as wide subgroupoids of G by **[16]**. Let m be the map $V * H \mapsto G$ defined by $m(v, h) = vh$ and let α and β be respectively the source and the target maps of the Lie groupoid G. Let $(X_v, Y_h) \in T_{(v,h)}(V * H)$ be such that

$$Tm(X_v, Y_h) = 0. \tag{11}$$

By taking each side of the relation (11) by $T\alpha$ and then by $T\beta$ we find $T\alpha Y_h = T\beta X_v = 0$. Since the vector tangents X_v and Y_h are tangent to the β fibers and the α fibers respectively, the tangent vectors $TL_v^{-1}X_v$ and $TR_h^{-1}Y_h$ are defined and are in the Lie algebroids AV and AH, respectively. If we take now simultaneously each side of the relation (11) by TL_v^{-1} and by TR_h^{-1}, we obtain

$$TL_v^{-1}X_v + TR_h^{-1}Y_h = 0.$$

Since the sum of $AV \oplus AH$ is direct we have

$$TL_v^{-1}X_v = TR_h^{-1}Y_h = 0.$$

It follows that $X_v = Y_b = 0$, and the map m is an immersion. Since $AV \oplus AH = AG$, we have $\dim G = \dim(V * H)$, hence m is a local diffeomorphism. Since $V * H$ is compact, m is a proper map, and hence a covering map. By the Lemma 6.3, $V * H$ is connected; now m is a covering map from the connected space $V * H$ to the simply connected space G, hence m is a diffeomorphism. □

ACKNOWLEDGEMENTS. This work was part of my Sheffield University PhD thesis. I would like to thank Dr K. Mackenzie for his help and encouragement in writing it. I have benefited greatly from several conversations with him during my stay at the University of Sheffield. I would like also to thank Professor Y. Kosmann-Schwarzbach for her many helpful comments. I acknowledge with gratitude a grant from the Algerian Ministry of Education and from the British Council.

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