A class of modules over a locally finite group III

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Let G be a locally finite group, k a field of characteristic
$p \geq 0$ and V a right kG-module. We say that V is an
$\underline{\underline{M}}_c$-module over kG, if each p'-subgroup H of G contains a
finite subgroup F with the same fixed points as H in V.
(By convention, $0'$ is taken as the set of all primes.) Such
modules arise as elementary abelian sections of $\underline{U}$-groups, a class
of locally finite groups similar in many ways to the class of
finite soluble groups.

The main theorem is that if V is an $\underline{\underline{M}}_c$-module over kG with
trivial Frattini submodule, and G is almost abelian, then every
composition factor of V is complemented. This is a crucial
ingredient in Tomkinson's theory of prefrattini subgroups in a
certain subclass of $\underline{U}$. An example is given to show that the
theorem breaks down for metabelian G. This leads to an example
of a $\underline{U}$-group in which there are no analogues of prefrattini
subgroups – the first situation where one of the standard
conjugacy classes of subgroups of finite soluble groups has no
decent analogue in the whole class $\underline{U}$.

1.   Introduction

Let G be a locally finite group, let k be a field of
characteristic $p \geq 0$, and let V be a (right) kG-module. In the
terminology of [5] and [6], V is an $\underline{\underline{M}}_c$-module over kG, if each

p'-subgroup H of G contains a finite subgroup F such that $C_V(F) = C_V(H)$. Here, $0'$ is interpreted as the set of all primes, and $C_V(H)$ denotes the set of all elements of V fixed by every element of H . If G itself is a p'-group, it follows easily that the set $\{C_V(H) : H \leq G\}$ satisfies the minimal condition under inclusion, and it is for this reason that the $\underline{\underline{M}}_c$ notation is used.

In [5], we were interested in conditions ensuring the complete reducibility of $\underline{\underline{M}}_c$-modules. Before passing to the main topic of this paper, we mention an improvement of the results of [5] which can be obtained on the basis of recent results of Richardson [10]. Let H be a locally finite group, and let k be a field with algebraic closure $\overline{k}$ and prime field k_0 . Let $k_0(H)$ be the subfield of $\overline{k}$ generated by all roots of the polynomials $X^n - 1$, as n runs over the orders of the elements of H . If $k_0(H) \cap k$ has finite dimension over k_0 , we shall say that k is *far enough from* H . In particular, this happens if H is a finite group, or if k is a finite algebraic extension of k_0 . By the arguments of [5], incorporating [10], 3.4 instead of [5], Lemma 2.3, we obtain

THEOREM A. *Let G be a Černikov group, let k be a field of characteristic $p \geq 0$, and let V be an $\underline{\underline{M}}_c$-module over kG . Suppose that k is far enough from G , and that either G is a p'-group or the Frattini submodule $\Phi(V) = 0$. Then V is completely reducible.*

By a Černikov group, we mean as usual a finite extension of a direct product of finitely many quasi-cyclic groups. Thus, Maschke's Theorem is a special case of the above.

If k is a finite field of characteristic p and G is an abelian p'-group which is not Černikov, then an $\underline{\underline{M}}_c$-module V over kG may well fail to be completely reducible (see [5], Example 4.1). But by a result of Kovács and Newman [8], which does not depend on the $\underline{\underline{M}}_c$-condition, the minimal submodules of V , if any, will be complemented. Since the same holds for factor modules of V , every composition factor A/B of V will

also be complemented, in the sense that $V = A + C$ and $A \cap C = B$ for some submodule C of V. In fact, we only need to assume that G is almost abelian (that is, has an abelian subgroup of finite index) here, by more recent work of Farkas and Snider [1]; see also [7].

The aim of this paper is to consider the complementation of the composition factors when $\Phi(V) = 0$ but G is not assumed to be a p'-group.

THEOREM B. *Let G be a periodic almost abelian group, let k be any field, and let V be an $\underline{\underline{M}}_c$-module over kG with $\Phi(V) = 0$. Then every composition factor of V is complemented.*

Here, the condition that V be an $\underline{\underline{M}}_c$-module is not superfluous. For let A be any infinite abelian group containing no elements of order 2, and let G be the semidirect product of A by an automorphism of order 2 which inverts the elements of A. If k is the field of two elements, the group ring kG is semisimple ([9], 21.1) and so, viewed as a right module over itself, has zero Frattini submodule. But kG has the group ring of a cyclic group of order 2 over k as homomorphic image, and this group ring contains a minimal right ideal which is not complemented.

The study of $\underline{\underline{M}}_c$-modules has developed from the investigation of Sylow subgroups of locally finite groups, and particularly from the study of a class $\underline{U}$ of locally finite-soluble groups whose properties are in many ways analogous to those of the class of finite soluble groups (see [2]). Theorem B has already been applied to the class $\underline{U}$ by Tomkinson [11] in developing the theory of prefrattini subgroups of $\underline{U}$-groups.

TOMKINSON'S THEOREM. *Let G be a $\underline{U}$-group with Hirsch-Plotkin radical $\rho(G)$, and suppose that $G/\rho(G)$ is almost abelian. Then each prefrattini subgroup of G avoids the complemented chief factors of G and covers the non-complemented chief factors of G.*

The second aim of this paper is to show that Tomkinson's Theorem breaks down in the class of $\underline{U}$-groups as a whole. Now the prefrattini subgroups are not characterized by the covering and avoiding properties described above, even in finite soluble groups [3], but in our discussion we shall not need to know precisely what the prefrattini subgroups are.

We prove

THEOREM C. *Let r be any prime. Then there exists a group $G \in \underline{U}$
and an $\underline{\underline{M}}_c$-module W over $Z_r G$ such that $\Phi(W) = 0$, but W has an
indecomposable quotient module of composition length two.*

Let H be the semidirect product $H = WG$, W being written
multiplicatively. Then $H \in \underline{U}$ ([4], Lemma 7.1). Let X be any maximal
submodule of W . In H , W/X is a chief factor complemented by GX/X ,
and so, if D is any subgroup of H which avoids all the complemented
chief factors, $D \cap W \le X$. Since $\Phi(W) = 0$, $D \cap W = 1$. However, there
exist submodules $T < S < W$ such that S/T is irreducible and not
complemented in W/T . In H , S/T is a non-complemented chief factor.
Thus, H has a non-complemented chief factor which D avoids, and Theorem
C has the

COROLLARY. *There exists a $\underline{U}$-group H such that every subgroup of H
which avoids all the complemented chief factors, avoids also a non-
complemented chief factor.*

The prefrattini subgroups seem to be the first example of one of the
standard conjugacy classes of subgroups from the theory of finite soluble
groups which have no well behaved analogue in the class $\underline{U}$ as a whole.
However, Tomkinson's Theorem shows that even they behave well in some of
the more familiar subclasses of $\underline{U}$, such as the class of images of
periodic soluble linear groups, or the class of periodic, locally soluble,
almost locally nilpotent groups.

2.   Local complete reducibility

Since it is difficult to see how the vanishing of the Frattini
submodule of a module influences its quotient modules, we shall prove
Theorem B by showing that the modules in which we are interested have
another property, which clearly does pass to quotient modules, and using
that to obtain the complementation of the composition factors. This is the
property of strong local complete reducibility.

DEFINITION. Let G be a locally finite group, k a field, and V a
kG-module. We say that V is *strongly locally completely reducible* if
there exists a finite subgroup F_1 of G such that the restricted module

V_F is completely reducible for all finite subgroups $F \geq F_1$ of G.

This, on the face of it, is stronger than demanding that G have a local system X of finite subgroups X such that V_X is completely reducible for all $X \in X$ — the natural concept of local complete reducibility.

We now study this notion, beginning with the case when k contains all roots of 1. Throughout the rest of this section, G denotes a locally finite group containing an abelian normal subgroup H of finite index, k denotes a field and V a kG-module.

LEMMA 2.1. *Let L be any subgroup of H. There exists a finite set X of elements of G such that, if F is any subgroup of G containing X, then*

(*) $F \cap C_G(H/L) = C_F\big((F{\cap}H)/(F{\cap}L)\big)$.

Here, if A and B are subgroups of a group Y with $B \lhd A$, we write

$$C_Y(A/B) = \big\{y \in Y : [a, y^{\pm 1}] \in B \text{ for all } a \in A\big\} .$$

Thus an element $y \in Y$ belongs to $C_Y(A/B)$ if and only if y normalizes both A and B and operates trivially by conjugation on A/B. Therefore $C_Y(A/B)$ is a subgroup of Y.

Proof. Let $C = C_G(H/L) \geq H$, and let $1 = s_1, s_2, \ldots, s_n$ be a left transversal to C in G. For each $i = 2, \ldots, n$, there exists an element $h_i \in H$ such that $[h_i, s_i] \notin L$ (recall that G is periodic). Let $X = \{s_2, \ldots, s_n, h_2, \ldots, h_n\}$ and let F be a subgroup of G containing X. Clearly $F = \bigcup_{i=1}^{n} s_i(F \cap C)$, and $F \cap C$ centralizes $(F{\cap}H)/(F{\cap}L)$, whereas $s_2, \ldots, s_n$ do not. Therefore $F \cap C = C_F\big((F{\cap}H)/(F{\cap}L)\big)$.

LEMMA 2.2. *Suppose that V is an irreducible kG-module, and k contains all roots of 1. Then the restricted module V_H is completely*

*reducible. Let Y be an irreducible submodule of V_H , let $L = C_H(Y)$,
and suppose F is a subgroup of G such that $G = HF$ and F has the
property* (*) *with respect to L . Then V_F is irreducible.*

Proof. Since H is a normal subgroup of finite index of G ,
Clifford's Theorem ([6], Lemma 2.11) shows that V_H is completely
reducible, and is the direct sum of finitely many irreducible
kH-submodules. Since H is periodic abelian and k contains all roots of
1 , each of these is one-dimensional, so that in fact $\dim_k V < \infty$. It
follows easily that there exist finite subgroups E of G such that V_E
is irreducible, since the image of kG in $\mathrm{end}_k V$ is finite-dimensional
and so can be spanned by finitely many elements of G . However we require
more precise information in the sequel.

Let

$$(1) \qquad\qquad V_H = Y_1 \oplus \ldots \oplus Y_s$$

be the decomposition of V_H into its homogeneous components, with
$Y \le Y_1$. Since $G = HF$, there exist elements $1 = x_1, x_2, \ldots, x_s \in F$
such that $Y_i = Y_1 x_i$. If $i \ge 2$, then $Y x_i$ is not kH-isomorphic to
Y , and so $x_i \notin C = C_G(H/L)$, and since F has the property (*),
$x_i \notin C_F\big((F\cap H)/(F\cap L)\big)$. Therefore Y and $Y x_i$ are non-isomorphic
$k(F \cap H)$-modules (see [6], Lemma 5.1), and (1) is the decomposition of
$V_{F\cap H}$ into its homogeneous components. Now let W be any non-zero
kF-submodule of V . Restricting W to $F \cap H$, we see that
$Z = W \cap Y_i \ne 0$ for some i . Since H operates by scalar multiplication
on Y_i , every subspace of Y_i , and Z in particular, is H-invariant.
Therefore, since $G = HF$, $\displaystyle\sum_{f\in F} Zf$ is a non-zero kG-submodule of V .
Since it lies in W , while V is irreducible, we obtain $W = V$.

Some of our subsequent arguments will involve carrying out field
extensions, and the following fact will be crucial.

LEMMA 2.3. *Let A be any k-algebra, let W be any A-module, and*

let K be a separable algebraic extension of k . If $\Phi(W) = 0$ and $W \otimes_k K$ is regarded as an $A \otimes_k K$-module in the usual way, then $\Phi(W \otimes_k K) = 0$.

Proof. If W_1 is any maximal submodule of W , then

$$(W \otimes_k K)/(W_1 \otimes_k K) \cong (W/W_1) \otimes_k K ,$$

and $\cap(W_1 \otimes_k K) = 0$ as W_1 runs over all maximal submodules of W , since $\Phi(W) = 0$. Therefore, we may assume that W is irreducible. Let v be any non-zero element of $W \otimes_k K$. There exists a finite separable algebraic extension k_1 of k , contained in K , such that $v \in W \otimes_k k_1$. From [9], Chapter III, see in particular Lemma 15.4, we see that $W \otimes_k k_1$ is completely reducible as $A \otimes_k k_1$-module, and so there exists a maximal submodule M of $W \otimes_k k_1$ not containing v . Let $\overline{M} = M \otimes_{k_1} K$, which is a proper $A \otimes_k K$-submodule of $W \otimes_k K$, and let N be a maximal submodule of $W \otimes_k K$ containing $\overline{M}$. Such a maximal submodule exists since $W \otimes_k K$ is a cyclic $A \otimes_k K$-module, being generated by any non-zero element of W . For this reason also, $N \cap (W \otimes_k k_1) = M$. Therefore $v \notin N$, and it follows that $\Phi(W \otimes_k K) = 0$.

The next result is not strictly relevant to the main theorems of the paper, but nevertheless seems of some interest.

PROPOSITION 2.4. *If V is irreducible then it is strongly locally completely reducible.*

Proof. Let K be a field containing and generated by k and all roots of 1 . Then K is a separable algebraic extension of k , and so by Lemma 2.3, if $\overline{V}$ denotes the KG-module $V \otimes_k K$, we have $\Phi(\overline{V}) = 0$.

Now V_H is completely reducible by Clifford's Theorem, and we can write

(2) $$V_H = U_1 \oplus \cdots \oplus U_s ,$$

where the U_i are irreducible kH-modules. If $\{\omega_\lambda\}$ is a basis of K
over k, then $\overline{V} = \bigoplus_\lambda V \otimes \omega_\lambda$, and we find that, as a kH-module, $\overline{V}$ is
completely reducible into a direct sum of irreducible submodules each
isomorphic to some U_i. The same holds for any kH-factor A/B of $\overline{V}$,
and in particular for any KH-composition factor X of $\overline{V}$. Now if U is
any irreducible kH-submodule of X, then $\sum\limits_{0 \neq \alpha \in K} U\alpha$ is a KH-submodule,
and is a direct sum of copies of U as kH-module. Therefore every
KH-composition factor X of $\overline{V}$ is a direct sum of copies of some U_i,
and if $L_i = C_H(U_i)$, then $L_i = C_H(X)$.

Let T be any maximal KG-submodule of $\overline{V}$. Then as in Lemma 2.2,
$\overline{V}/T$ contains an irreducible KH-submodule Y/T, and, by the above,
$C_H(Y/T)$ is one of $L_1, \ldots, L_s$. By Lemma 2.1, there exists a finite
subgroup F_1 of G such that every finite subgroup $F \geq F_1$ of G has
the property (*) with respect to each of $L_1, \ldots, L_s$, and such that
$HF_1 = G$. By Lemma 2.2, $\overline{V}/T$ is an irreducible KF-module for all such
F. Therefore $\Phi\left(\overline{V}_F\right) = 0$.

It follows that the Jacobson radical of KF annihilates $\overline{V}$, and
hence that the Jacobson radical of kF, which, being nilpotent, lies in
that of KF, annihilates V. Therefore V_F is completely reducible, and
the result is established.

Passman has pointed out that Proposition 2.4 may also be proved rather
easily using the fact that V is finite-dimensional over its endomorphism
ring ([9], Chapter II, 5.1 and 6.4, or [7]).

The next result provides the key to the proof of Theorem B.

LEMMA 2.5. *If V is an $\underline{\underline{M}}_c$-module over kG and $\Phi(V) = 0$, then V
is strongly locally completely reducible.*

Proof. Let K be a field containing and generated by k and all
roots of 1, and let $\overline{V}$ be the KG-module $V \otimes_k K$. By Lemma 2.3,
$\Phi(\overline{V}) = 0$. Also, $\overline{V}$ is an $\underline{\underline{M}}_c$-module over KG. For let $p \geq 0$ be the

characteristic of k , and let H be any p'-subgroup of G . Then as V
is an $\underline{M}_c$-module, $C_V(H) = C_V(E)$ for some finite subgroup E of H . But
as kG-module, $\overline{V}$ is a direct sum of copies of V . Therefore
$C_{\overline{V}}(H) = C_{\overline{V}}(E)$, and $\overline{V}$ is an $\underline{M}_c$-module over KG . Finally, if F is a
finite subgroup of G such that $\overline{V}_F$ is completely reducible, then V_F is
completely reducible, by the argument at the end of the proof of
Proposition 2.4. Therefore we may assume that k contains all roots of
1 .

We may also assume that V is faithful for G . Since $O_q(G)$
operates trivially on every irreducible module for G over a field of
prime characteristic q , and $\Phi(V) = 0$, it follows that $O_p(G) = 1$ if
the characteristic p of k is a prime. In particular, H is a
p'-group. Also, H has finite (Mal'cev special) rank, and in particular
its Sylow subgroup H_q satisfies min for each prime q . This follows
from $[6]$, Lemma 3.4, or $[4]$.

Let $\{M_\lambda : \lambda \in \Lambda\}$ be the set of all maximal submodules of V . Then
as in Lemma 2.2, $V/M_\lambda = V_\lambda$ is an irreducible kG-module of finite
dimension over k . Let Y_λ be an irreducible kH-submodule of V_λ and
let $N_\lambda = C_H(Y_\lambda)$. In order to apply the methods used in proving
Proposition 2.4, we need to study the family $\{N_\lambda\}$ of subgroups of H .

Let Y be a finite subgroup of G such that $G = HY$ and let π be
the set of prime divisors of $|Y|$. Then π is finite. Let $y \in Y$.
Since V is an $\underline{M}_c$-module and H is a p'-group, $C_V([H_{\pi'}, y]) = C_V(E_y)$
for some finite subgroup E_y of $[H_{\pi'}, y]$, which may be taken to be
$\langle y \rangle$-invariant. Now we trivially have

(3) $C_{V_\lambda}(E_y) \geq C_{V_\lambda}([H_{\pi'}, y])$.

But since E_y is a finite p'-group, a simple local argument allows us to
deduce from the finite dimensional case that $C_{V_\lambda}(E_y)$ is the natural image
in V_λ of $C_V(E_y)$. Since this is centralized by $[H_{\pi'}, y]$, the

inclusion in (3) can be replaced by equality. In particular,

(4) $N_\lambda \geq E_y \Longleftrightarrow N_\lambda \geq [H_{\pi'}, y]$.

By Fitting's Lemma, y fixes no non-trivial element of $[H_{\pi'}, y]$,
and hence $E_y = [E_y, y]$. Since H satisfies min-q for all primes q ,
E_y lies in a finite characteristic π'-subgroup D_y of H . Let
$D = \prod_{y \in Y} D_y$. Then D is a finite characteristic subgroup of H , and we
claim that

(5) $C_G(H_{\pi'}/(H_{\pi'} \cap N_\lambda)) = C_G(D/(D \cap N_\lambda))$

for all λ . In fact, suppose that an element $g \in G$ centralizes
$D/(D \cap N_\lambda)$. Write $g = hy$ with $y \in Y$. Then y centralizes $D/(D \cap N_\lambda)$.
Therefore $E_y = [E_y, y] \leq D \cap N_\lambda$. By (4), $[H_{\pi'}, y] \leq N_\lambda$, and so g
centralizes $H_{\pi'}/(H_{\pi'} \cap N_\lambda)$. Thus (5) holds.

Now consider the subgroups $H_\pi \cap N_\lambda$ $(\lambda \in \Lambda)$. If infinitely many of
these are distinct, then it is not hard to see ([5], Lemma 3.2) that there
exists a subgroup B of H_π such that $B = \bigcup_{i=0}^{\infty} B_i$ is the union of a
tower $B_0 \leq B_1 \leq \dots$ of subgroups, each of which lies in infinitely many
distinct subgroups $H_\pi \cap N_\lambda$, while B itself lies in only finitely many.
Let A be a finite subgroup of B such that $C_V(A) = C_V(B)$. As above,
we find that $N_\lambda \geq A$ if and only if $N_\lambda \geq B$. Since A lies in one of
the subgroups B_i , we have a contradiction. Thus, only finitely many of
the subgroups $H_\pi \cap N_\lambda$ are distinct, and by Lemma 2.1, there exists a
finite subgroup F_2 of $H_\pi Y$ such that every subgroup of $H_\pi Y$ containing
F_2 has the property (*), with H_π playing the role of H and each of
$H_\pi \cap N_\lambda$ playing the role of L . Here F_2 is independent of λ , and we
may suppose $F_2 \geq Y$. Let $F_1 = DF_2$ and let F be a finite subgroup of
G containing F_1 . We claim that F has the property (*) with respect to

each of the N_λ as L . For if $f \in F$ centralizes $(F \cap H)/(F \cap N_\lambda)$, write
$f = hy$ with $h \in H$, $y \in Y$. Then y centralizes $(F \cap H)/(F \cap N_\lambda)$, and so
centralizes $D/(D \cap N_\lambda)$. By (5), y centralizes $H_{\pi'}/(H_{\pi'} \cap N_\lambda)$. The
property of F_2 ensures that y centralizes $H_\pi/(H_\pi \cap N_\lambda)$. Hence y
centralizes $H/(H \cap N_\lambda)$ and so does f , as claimed.

It now follows from Lemma 2.2 that each $V_\lambda = V/M_\lambda$ remains
irreducible when restricted to any finite subgroup $F \geq F_1$ of G .
Therefore $\Phi(V_F) = 0$. But then all submodules of V_F , and in particular
those finite-dimensional over k , have trivial Frattini submodule. Since
V_F is a sum of finite-dimensional kF-modules, and these are completely
reducible since their Frattini submodules vanish, V_F is complete
reducible. This establishes Lemma 2.5.

Theorem B follows from the somewhat stronger version below.

THEOREM B'. *Let G be a periodic almost abelian group, k a field
and V an $\underline{\underline{M}}_c$-module over kG . The following three conditions are
equivalent:*

(i) $\Phi(V) = 0$;

(ii) V is strongly locally completely reducible;

(iii) every composition factor of V is complemented.

It will be seen that some implications between these conditions can be
obtained without the $\underline{\underline{M}}_c$-condition.

Proof. *(i)* $\Rightarrow$ *(ii)* This is given by Lemma 2.5.

(ii) $\Rightarrow$ *(iii)* We shall prove this implication without using the
$\underline{\underline{M}}_c$-condition. Since the property of being strongly locally completely
reducible is inherited by quotient modules of V , it suffices to show that
every minimal submodule U of V is complemented. Let W be a submodule
of V maximal subject to $U \cap W = 0$. Then V/W is monolithic, with
$(U+W)/W$ as its unique minimal submodule, and it is sufficient to show that
$V/W = (U+W)/W$. Therefore, we may assume without loss of generality that

U is the monolith of V .

Let $A(H)$, $A(G)$ be the images of kH and kG respectively in $\text{end}_k V$. We shall show that $A(G)$ is a semisimple artinian ring. It will follow that U is complemented in V , and hence that $U = V$. Consider first $A(H)$. Let $a \in A(H)$, and suppose that $Ua = 0$. Since V is strongly locally completely reducible, there exists a finite subgroup F of G such that V_F is completely reducible, a is induced by an element of kF , and $G = HF$. We have $V = U \oplus U^*$ for some kF-module U^* . Then $Va \le U^*$. Clearly Va is a kH-submodule, and it is not hard to deduce that $\sum_{f \in F} Vaf$ is a kG-submodule of V . Since this submodule lies in U^* , it intersects U trivially, and must be zero. In particular, $Va = 0$ and $a = 0$.

Therefore U is a faithful $A(H)$-module. But U_H is the direct sum of finitely many irreducible $A(H)$-modules, by Clifford's Theorem. Consequently $A(H)$ is a commutative semisimple artinian ring. Since $A(G)$ is a finitely generated right or left module over its subring $A(H)$, it too is artinian.

Suppose now, if possible, that I is a non-zero nilpotent ideal of $A(G)$. Since V is strongly locally completely reducible, there exists a finite subgroup F of G such that V_F is completely reducible and $I \cap A(F) \ne 0$, where $A(F)$ is the image of kF in $\text{end}_k V$. But a nilpotent ideal of $A(F)$ annihilates every completely reducible $A(F)$-module. Hence $I \cap A(F)$ annihilates V , and is zero. We have shown that $A(G)$ is semisimple artinian.

(iii) $\Rightarrow$ *(i)* Suppose that every composition factor of V is complemented, but $\Phi(V) \ne 0$. Let $0 \ne v \in \Phi(V)$. Let B be a submodule of V maximal subject to $v \notin B$, and let A/B be the submodule of V/B generated by $v + B$. Then A/B is the unique minimal submodule of V/B , and in particular is a composition factor of V . Therefore there exists a submodule C of V such that $V = A + C$ and $A \cap C = B$. Clearly C is a maximal submodule of V . Since $v \notin C$, we have a contradiction.

3. An example

The construction needed for the proof of Theorem C is a special case of another construction given in [7]. The ingredients will consist of a $\underline{U}$-group G containing no elements of order r (r being the prime assigned in the statement of the theorem), and an irreducible $\underline{\underline{M}}_C$-module V over $Z_r G$ such that V is infinite-dimensional over its endomorphism ring E . The latter is of course a division ring, by Schur's Lemma. Such a group and module were in fact constructed in [4], Section 7. We briefly recall the details of the construction.

First, using Dirichlet's Theorem on the primes in an arithmetic progression, we construct two disjoint infinite sequences $\{p_1, p_2, \ldots\}$, $\{q_1, q_2, \ldots\}$ of distinct primes, none of which is equal to r , such that

(6) $q_1 \cdots q_i \mid p_i - 1$

for all i .

Let A_i be a cyclic group of order q_i^2 , B_i be a group of order p_i , and $A = A_1 \times A_2 \times \ldots$, $B = B_1 \times B_2 \times \ldots$. We can represent A by automorphisms of B in such a manner that $[A_i, B_j] = 1$ if $j < i$, while $|A_i/C_{A_i}(B_j)| = q_i$ if $j \geq i$. Then the semidirect product $G = BA$ is a $\underline{U}$-group, and if $\overline{A}$ denotes the subgroup of A consisting of the elements of square-free order, then $C = B\overline{A}$ is a locally cyclic subgroup of G containing all the elements of prime order of G . Now there is a monomorphism of C into the multiplicative group of the algebraic closure $\overline{Z}_r$ of Z_r , and by using this monomorphism we can view the additive group of $\overline{Z}_r$ as a $Z_r C$-module X on which each non-trivial element of C operates fixed point freely. Let V be any composition factor of X^G . Since $C \triangleleft G$, every non-identity element of C is fixed point free on X^G and hence on V . For C is an r'-group, and so the fixed points of any element of C in a $Z_r C$-module are preserved by homomorphisms of the module.

It follows that every element of prime order in G operates fixed

point freely on V , so that V is clearly an $\underline{\underline{M}}_c$-module over Z_rG , and
is furthermore faithful for G . Let E be the endomorphism ring of V .
By the Jacobson Density Theorem, E is a homomorphic image of a subring of
Z_rG . Now every finite set of elements of Z_rG generates a finite
subring, and hence the same holds for E . By Wedderburn's Theorem on
finite division rings, E is a field. If V were finite-dimensional over
E , it would now follow from a modular analogue of Jordan's Theorem on
periodic linear groups ([12], 9.4), and the fact that G operates
faithfully on V , that G were almost abelian. This is not the case, and
so we now have

   LEMMA 3.1. *With the above notation,* G *is a* $\underline{U}$*-group,* V *is an
irreducible* $\underline{\underline{M}}_c$*-module over* Z_rG , E *is the endomorphism ring of* V *and*
$\dim_E V$ *is infinite.*

   Theorem C now follows immediately from

   LEMMA 3.2. *Let* k *be a field of characteristic* $r \geq 0$, *let* G *be
a countable locally finite* r'*-group, and let* V *be an irreducible*
$\underline{\underline{M}}_c$*-module over* kG *of infinite dimension over its endomorphism ring. Then
there exists an indecomposable* kG*-module* $T \geq V$ *and an* $\underline{\underline{M}}_c$*-module* W
over kG *such that* T/V *is irreducible,* $\Phi(W) = 0$, *and* T *is an
epimorphic image of* W .

   Proof. Let A be the natural image of kG in $\mathrm{end}_k V$. Then A is
a locally Wedderburn algebra over k in the sense that every finite set of
elements of A lies in a semisimple k-subalgebra of A of finite
dimension over k . For every such set of elements lies in the image of
the group algebra kF of some finite subgroup F of G . The group
algebra kF is semisimple by Maschke's Theorem, and so are its homomorphic
images. Clearly, A has countable dimension over k .

   If $0 \neq v \in V$, then the map $a \rightarrow va$ $(a \in A)$ determines an
epimorphism of right A-modules from A to V with kernel the annihilator
$M(v)$ of v in A . Thus $M(v)$ is a maximal right ideal of A , and

(7) $A/M(v) \cong V$

as right A-modules. Clearly

(8)
$$\bigcap_{0 \neq v \in V} M(v) = 0 .$$

Since V has infinite dimension over its endomorphism ring, Theorem Cl of [7] shows that there exists an indecomposable A-module T containing V such that T/V is irreducible. Clearly T is cyclic, being generated by any element not lying in V, and so there exists an epimorphism of right A-modules from A to T.

Since we have an obvious epimorphism from kG to A, we may use that epimorphism to view the A-modules above as kG-modules. Writing W for the kG-module A, (7) and (8) tell us that $\Phi(W) = 0$. Also, the kG-module T is an epimorphic image of W.

It remains to see that W is an $\underline{\underline{M}}_c$-module over kG. Let H be a subgroup of G. Since V is an $\underline{\underline{M}}_c$-module and G has no elements of order $r = \operatorname{char} k$, H contains a finite subgroup F such that $C_V(H) = C_V(F)$. Let $Y = C_W(F)$. Then, for each $0 \neq v \in V$, $\big(Y + M(v)\big)/M(v)$ lies in the centralizer of F in $W/M(v)$. Because of the isomorphism (7), which now becomes $W/M(v) \cong V$, H operates trivially on $\big(Y + M(v)\big)/M(v)$. Thst is, if $y \in Y$ and $h \in H$, then $y(h-1) \in M(v)$. By (8), $y(h-1) = 0$. Thus $C_W(F) \leq C_W(H)$. The reverse inclusion being obvious, it follows that W is an $\underline{\underline{M}}_c$-module.

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