ON THE PROPERTY C AND A PROBLEM OF HAUSDORFF

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1. Introduction. In an earlier paper [3] I studied the property C and related properties C′ and C″; but the principal problem, viz, to prove, with the axiom of choice only (without any other hypothesis), the existence of a non-denumerable set of property C, remains open.

In another paper [4] I studied Hausdorff's problem [1] of the existence of Ω-limits for (transfinite) sequences of dyadic sequences, and we have some conditional results; but again the main problem remains open, viz, the problem of proving (with the axiom of choice only) the existence of such Ω-limits.

In the present paper we are going to solve, in a certain sense, a compound of these two problems. We are going to show that: either there exist Ω-limits, or non-denumerable C-sets, or both (Theorem I). We also prove two other theorems which are related.

For the definitions and the general theory we refer the reader to the two papers mentioned above. We shall, however, repeat here those theorems which we are going to use explicitly, and those definitions where more than just the name occurs.

We denote generically a finite set by Λ. Individual finite sets will be indicated with a superscript, such as $\Lambda^1, \Lambda^\alpha$. If $E \subset F + \Lambda$, we shall write $E < F$ (E is almost-contained in F). Whereas, in [4], these definitions were used for sets of natural numbers only, we shall use them here for other sets also, but only for subsets of a fixed denumerable set (e.g., the set of all rational numbers) and thus we shall still have the same theorems, *mutatis mutandis.*

A set E is said to have the property C″ if every double sequence of intervals J_{mn} satisfying the conditions

$$E \subset \sum_n J_{mn} \qquad (\text{for all } m),$$

contains a diagonal sequence

$$J_{1n_1}, J_{2n_2}, \ldots, J_{mn_m}, \ldots$$

such that

$$E \subset \sum_m J_{mn_m}.$$

It can easily be shown that *every* C″-set is a C-set [cf. 2].

THEOREM I. *The non-existence of Ω-limits (for dyadic sequences) implies that every linear set of power $\aleph_1$ has property C (and also C″).*

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We shall actually prove it for $\mathbf{C}''$ and we shall give two proofs.

THEOREM II. *The non-existence of Ω-limits implies the following proposition: Given any family of $\aleph_1$ infinite sets of natural numbers E^α ($\alpha < \Omega$), there exists a set D such that $E^\alpha \cdot D$ and $E^\alpha \cdot CD$ are infinite sets, for all α.* (CD means: complement of D.)

There is a stronger theorem, from which the above two follow, namely:

THEOREM III. *The non-existence of Ω-limits implies the following proposition: The sum of $\aleph_1$ (linear) sets of first category is again of first category.*

For the proofs of these theorems we need a few preliminaries.

The abbreviation "of n.n." means "of natural numbers." The letters μ, ν, m, n, r, s, t, (without or with subscripts) will always denote natural numbers; and the letters a, b, c, d, (without or with subscripts) will denote "segments," to be defined presently. Sets of segments, and also other sets, will be denoted by capitals, A, B, ...

A finite sequence of n.n. $(r_1, r_2, \ldots, r_n)$ will be called a *segment*. The first m terms ($m < n$) of a segment form a *subsegment*. Example: (1, 3, 5) is a subsegment of (1, 3, 5, 7, 9), but (1, 5, 9) is not a subsegment of it in our sense.

The word "sequence" shall always mean "infinite sequence."

Two sequences of n.n., $\{s_n\}$ and $\{t_n\}$, will be said to *intersect* if we have $s_n = t_n$ for some value of n. (In [**3**, p. 118, Lemme 5], two sequences were said to be "tout-à-fait différentes" if and only if, in this sense, they do not intersect.)

A sequence s_n and a segment $(r_1, r_2, \ldots, r_m)$ *intersect*, if $s_n = r_n$ for some $n \leqslant m$.

For a given set $\mathscr{S}$ of sequences of n.n., a *diagonal sequence* is a sequence (not necessarily in $\mathscr{S}$) which intersects each element of $\mathscr{S}$. If such a sequence exists, we shall say that $\mathscr{S}$ *admits a diagonal.*

We quote five theorems from the other papers, for later use:

(1) [**3**, p. 119, Lemme 6]. *The proposition 'Every linear set of power $\aleph_1$ has property $\mathbf{C}''$' is equivalent to the following: 'Every set of sequences (of* n.n.*) of power $\aleph_1$ admits a diagonal.'*

(2) [**3**, p. 120, Lemme 8]. *The existence of a non-$\mathbf{C}''$ set of power $\aleph_1$ implies that the interval* (0, 1) *is the sum of $\aleph_1$ sets of first category.*

(3) [**4**, p. 34, Theorem 3^a]. *The non-existence of Ω-limits implies the proposition $\mathbf{B}(\aleph_1)$, i.e., the non-existence of (Ω, ω^*)-gaps.*

(4) [**4**, p. 37, Lemma 5]. *$\mathbf{B}(\aleph_1)$ is equivalent to the following proposition: 'If $Y_n < X_\alpha$ for all $n < \omega$, $\alpha < \Omega$ (X's and Y's are sets of* n.n.*), then there exists a set D such that $Y_n < D < X_\alpha$ for all n and α.'*

(5) [**4**, p. 38, Lemma 7]. *The non-existence of Ω-limits (for dyadic sequences) implies the following proposition: 'Given $\aleph_1$ sets (of* n.n.*) X_α, if every finite product of X's is an infinite set, then there exists an infinite set D, such that $D < X_\alpha$ (for all α).'* ("Finite product" means a product of a finite number of sets.)

The last two theorems, i.e., (4) and (5), are the clue to the proofs of this paper; they also contain, in a sense, the clue to [**4**, Chapter III]. We shall, however, have to replace, in (4) and (5), the words "of n.n." by "of segments" and later on by "of rational numbers." This is permissible, since, in theorems of this type, the set of all natural numbers may be replaced by any other denumerable set, e.g., the set of all segments.

(4′), (5′). Same as (4), (5), with "segments" or "rational numbers" in place of "n.n."

2. Proof of Theorem I. We need the following two new lemmas, which are obvious:

LEMMA (i). *Given a finite set of sequences* (*of* n.n.), *there exist infinitely many diagonal segments, a diagonal segment being a segment intersecting each of the given sequences.*

LEMMA (ii). *Given a segment* $b = (r_1, r_2, \ldots, r_m)$ *and a finite set of sequences, there exist infinitely many diagonal segments starting with* $(r_1, r_2, \ldots, r_m)$, *i.e., having* b *as a subsegment.*

Proof of Theorem I. Let

$$\{s_n^1\}, \{s_n^2\}, \ldots, \{s_n^\omega\}, \ldots, \{s_n^\alpha\}, \ldots \qquad (\alpha < \Omega)$$

be a fixed, but arbitrary, set of $\aleph_1$ sequences of n.n.

Assuming that no Ω-limits exist, it is sufficient to show that the above set admits a diagonal sequence, cf. (1).

Let A^α be the set of all segments intersecting $\{s_n^\alpha\}$. It follows from Lemma (i) that every finite product

$$\prod_{\nu=1}^{n} A^{\alpha_\nu}$$

is an infinite set, hence, by (5′), there exists an infinite set (of segments) D_0 such that $D_0 < A^\alpha$ (all $\alpha < \Omega$).

More generally, let A_b^α be the set of all those segments which have b as a (proper) subsegment and intersect $\{s_n^\alpha\}$. Just as before, but using Lemma (ii), we see that every finite product is an infinite set, hence there exists an infinite set D_b, such that $D_b < A_b^\alpha$ (all α).

Since obviously $A_b^\alpha < A^\alpha$, we have $D_b < A^\alpha$, for all α and all segments b. Now, the b's form a denumerable set, and the α's a set of power $\aleph_1$, hence, by (3) and (4′), there exists a set D such that

$$(6) \qquad D_b < D < A^\alpha, \quad \text{for all } \alpha \text{ and all } b.$$

We shall use this set D to construct the required diagonal sequence.

Let $b_1 \in D$. Next, let

$$b_2 \in D_{b_1} \cdot D.$$

(Such a segment exists, because $D_b \cdot D$ is an infinite set for any b.) Note that b_1 is a subsegment of b_2. Next, let

$$b_3 \in D_{b_2} \cdot D,$$

and, generally, let

$$b_{n+1} \in D_{b_n} \cdot D, \ldots .$$

We see that b_n is always a (proper) subsegment of b_{n+1}, hence all b_n's are subsegments of one common sequence. More explicitly, we can write:

$$b_1 = (r_1, r_2, \ldots, r_{\nu_1}),$$

$$b_2 = (r_1, r_2, \ldots, r_{\nu_1}, \ldots, r_{\nu_2}),$$

$$\ldots \qquad \ldots$$

$$b_n = (r_1, r_2, \ldots, r_{\nu_1}, \ldots, r_{\nu_2}, \ldots, r_{\nu_n}),$$

$$\ldots \qquad \ldots$$

In order to show that $\{r_n\}$ is the required diagonal sequence, it is sufficient to notice that, by definition, $b_n \in D$ for all n, and that, by (6), $D < A^\alpha$ for all α. Thus $b_n \in A^\alpha$ for any given α and almost all n. Therefore, for any given α, almost all b_n intersect $\{s_n^\alpha\}$, and hence $\{r_n\}$ intersects $\{s_n^\alpha\}$ for all α.

Theorem II can be proved in a similar way, but we shall rather deduce it from Theorem III.

3. Proof of Theorem III. We need the following result:

(7) [**3**, p. 112, Théorème 1,B$_3$]. $\mathbf{B}(\aleph_1)$ *is equivalent to the following proposition: 'The sum of* $\aleph_1 F_\sigma$*'s disjoint from* $\mathfrak{R}$ *is contained in an* F_σ *disjoint from* $\mathfrak{R}$*, where* $\mathfrak{R}$ *is the set of all rational numbers.'*

Now, the set $\mathfrak{R}$ may be replaced by any other dense denumerable set $\mathfrak{D}$; also, the sum of $\aleph_1 F_\sigma$'s is equal to the sum of $\aleph_1$ closed sets (because $\aleph_1 \cdot \aleph_0 = \aleph_1$). From this, together with (3) and (7), we have the following:

LEMMA (iii). *The non-existence of* Ω*-limits implies the proposition: 'The sum of* $\aleph_1$ *closed sets disjoint from* $\mathfrak{D}$*, is contained in an* F_σ *disjoint from* $\mathfrak{D}$*; where* $\mathfrak{D}$ *is any everywhere-dense denumerable set.'*

Taking complements, we get the following:

LEMMA (iv). *The non-existence of* Ω*-limits implies that the product of* $\aleph_1$ *open sets or* G_δ*'s containing* $\mathfrak{D}$*, contains a* G_δ *containing* $\mathfrak{D}$ *(where* $\mathfrak{D}$ *is everywhere dense and denumerable).*

Proof of Theorem III. Without loss of generality, the sets of first category in the proposition in the theorem may be replaced by F_σ's of first category, i.e., by non-dense F_σ's. Then, by the same argument as above, the proposition may be changed to the following one:

The product of $\aleph_1$ *everywhere-dense open sets contains an everywhere-dense* G_δ.

Let $G^1, G^2, \ldots, G^\omega, \ldots, G^\alpha, \ldots$ $(\alpha < \Omega)$ be $\aleph_1$ everywhere-dense open sets. It is sufficient to prove that they contain an everywhere-dense G_δ, assuming the non-existence of Ω-limits.

Let A^α be the set of all rational numbers contained in G^α. Since every finite product of G^α's is again an open set, every finite product of A^α's is an infinite set. Hence, by (5′), there exists an infinite set D_0 with $D_0 < A^\alpha$ (for all α).

Now let J be any interval with rational endpoints. Then $J \cdot A^\alpha$ is the set of rational points in $J \cdot G^\alpha$. Again, any finite product of these sets $J \cdot A^\alpha$ is an infinite set. Hence there is an infinite set D_J with $D_J < J \cdot A^\alpha < A^\alpha$, for all α. Thus we have:

(8) $\quad D_J \subset J,\qquad$ for all J's,

(9) $\quad D_J < A^\alpha,\qquad$ for any J and any α. (There are $\aleph_0$ J's and $\aleph_1$ α's.)

Therefore, by (3), (4′), and (9), there is a set D with $D_J < D < A^\alpha$, for all J and α.

Now, since D_J is an infinite set, it has, by (8), at least one accumulation point in the closure of J, and this accumulation point is necessarily an accumulation point of D, because almost all[1] elements of D_J are elements of D. Thus we see that D has accumulation points in every interval J, therefore D is everywhere-dense (and denumerable).

Also, since $D < A^\alpha$, we have $D \subset A^\alpha + \Lambda^\alpha$, where the Λ^α's are certain subsets of $\mathfrak{R}$; and since $A^\alpha \subset G^\alpha$, we finally have

$$D \subset G^\alpha + \Lambda^\alpha, \quad \text{for all } \alpha. \tag{10}$$

We may now apply Lemma (iv), for the left hand side of (10) is everywhere-dense and denumerable, whereas the right hand side is a G_δ (because the sum of an open set and a finite set is always a G_δ).

It follows, from Lemma (iv), that there exists a G_δ, say E, such that

$$D \subset E \subset \prod_\alpha (G^\alpha + \Lambda^\alpha) \subset \prod_\alpha G^\alpha + \mathfrak{R}. \tag{11}$$

From $D \subset E$ it follows that E is everywhere-dense, therefore it is everywhere of second category (because it is a G_δ). Therefore, $E - \mathfrak{R}$ is still everywhere-dense and is obviously still a G_δ. Finally, we see from (11), that $E - \mathfrak{R}$ is contained in all G^α. Thus $E - \mathfrak{R}$ is the G_δ which we set out to find.

4. Proof of Theorem II. To every set of n.n. there corresponds a dyadic "decimal" representation of a real number belonging to the interval [0,1]. A set of sets of n.n. is said to be non-dense, or of first category, if the corresponding set of real numbers is non-dense, or of first category. Let E be an infinite set of n.n. Then the linear set corresponding to the set of X's such that $E \subset X$ is a Cantor discontinuum, and thus non-dense, and the set of X's such that $E < X$ is of first category, being the sum of $\aleph_0$ non-dense sets.

[1]"almost all" means "all but a finite number of."

Similarly, the set of all X's such that $E < CX$ is also of first category. Hence, the set of all X's such that

$$\text{(12)} \qquad \text{either } E \cdot X = \Lambda \text{ or } E \cdot CX = \Lambda,$$

is of first category. Therefore, given a set of $\aleph_1$ infinite sets E^α ($\alpha < \Omega$), and assuming that there are no Ω-limits, it follows from Theorem III that the set of all X's such that

$$\text{(13)} \qquad \text{for } \textit{some } \alpha, \text{ either } E^\alpha \cdot X = \Lambda \text{ or } E^\alpha \cdot CX = \Lambda,$$

is likewise of first category. Hence the complement of this set of X's is not empty (because of second category), so that there exists an infinite set D (belonging to the said complement and thus satisfying the negation of (13)), such that

$$\text{(14)} \qquad \text{for } \textit{all } \alpha, \text{ both } E^\alpha \cdot D \text{ and } E^\alpha \cdot CD \text{ are infinite sets,}$$

which proves the theorem.

5. Alternative proof of Theorem I. It follows from (2) (reversing the implication) that, if the sum of $\aleph_1$ sets of first category is always also of first category, then every set of power $\aleph_1$ has property $\mathbf{C}''$. Combining this with Theorem III, we have our theorem.

References

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