# ON THE DIMENSION OF MODULES AND ALGEBRAS, VII
## ALGEBRAS WITH FINITE-DIMENSIONAL RESIDUE-ALGEBRAS

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It was shown in Eilenberg-Nagao-Nakayama [3] (Theorem 8 and § 4) that if $\Omega$ is an algebra (with unit element) over a field $K$ with $(\Omega : K) < \infty$ and if the cohomolgical dimension of $\Omega$, $\dim \Omega$, is $\leqq 1$, then every residue-algebra of $\Omega$ has a finite cohomological dimension. In the present note we prove a theorem of converse type, which gives, when combined with the cited result, a rather complete general picture of algebras whose residue-algebras are all of finite cohomological dimension. Namely, if $\Lambda$ is an algebra over a field $K$ with $(\Lambda : K) < \infty$ and if

$$\dim (\Lambda/N^2) < \infty,$$

where $N$ is the radical of $\Lambda$, then $\Lambda$ is a homomorphic image of an algebra $\Omega$ over $K$ with $(\Omega : K) < \infty$ such that

$$\dim \Omega \leqq 1.$$

We may further impose the condition

$$\Omega/M^2 \approx \Lambda/N^2$$

where $M$ is the radical of $\Omega$, and with this additional condition the algebra $\Omega$ and the homomorphism $\Omega \to \Lambda$ are determined uniquely up to an isomorphism.

Thus, algebras with cohomological dimension $\leqq 1$ are in a sense "prototypes" for algebras with finite-dimensional residue-algebras. The construction of $\Omega$ and the homomorphism $\Omega \to \Lambda$ is essentially what was employed by Hochschild [5, 6] in connection with his notion of "maximal algebra" and by Jans [3] as free algebras.

We shall start with semi-primary rings (in the sense in [3]). For them and for their global dimensions we shall prove a theorem which is quite similar as above but which assumes an additional condition on "splitting".

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## § 1. Rings with $N^2 = 0$

In this section $\Lambda$ will denote a semi-primary ring with radical $N$ such that $N^2 = 0$. The quotient ring $\Gamma = \Lambda/N$ is then semi-simple and $N$ is a two-sided $\Gamma$-module.

LEMMA 1. *Let $e$, $e'$ be primitive idempotents in $\Gamma$ such that*

$$Ne \neq 0 \neq eNe'.$$

*Then*

$$0 \leqq \mathrm{l.dim}_{\Lambda} Ne < \mathrm{l.dim}_{\Lambda} Ne'.$$

*Proof.* Our lemma (as well as Proposition 2 below) follows readily from the consideration of "minimal resolution" (i.e. a projective resolution consisting of "minimal homomorphisms") (Eilenberg-Nakayama [4], Eilenberg [2]). But, since we are dealing here with a very simple situation, we shall give a direct proof. Since $NNe' = 0$, the left $\Lambda$-module $Ne'$ is semi-simple and thus $Ne' \approx \Sigma \Gamma e_\alpha$ where the sum is direct and $\{e_\alpha\}$ is an indexed family of primitive idempotents in $\Gamma$. Since $eNe' \neq 0$ we have $e\Gamma e_\alpha \neq 0$ for at least one index $\alpha$. Thus $e_\alpha \approx e$ (meaning $\Gamma e_\alpha \approx \Gamma e$) and $Ne'$ has a direct factor isomorphic with $\Gamma e$. Thus

$$\mathrm{l.dim}_{\Lambda} \Gamma e \leqq \mathrm{l.dim}_{\Lambda} Ne'$$

Next consider the exact sequence $0 \to Ne \to \Lambda e \to \Gamma e \to 0$. If $\Gamma e$ is not $\Lambda$-projective, then

$$\mathrm{l.dim}_{\Lambda} \Gamma e = 1 + \mathrm{l.dim}_{\Lambda} Ne \geqq 1$$

which implies the desired result. If $\Gamma e$ is $\Lambda$-projective, then the exact sequence splits and we have a direct sum $\Lambda e = Ne + \mathfrak{l}$ where $\mathfrak{l}$ is a left ideal of $\Lambda$. Multiplying by $N$ we find $Ne = N^2 e + N\mathfrak{l} = N\mathfrak{l} \subset \mathfrak{l}$. Thus $Ne = 0$ contrary to hypothesis.

A sequence $(e_0, \ldots, e_n)$ of primitive idempotents in $\Gamma$ is called *connected* if $e_{i-1}Ne_i \neq 0$ for $i = 1, \ldots, n$. The number $n$ is called the *length* of the connected sequence. It is clear that if in a connected sequence an idempotent is replaced by an isomorphic one, the sequence remains connected.

PROPOSITION 2. *A connected sequence of length $n$ exists if and only if* $\mathrm{gl.dim}\, \Lambda \geqq n$.

*Proof.* We may assume $n \geqq 1$. The condition $\mathrm{gl.dim}\, \Lambda \geqq n$ is equivalent to

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$\mathrm{l.dim}_\Lambda n \geqq n-1$. Let $(e_0, \ldots, e_n)$ be a connected sequence. Then, by Lemma 1,

$$0 \leqq \mathrm{l.dim}_\Lambda Ne_i < \mathrm{l.dim}_\Lambda Ne_{i+1} \qquad \text{for } i=1, \ldots, n-1.$$

Thus $\mathrm{l.dim}_\Lambda Ne \geqq n-1$, whence $\mathrm{l.dim}_\Lambda N \geqq n-1$.

Suppose conversely $\mathrm{l.dim}_\Lambda N \geqq n-1$. Since $N$ is the direct sum of modules of form $Ne$, where $e$ is a primitive idempotent in $\Gamma$, there exists a primitive idempotent $e_n$ in $\Gamma$ such that $\mathrm{l.dim}_\Lambda Ne_n \geqq n-1$. Since $NNe_n = 0$, the $\Lambda$-module $Ne_n$ is semi-simple and is therefore the direct sum of modules $\Gamma e$. Thus there exists a primitive idempotent $e_{n-1}$ in $\Gamma$ such that

(i) $\Gamma e_{n-1}$ is isomorphic with a direct suumand of $Ne_n$,

(ii) $\mathrm{l.dim}_\Lambda \Gamma e_{n-1} \geqq n-1$.

Since $e_{n-1}\Gamma e_{n-1} \neq 0$ we have $e_{n-1}Ne \neq 0$. Further, from the exact sequence $0 \to Ne_{n-1} \to \Lambda e_{n-1} \to \Gamma e_{n-1} \to 0$ we deduce that $\mathrm{l.dim}_\Lambda Ne_{n-1} \geqq n-2$. Continuing in this fashion we obtain a connected sequence $(e_1, \ldots, e_n)$ such that $\mathrm{l.dim}_\Lambda Ne_i \geqq i-1$. In particular, $\mathrm{l.dim}_\Lambda Ne_1 \geqq 0$ i.e. $Ne_1 \neq 0$. There exists therefore a primitive idempotent $e_0$ in $\Gamma$ such that $e_0 Ne_1 \neq 0$. Thus $(e_0, \ldots, e_n)$ is a connected sequence of length $n$ as desired.

COROLLARY 3. *Let $\Lambda$ be a semi-primary ring with radical $N$ such that $N^2 = 0$. Let $l$ be the number of simple components of the semi-simple ring $\Gamma = \Lambda/N$. Then*

$$\mathrm{gl.dim}\,\Lambda < l \qquad \text{or} \ = \infty.$$

*Proof.* Assume $\mathrm{gl.dim}\,\Lambda \geqq l$. Then there exists a connected sequence $(e_0, \ldots, e_l)$ of primitive idempotents in $\Gamma$. At least two of these idempotents must be isomorphic and therefore there exists a connected sequence $(e_0', \ldots, e_n')$ with $e_0' = e_n'$. This implies the existence of connected sequences of any length. Thus $\mathrm{gl.dim}\,\Lambda = \infty$.

## § 2. The "maximal" ring $\Omega$

Let $\Gamma$ be a semi-simple ring and $A$ a two-sided $\Gamma$-module. Define $A^{(0)} = \Gamma$, $A^{(n+1)} = A^{(n)} \otimes_\Gamma A$. Then define the (graded) ring

$$\Omega = \sum_{i=0}^{\infty} A^{(i)} \qquad \text{(restriced direct sum)}$$

with multiplication defined by the obvious mapping $A^{(p)} \times A^{(q)} \to A^{(p+q)}$. Set

$M = \sum_{i=1}^{\infty} A^{(i)}$. Then

$$\Omega = \Gamma + M = \Gamma + A + M^2,$$
$$M^k = \sum_{i=0}^{\infty} A^{(k+i)}.$$

The ring $\Sigma = \Omega/M^2$ may be identified with the split extension $\Gamma + A$ (in which $A^2 = 0$). Clearly

$$M = \Omega \otimes_\Gamma A.$$

Since $A$ is projective as a left $\Gamma$-module, it follows that $M$ is projective as a left $\Omega$-module.

PROPOSITION 4. *The following conditions are equivalent:*

(a) $$\mathrm{gl.dim}\, \Sigma = n,$$
(b) $$A^{(n+1)} = 0, \ A^{(n)} \neq 0.$$

*If these conditions hold then* $\Omega$ *is a hereditary* (*i.e.* gl.dim $\Omega \leqq 1$) *semi-primary ring with radical* $M$ *such that* $M^{n+1} = 0$, $M^n \neq 0$.

*Proof.* Assume $A^{(n)} \neq 0$. Then there exist elements $a_1, \ldots, a_n \in A$ and primitive idempotents $e_1, f_1, \ldots, e_n, f_n \in \Gamma$ such that

$$e_1 a_1 f_1 \otimes \ldots \otimes e_n a_n f_n \neq 0$$

in $A^{(n)}$. Since $e_i a_i f_i \otimes e_{i+1} a_{i+1} f_{i+1} = e_i a_i \otimes f_i e_{i+1} a_{i+1} f_{i+1}$ it follows that $f_i e_{i+1} \neq 0$ for $i = 1, \ldots, n-1$. Thus $f_i \approx e_{i+1}$ for $i = 1, \ldots, n-1$ and therefore $(e_1, f_1, f_2, \ldots, f_n)$ is a connected sequence of idempotents in $\Gamma$, in the sense of the preceding section (with $\Lambda$ replaced by $\Sigma$). Thus, by Proposition 2, gl.dim $\Sigma \geqq n$.

Now assume $A^{(n+1)} = 0$. Then $\Omega$ is semi-primary with radical $M$ and $M^{n+1} = 0$. Since $M$ is projective as a left $\Omega$-module it follows that gl.dim $\Omega \leqq 1$, i.e. $\Omega$ is hereditary. By Corollary 11 of [3] we have gl.dim $\Sigma$ = gl.dim $(\Omega/M^2)$ $\leqq n$. This concludes the proof.

## § 3. Ring in split form

Let $\Lambda$ be a semi-primary ring with radical $N$. A *splitting* for $\Lambda$ is a direct sum decomposition

$$\Lambda = \Gamma + A + N^2$$

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such that

$$\Gamma\Gamma \subset \Gamma,\ \Gamma A \subset A,\ A\Gamma \subset A,\ A + N^2 = N.$$

We have $1 \in \Gamma$. Indeed let $1 = \gamma + (1-\gamma)$ with $\gamma \in \Gamma$, $1-\gamma \in N$. Then $\gamma = 1\gamma = \gamma^2 + (1-\gamma)\gamma$ with $\gamma^2 \in \Gamma$ and $(1-\gamma)\gamma \in N$. Thus $(1-\gamma)\gamma = 0$. Consequently $(1-\gamma)^2 = 1-\gamma$. Since $1-\gamma \in N$ it follows that $1-\gamma = 0$ i.e. $1 = \gamma \in \Gamma$. Thus $\Gamma$ is a subring of $\Lambda$ which may be identified with the semi-simple ring $\Lambda/N$, and $A$ is a two-sided $\Gamma$-module which may be identified with $N/N^2$. The ring $\Lambda/N^2$ may be identified with the split extension $\Sigma = \Gamma + A$.

THEOREM 5. *Let $\Lambda$ be a semi-primary ring with radical N such that $\Lambda$ admits a splitting and*

$$\text{gl.dim}\,(\Lambda/N^2) = n < \infty.$$

*Then there exist a hereditary semi-primary ring $\Omega$ with radical $M$ and a ring epimorphism $\varphi : \Omega \to \Lambda$ such that $\varphi^{-1}(N^2) = M^2$ i.e. $\varphi$ induces an isomorphism*

$$\Omega/M^2 \approx \Lambda/N^2.$$

*The pair $(\Omega, \varphi)$ is determined uniquely up to an isomorphism. Moreover, the ring $\Omega$ admits a splitting, $M^{n+1} = 0$, and $N^{n+1} = 0$.*

COROLLARY 6. *With $\Lambda$ as in Theorem 5*

$$\text{gl.dim}\,(\Lambda/a) < \infty$$

*for every two-sided ideal $a$ in $\Lambda$. If $a \subset N^2$ then*

$$\text{gl.dim}\,(\Lambda/a) \leqq n.$$

*Inparticular,*

$$\text{gl.dim}\,\Lambda \leqq n.$$

*If $l$ is the number of simple components of $\Gamma = \Lambda/N$ then $n < l$.*

*Proof.* Let $\Lambda = \Gamma + A + N^2$ be a splitting for $\Lambda$. Let $\Omega$ be the ring constructed in § 2 using the ring $\Gamma$ and the two-sided $\Gamma$-module $A$. Since $\Sigma = \Lambda/N^2$ we have $\text{gl.dim}\,\Sigma = n < \infty$. Thus, by Proposition 4, $\Omega$ is a semi-primary ring with radical $M$ and $M^{n+1} = 0$. Define the ring homomorphism $\varphi : \Omega \to \Lambda$ by setting $\varphi(\gamma) = \gamma$ for $\gamma \in \Gamma$ and $\varphi(a_1 \otimes \dots \otimes a_k) = a_1 \dots a_k$ for $a_1 \otimes \dots \otimes a_k \in A^{(k)}$, $k > 0$. We have $A \subset \varphi(M) \subset N$. It follows that $N = \varphi(M) + N^2$. Therefore $N=\varphi(M)$ and $\varphi$ is an epimorphism. Clearly $\Omega$ admits a splitting $\Omega=\Gamma+A+M^2$, and $\varphi^{-1}(N^2)=M^2$.

Let $\Omega'$ be another hereditary semi-primary ring with radical $M'$ and let $\varphi':\Omega'\to\Lambda$ be a ring epimorphism such that $\varphi'^{-1}(N^2)=M'^2$. There results for $\Omega'$ a splitting $\Omega'=\varphi'^{-1}(\Gamma)+\varphi'^{-1}(A)+M'^2$. If we identify $\varphi'^{-1}(\Gamma)$ with $\Gamma$ and $\varphi'^{-1}(A)$ with $A$ using the mapping $\varphi'$ we obtain a splitting $\Omega'=\Gamma+A+M'$ and $\varphi'$ is the identity on $\Gamma+A$. If we replace $\Lambda$ by $\Omega'$ in the construction above we obtain an epimorphism $\psi:\Omega\to\Omega'$ such that $\psi^{-1}(M'^2)=M^2$. Since the ring homomorphisms $\varphi$, $\varphi'\psi:\Omega\to\Lambda$ coincide on $\Gamma+A$, it follows that $\varphi=\varphi'\psi$. There remains to be shown that $\psi$ is an isomorphism. Let $a$ be the kernel of $\psi$. Then $\Omega/a\approx\Omega'$ and $a\subset M^2$. It follows then from Theorem I of [4] (or [3], Proposition 10 and Remark there) that $a=0$. Since $M^{n+1}=0$ and $N=\varphi(M)$ we have $N^{n+1}=0$. This concludes the proof of the theorem.

The last statement of the corollary follows from Corollary 3 applied to the ring $\Sigma=\Gamma+A=\Lambda/N^2$.

Let $a$ be any two-sided ideal in $\Lambda$ and let $b=\varphi^{-1}(a)$. Then $\Lambda/a\approx\Omega/b$ so that by [3], Theorem 8, $\text{gl.dim}(\Lambda/a)<\infty$.

If $a\subset N^2$ then $b\subset M^2$ and the conclusion that $\text{gl.dim}(\Lambda/a)\leqq n$ is then a consequence of

PROPOSITION 7. *Let $\Omega$ be a hereditary semi-primary ring with radical $M$ such that $M^{n+1}=0$. For any two-sided ideal $b\subset M^2$*

$$\text{gl.dim}(\Omega/b)\leqq n.$$

*Proof.* Assume $n$ even, $n=2i$. We may assume $i>0$ since if $i=0$ then $M=0$, $b=0$ and $\Omega=\Omega/b$ is semi-simple. Since $b\subset M^2$ and $M^{2i+1}=0$ it follows that $b^iM=b^{i+1}=0$. Thus [3] Proposition 9, condition (iii′) implies $\text{gl.dim}(\Omega/b)\leqq n$.

Let $n$ be odd, $n=2i+1$. We may assume $i>0$ since if $i=0$ then $n=1$, $M^2=0$, $b=0$ and $\text{gl.dim}(\Omega/b)=\text{gl.dim}\,\Omega\leqq 1$ by hypothesis. Since $b\subset M^2$ and $M^{2i+2}=0$ it follows that $b^{i+1}=0$. Thus [3] Proposition 9, condition (iii) implies $\text{gl.dim}(\Omega/b)\leqq n$.

Next we consider a semi-primary ring $\Lambda$, with radical $N$ and admitting a splitting $\Lambda=\Gamma+A+N^2$, which satisfies

$$\text{gl.dim}(\Lambda/N^2)=\infty$$

contrary to Theorem 5. Again construct $\Omega$ as in §2 using the ring $\Lambda$ and the two-sided $\Lambda$-module $A$, and let $M$ have the same significance as before. Let $N^h = 0$. Then $\Lambda$ is a homomorphic image of $\Omega/M^m$ for every $m \geqq h$. We want to show

PROPOSITION 8. (*Under our assumption* gl.dim$(\Lambda/N^2) = \infty$) *the semi-primary ring* $\Omega/M^m$ *has gl.dimension* $\infty$ *for infinitely many* $m$.

*Proof.* By our assumption gl.dim$(\Lambda/N^2) = \infty$, there exists a connected sequence $(e_0, e_1, \ldots, e_{k-1}, e_0)$ $(k \neq 0)$ of primitive idempotents in $\Gamma$, with respect to $\Lambda/N^2$, whose first and last terms coincide. We contend that gl.dim $(\Omega/M^{2k}) = \infty$. To see this, consider the left $(\Omega/M^{2k})$-module $(\Omega/M^k)e_0$. We have the exact sequence

$$0 \longrightarrow (M^k/M^{2k})e_0 \longrightarrow (\Omega/M^{2k})e_0 \longrightarrow (\Omega/M^k)e_0 \longrightarrow 0.$$

Let $1 = e_0 + \Sigma f_\nu$ be a decomposition of 1 into mutually orthogonal primitive idempotents in $\Gamma$. We have $M^k = \Omega \otimes_\Gamma A^{(k)} = \Omega \otimes_\Gamma e_0 A^{(k)} + \Sigma \Omega \otimes_\Gamma f_\nu A^{(k)}$ (direct). Hence $M^k e_0 = \Omega \otimes_\Gamma e_0 A^{(k)} e_0 + \Sigma \Omega \otimes_\Gamma f_\nu A^{(k)} e_0$ (direct). As $M^{2k} = M^k \otimes_\Gamma A^{(k)}$, we have similarly $M^{2k} e_0 = M^k \otimes_\Gamma e_0 A^{(k)} e_0 + \Sigma M^k \otimes_\Gamma f_\nu A^{(k)} e_0$ (direct). Then we obtain readily

$$(M^k/M^{2k})e_0 \approx (\Omega/M^k) \otimes_\Gamma e_0 A^{(k)} e_0 + \Sigma (\Omega/M^k) \otimes_\Gamma f_\nu A^{(k)} e_0 \quad \text{(direct)}.$$

Since $(e_0, e_1, \ldots, e_{k-1}, e_0)$ is connected, we have here $e_0 A^{(k)} e_0 \neq 0$. On taking a left $e_0 \Gamma e_0$-basis of $e_0 A^{(k)} e_0$ we then obtain an isomorphism

$$(M^k/M^{2k})e_0 \approx (\Omega/M^k)e_0 + W \quad \text{(direct)}$$

where $W$ is a left $(\Omega/M^{2k})$-module whose structure does not concern us. Thus we have the exact sequence

$$0 \longrightarrow (\Omega/M^k)e_0 + W \longrightarrow (\Omega/M^{2k})e_0 \longrightarrow (\Omega/M^k)e_0 \longrightarrow 0.$$

Now, suppose $r = \text{l.dim}_{\Omega/M^{2k}}(\Omega/M^k)e_0 < \infty$ and let

$$0 \longrightarrow X_r \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow (\Omega/M^k)e_0 \longrightarrow 0$$

be a shortest $(\Omega/M^{2k})$-projective resolution of $(\Omega/M^k)e_0$; we have $r > 0$ since $(\Omega/M^k)e_0$ is not $(\Omega/M^{2k})$-projective. We have then an exact sequence

$$0 \longrightarrow X_r + Y_r \longrightarrow \ldots \longrightarrow X_0 + Y_0 \longrightarrow (\Omega/M^{2k})e_0 \longrightarrow (\Omega/M^k)e_0 \longrightarrow 0,$$

where sums are all direct and where $0 \longrightarrow Y_r \longrightarrow \ldots \longrightarrow Y_0 \longrightarrow W \longrightarrow 0$ is an exact sequence such that all $Y_\mu$ except $Y_r$, perhaps, are $(\Omega/M^{2k})$-projective. Since $\mathrm{l.dim}_{\Omega/M^{2k}}(\Omega/M^k)e_0 = r$, then necessarily the image of $X_r + Y_r$ in $X_{r-1} + Y_{r-1}$ is a direct summand. Hence the image of $X_r$ in $X_{r-1}$ is a direct summand. This in turn implies that $(\Omega/M^k)e_0$ has a projective resolution, with respect to $\Omega/M^{2k}$, of length $r-1$, contradicting the above assumption. Hence $\mathrm{l.dim}_{\Omega/M^{2k}}(\Omega/M^k)e_0 = \infty$ and $\mathrm{gl.dim}(\Omega/M^{2k}) = \infty$.

Here we may assume that $k$ is arbitrarily large, since otherwise we have simply to repeat the given connected sequence of idempotents sufficiently many times. So this proves our proposition.

## §4. Algebras

Let $\Lambda$ be a semi-primary algebra over a field $K$, let $N$ be the radical of $\Lambda$ and let $\Gamma = \Lambda/N$. Assume $\dim \Gamma = 0$, or equivalently that $\Gamma \otimes_K \Gamma^*$ is semi-simple. Then (Rosenberg-Zelinsky [8]) necessarily $(\Gamma : K) < \infty$ and $\Gamma$ is separable. It follows readily that $\Lambda$ admits a splitting $\Lambda = \Gamma + A + N^2$, $A \approx N/N^2$. It is further known (Eilenberg [1]) that $\dim \Lambda = \mathrm{gl.dim}\,\Lambda$. Similarly if $\mathfrak{a}$ is any two-sided ideal in $\Lambda$ then $\dim(\Lambda/\mathfrak{a}) = \mathrm{gl.dim}(\Lambda/\mathfrak{a})$.

The same comments apply to the algebra $\Omega$ constructed in §2, provided $M$ is nilpotent. The results of §3 may now be restated with "dim" replacing "gl.dim".

If we assume that $(\Lambda : K) < \infty$ then clearly $\Lambda$ is semi-primary and the assumption $\dim \Gamma = 0$ (i.e. the separability of $\Gamma$) follows automatically from $\dim(\Lambda/N^2) < \infty$ (Ikeda-Nagao-Nakayama [7], Eilenberg [1]). It is further clear that in the splitting $\Lambda = \Gamma + A + N^2$ of $\Lambda$ we have $(A : K) < \infty$. Since $\Omega = \Gamma + M$ we deduce that $(\Omega : K) < \infty$. Thus we have

THEOREM 9. *Let $\Lambda$ be an algebra over a field $K$ with $(\Lambda : K) < \infty$. Let $N$ be the radical of $\Lambda$. Suppose*

$$\dim(\Lambda/N^2) = n < \infty.$$

*Then there exist an algebra $\Omega$ over $K$ with radical $M$ and an algebra epimorphism $\varphi : \Omega \longrightarrow \Lambda$ such that $(\Omega : K) < \infty$, $\varphi^{-1}(N^2) = M^2$ and*

$$\dim \Omega \leqq 1.$$

*The pair $(\Omega, \varphi)$ is determined uniquely up to an isomorphism. $M^{n+1} = 0$ and $N^{n+1} = 0$. If $\mathfrak{a}$ is a two-sided ideal of $\Lambda$ then $\dim(\Lambda/\mathfrak{a}) < \infty$, and indeed $\leqq n$ if*

$a \subset N^2$. *If $l$ is the number of simple components in $\Gamma = \Lambda/N$ then $n < l$.*

We close our note with a remark on Cartan matrices. Starting again with a semi-primary ring $\Lambda$, with radical $N$, let $e_1, \ldots, e_l$ be a maximal set of non-isomorphic primitive idempotents in $\Lambda$. For each pair $(i, j)$ of indices $1, 2, \ldots, l$ we choose a non-negative real number $\beta(i, j)$ so that

$$\beta(i, j) = 0 \text{ or } > 0 \text{ according as } e_i N e_j = 0 \text{ or } \neq 0,$$

and otherwise arbitrarily. Let us call the matrix $C(\Lambda) = I + (\beta(i, j))$ a generalized Cartan matrix of $\Lambda$, where $I$ is the identity matrix of degree $l$.

PROPOSITION 10. *The matrix $(C(\Lambda) - I)^{n+1} = (\beta(i, j))^{n+1}$ vanishes if and only if* $\operatorname{gl.dim}(\Lambda/N^2) \leqq n$.

*Proof.* Since the entries $\beta(i, j)$ of $C(\Lambda) - I$ are all non-negative, that $(C(\Lambda) - I)^{n+1} \neq 0$ is equivalent to the existence of $n+1$ pairs $(i_0, j_0), \ldots, (i_n, j_n)$ such that

$$\text{(i)} \qquad j_\nu = i_{\nu+1} (\nu = 0, \ldots, n-1), \ \beta(i_\nu, j_\nu) \neq 0 \quad (\nu = 0, \ldots, n).$$

By the definition of $\beta(i, j)$, this is equivalent to

$$\text{(ii)} \qquad j_\nu = i_{\nu+1} (\nu = 0, \ldots, n-1), \ e_{i_\nu} N e_{j_\nu} \neq 0 \quad (\nu = 0, \ldots, n).$$

Now, if $eN^t f \neq 0$ but $eN^{t+1} f = 0$, with a pair of primitive idempotents $e, f$ in $\Lambda$, take $t-1$ primitive idempotents $g_1, \ldots, g_{t-1}$ such that $N g_1 N g_2 \ldots N g_{t-1} N f \neq 0$. Since $eN^{t+1} f = 0$, it follows that $g_\mu N g_{\mu+1} \not\subseteq N^2$ for $\mu = 0, \ldots, t-1$, where we put $g_0 = e$, $g_t = f$. This observation shows that the existence of $n+1$ pairs $(i_\nu, j_\nu)$ satisfying (ii) is equivalent to the existence of a connected sequence of length at least $n+1$ of primitive idempotents in $\Gamma = \Lambda/N$, with respect to $\Lambda/N^2$, in the sense of §1. This is in turn equivalent to $\operatorname{gl.dim}(\Lambda/N^2) \geqq n+1$ by Proposition 2.

In case of an algebra $\Lambda$ over a field $K$ with $(\Lambda : K) < \infty$, the ordinary Cartan matrix of $\Lambda$ is clearly a generalized Cartan matrix in the above sense.

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