J. Austral. Math. Soc. (*Series A*) **68** (2000), 261–277

LOOMIS-SIKORSKI THEOREM FOR σ-COMPLETE MV-ALGEBRAS AND ℓ-GROUPS

ANATOLIJ DVUREČENSKIJ

(Received 22 February 1999; revised 27 August 1999)

Communicated by B. A. Davey

Abstract

We show that every σ-complete MV-algebra is an MV-σ-homomorphic image of some σ-complete MV-algebra of fuzzy sets, called a tribe, which is a system of fuzzy sets of a crisp set Ω containing 1_Ω and closed under fuzzy complementation and formation of $\min\{\sum_n f_n, 1\}$. Since a tribe is a direct generalization of a σ-algebra of crisp subsets, the representation theorem is an analogue of the Loomis-Sikorski theorem for MV-algebras. In addition, this result will be extended also for Dedekind σ-complete ℓ-groups with strong unit.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 03B50, 03G12.
Keywords and phrases: MV-algebra, maximal ideal, state, σ-complete MV-algebra, basically disconnected space, tribe, Loomis-Sikorski theorem, ℓ-group, g-tribe.

1. Introduction

The concept of MV-algebras was introduced by Chang [Cha] in 1958 as a non-latticeal generalization of Boolean algebras and they arise from many valued logic of Łukasiewicz in the same manner as Boolean algebras arise from classical two-valued logic.

σ-complete MV-algebras are MV-algebras which are σ-complete lattices. Such MV-algebras are always semisimple algebras, and they are exactly those for which there exists an MV-isomorphism with a *Bold algebra*, that is, with an algebra of fuzzy sets of a crisp set Ω which contains 1_Ω, and which is closed under the fuzzy complementation and formation of $\min\{f + g, 1\}$. Belluce [Bel] showed that every semisimple MV-algebra M can be always represented as a Bold algebra of continuous

https://doi.org/10.1017/S1446788700001993 Published online by Cambridge University Press

fuzzy sets on the compact Hausdorff space of all maximal ideals of M. And this is an analogue of Stone's representation theorem for Boolean algebras.

Tribes are Bold algebras of fuzzy sets which are roughly speaking closed under pointwise suprema, and they are a direct generalization of a σ-algebra of crisp subsets.

The famous Loomis-Sikorski theorem [Sik] plays a crucial rôle for analysis of Boolean σ-algebras, and it says that every Boolean σ-algebra is a σ-homomorphic image of a σ-algebra of subsets.

In the present paper, we show that every σ-complete MV-algebra is an MV-σ-homomorphic image of a tribe, which gives a generalization of the Loomis-Sikorski theorem for σ-complete MV-algebras. In addition, we extended this result also for Dedekind σ-complete ℓ-groups with strong unit.

2. MV-algebras

An MV-*algebra* is a non-empty set M with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : M \times M \to M$ and with a unary operation $^* : M \to M$ such that, for all $a, b, c \in M$, we have

(MVi) $a \oplus b = b \oplus a$ (commutativity);

(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);

(MViii) $a \oplus 0 = a$;

(MViv) $a \oplus 1 = 1$;

(MVv) $(a^*)^* = a$;

(MVvi) $a \oplus a^* = 1$;

(MVvii) $0^* = 1$;

(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We define the following binary operations $\odot, \vee, \wedge$ as follows:

$$a \odot b := (a^* \oplus b^*)^*, \quad a, b \in M, \tag{2.1}$$

$$a \vee b := (a^* \oplus b)^* \oplus b, \qquad a \wedge b := (a^* \vee b^*)^*, \quad a, b \in M.$$

Then $(M; \odot, 1)$ is a semigroup written 'multiplicatively' with the neutral element 1.

If, for $a, b \in M$, we define

$$a \leq b \Leftrightarrow a = a \wedge b,$$

then $\leq$ is a partial order on M, and $(M; \vee, \wedge, 0, 1)$ is a distributive lattice with the least and greatest elements 0 and 1, respectively, [Cha]. We recall that $a \leq b$ if and only if $b \oplus a^* = 1$.

Without loss of generality we write for MV-algebras $M = (M; \oplus, \odot, ^*, 0, 1)$, where $\odot$ is defined by (2.1).

Similarly, we can define a (total) binary operation $*$ on M by

$$a * b := a \odot b^*, \quad a, b \in M.$$

Then

$$a * b = a * (a \wedge b),$$

and $1 * a = a^*$ for any $a \in M$.

Let $(G; +, 0, \leq)$ be an Abelian ℓ-group with a strong unit u, that is, given $v \in G$, there is an integer $n \geq 1$ such that $-nu \leq v \leq nu$.

Define

$$\Gamma(G, u) = \{g \in G : 0 \leq g \leq u\} \tag{2.2}$$

and set for all $g_1, g_2, g \in \Gamma(G, u)$

$$\begin{aligned} g_1 \oplus g_2 &= (g_1 + g_2) \wedge u, \\ g_1 \odot g_2 &= (g_1 + g_2 - u) \vee 0, \\ g^* &= u - g. \end{aligned}$$

Then $(\Gamma(G, u); \oplus, \odot, ^*, 0, u)$ is an MV-algebra. The famous Mundici result [Mun] says that given an MV-algebra M there is an Abelian ℓ-group G with a strong unit u such that M is isomorphic with some $\Gamma(G, u)$, in addition, Γ defines a categorical equivalence between the category of unital ℓ-groups and the category of MV-algebras. We denote any representation ℓ-group $(G; +, 0, \leq)$ with a strong unit u of an MV-algebra as a unital ℓ-group $G = (G; +, 0, \leq, u)$, or simply $G = (G, u)$.

The case $M = [0, 1]$, the real interval of the ℓ-group $\mathbb{R}$, that is, $[0, 1] = \Gamma(\mathbb{R}, 1)$, is of a particular importance for the study of MV-algebras.

Given an MV-algebra M, we can introduce a partial binary operation $+$ in the following way: $a + b$ is defined if and only if $a \leq b^*$, and in this case we put

$$a + b := a \oplus b.$$

It is easy to see that $a + 0 = 0 + a = a$ for any $a \in M$, and $+$ is commutative, that is, if $a + b$ is defined in M, so is $b + a$, and $a + b = b + a$; $+$ is associative, that is, if $(a + b)$, $(a + b) + c$ are in M so are defined $b + c$ and $a + (b + c)$, and $(a + b) + c = a + (b + c)$. Identifying the MV-algebra M with $\Gamma(G, u)$ via (2.2), we can see that our partial operation $+$ coincides with the group addition $+$.

In addition, we can define a subtraction: $a - b$ is defined in M if and only if $b \leq a$, and $a - b = c$ whenever $b + c = a$.

A nonvoid subset I of M is said to be an *ideal* of M if

(i) $x, y \in I$ imply $x \oplus y \in I$;
(ii) $x \in I, y \leq x$ imply $y \in I$.

A proper ideal A of M is said to be (i) *maximal* if there is no proper ideal of M containing A as a proper subset, and (ii) *prime* if $a \wedge b \in A$ entails $a \in A$ or $b \in A$. Every maximal ideal is prime. Let $\mathscr{M}(M)$ denote the set of all maximal ideals of M. Then $\mathscr{M}(M) \neq \emptyset$. Denote by

$$\mathrm{Rad}(M) := \bigcap\{A : A \in \mathscr{M}(M)\},$$

and we call Rad(M) the *radical* of M. A nonzero element a of M is said to be *infinitesimal* if na exists in M for any integer $n \geq 1$. The set of all infinitesimals in M is denoted by Infinit(M). Then according to [CDM, Proposition 3.6.4], we have

$$\mathrm{Rad}(M) = \mathrm{Infinit}(M) \cup \{0\}.$$

Let a be an element in M and n an integer. We define $na := a_1 + \cdots + a_n$, where $a_1 = \cdots = a_n = a$. An MV-algebra M is said to be (i) *semisimple* if Rad$(M) = \{0\}$; (ii) *Archimedean* if existence of na for any $n \geq 1$ implies $a = 0$; and (iii) *Archimedean in the sense of Belluce* [Bel] if, for each $a, b \in M$, if $n \odot a = a \oplus \cdots \oplus a \leq b$ for all $n \geq 1$ then $a \odot b = a$.

We have the following characterization of the Archimedeanicity of M [DvGr].

PROPOSITION 2.1. *An* MV*-algebra M is Archimedean if and only if its representation ℓ-group (G, u) is an Archimedean ℓ-group.*

According to Belluce [Bel], we say that a subset $\mathscr{F} \subseteq [0, 1]^\Omega$, where $\Omega \neq \emptyset$, is a *Bold algebra* if

(i) $0_\Omega \in \mathscr{F}$;
(ii) $f \in \mathscr{F}$ entails $1_\Omega - f \in \mathscr{F}$;
(iii) $f, g \in \mathscr{F}$ imply $f \oplus g \in \mathscr{F}$, where

$$(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), 1\}, \quad \omega \in \Omega. \tag{2.3}$$

Then $\mathscr{F}$ with $f^* := 1_\Omega - f$ and with 0_Ω and 1_Ω is an MV-algebra which is semisimple and Archimedean. In particular, if X is a topological space, by $C(X)$ we denote the set of all continuous fuzzy subsets on X, and any Bold algebra of all continuous fuzzy subsets is of special interest (see (viii) in Theorem 2.3).

An MV-*homomorphism* of two MV-algebras M_1 and M_2 is any mapping $h : M_1 \to M_2$ preserving 0, 1, $\oplus$, and *.

A *state* on MV-algebra M is a mapping $m : M \to [0, 1]$ such that $m(1) = 1$, and $m(a + b) = m(a) + m(b)$, whenever $a + b$ is defined in M. Denote by $\mathscr{S}(M)$ the set of all states on M, then $\mathscr{S}(M) \neq \emptyset$.

A state m on M is said to be *σ-additive* if $a_n \nearrow a$ entails $m(a_n) \nearrow m(a)$.

A *state-homomorphism* is a mapping $m : M \to [0, 1]$ such that $m(1) = 1$, and $m(a \oplus b) = \min\{1, m(a) + m(b)\}$, $a, b \in M$. Any state-homomorphism is a state, but the converse does not hold, in general. There is a one-to-one correspondence between state-homomorphisms and maximal ideals [Mun1, Theorem 2.4], [Goo, Theorem 12.18].

THEOREM 2.1. (1) *A state m on M is a state-homomorphism if and only if* $\mathrm{Ker}_m := \{a \in M : m(a) = 0\}$ *is a maximal ideal.*

(2) *Given a maximal ideal A of M, the mapping $x \mapsto x/A$ is a state-homomorphism.*

(3) *The mapping $m \mapsto \mathrm{Ker}_m$ is a one-to-one correspondence from the set of all state-homomorphisms on the set of all maximal ideals of M.*

(4) *A state m on M is an extremal point (an extremal state) of the set $\mathscr{S}(M)$ if and only if m is a state-homomorphism.*

Denote by $\mathrm{Ext}(\mathscr{S}(M))$ the set of all extremal states (state-homomorphisms) on M. Then [Mun1, Theorem 2.5] $\mathrm{Ext}(\mathscr{S}(M)) \neq \emptyset$ and it is a compact Hausdorff space with respect to the weak topology of states (that is, $m_\alpha \to m$ if and only if $m_\alpha(a) \to m(a)$ for any $a \in M$), and any state m on M is in the closure of the convex hull of $\mathrm{Ext}(\mathscr{S}(M))$.

We introduce a topology $\mathscr{T}_{\mathscr{M}}$ on the set $\mathscr{M}(M)$ of all maximal ideals of M. Given an ideal I of M, let

$$O(I) := \{A \in \mathscr{M}(M) : A \not\supseteq I\},$$

and let $\mathscr{T}_{\mathscr{M}}$ be the collection of all subsets of the above form. It is possible to show that $\mathscr{T}_{\mathscr{M}}$ gives a compact Hausdorff topological space. Moreover, any closed subset of $\mathscr{M}(X)$ is of the form

$$C(I) = \{A \in \mathscr{M}(M) : A \supseteq I\},$$

where I is any ideal of M. It is worth recalling that $\mathscr{M}(M)$ and $\mathrm{Ext}(\mathscr{S}(M))$ are homeomorphic spaces; the homeomorphism is given by $m \mapsto \mathrm{Ker}_m$, $m \in \mathrm{Ext}(\mathscr{S}(M))$, [Goo, Theorem 15.32].

A nonvoid subset $\mathscr{S}$ of $\mathscr{S}(M)$ is said to be *order-determining* if $m(a) \leq m(b)$ for any $m \in \mathscr{S}$ imply $a \leq b$.

The following characterizations of semisimple MV-algebras can be found in [DvGr, Bel, Mun1].

THEOREM 2.2. *Let M be an* MV-*algebra. The following statements are equivalent.*

(i) *M is semisimple;*

https://doi.org/10.1017/S1446788700001993 Published online by Cambridge University Press

(ii) *M is Archimedean;*
(iii) *M is Archimedean in the sense of Belluce;*
(iv) *There exists an order-determining system of state-homomorphisms on M;*
(v) *There exists an order-determining system of states on M;*
(vi) *M is isomorphic to some Bold algebra of fuzzy subsets of some $\Omega \neq \emptyset$;*
(vii) *M is isomorphic to some Bold algebra of continuous functions defined on some compact Hausdorff spaces X;*
(viii) *M is isomorphic to some Bold algebra of $C(\mathscr{M}(M))$, the set of all continuous fuzzy subsets defined on $\mathscr{M}(M)$.*

Let M be semisimple. The property (vi) means the following: Let $a \in M$ and $A \in \mathscr{M}(M)$. Then $a \mapsto \bar{a}$, where $\bar{a} \in [0,1]^{\mathscr{M}(M)}$ is defined as follows

$$\bar{a}(A) := a/A,\ A \in \mathscr{M}(M), \tag{2.4}$$

is an MV-isomorphism between M and $\{\bar{a} : a \in M\}$. We recall that if A is a maximal ideal of M, then using Hölder's theorem, [Bir, Theorem XIII.12] a/A, $a \in M$, can be represented as a number in [0, 1].

Or, equivalently, since there is a one-to-one correspondence between $\operatorname{Ext}(\mathscr{S}(M))$ and $\mathscr{M}(M)$ given by the homeomorphism $m \mapsto \operatorname{Ker}_m$, we have the embedding $a \mapsto \hat{a}$, of M into $[0,1]^M$, where $\hat{a}$ is defined as follows

$$\hat{a}(m) := m(a),\ m \in \operatorname{Ext}(\mathscr{S}(M)). \tag{2.5}$$

3. σ-complete MV-algebras and tribes

The following forms of distributive laws are known.

PROPOSITION 3.1. *Let $\bigvee_i a_i$ be defined in M. Then for any $b \in M$, we have*

$$b \wedge \left(\bigvee_i a_i\right) = \bigvee_i (b \wedge a_i),$$

$$\left(\bigvee_i a_i\right) \odot b^* = \bigvee_i (a_i \odot b^*),$$

$$b \odot \left(\bigvee_i a_i\right)^* = \bigwedge_i (b \odot a_i^*).$$

The equalities hold in the sense that the expressions on the right-hand side exist, and are equal to the left-hand ones.

https://doi.org/10.1017/S1446788700001993 Published online by Cambridge University Press

We recall that an ℓ-group G is *Dedekind σ-complete* if, for any sequence $\{g_n\}_n$ of elements of G with an upper bound in G, $\bigvee_n g_n \in G$; similarly for Dedekind complete.

We say that an MV-algebra M is *σ-complete* (*complete*) if M is a σ-complete (complete) lattice. M is σ-complete (complete) if and only if G is Dedekind σ-complete (complete) ℓ-group, where $M = \Gamma(G, u)$, [Jak], [Goo, Proposition 16.9]. An MV-*σ-homomorphism* is any MV-homomorphism of σ-complete MV-algebras preserving also countable joins (and meets).

We give another proof of the following statement (see for example [Cig, Lemma 2.1]).

PROPOSITION 3.2. *Any σ-complete* MV-*algebra is Archimedean.*

PROOF. Assume that na is defined in M for any integer $n \geq 1$. For any integer k, we have $(k+1)a \leq \bigvee_{n=1}^{\infty} na$ so that $ka \leq \left(\bigvee_{n=1}^{\infty} na\right) - a$ which entails $\bigvee_{k=1}^{\infty} ka \leq \left(\bigvee_{n=1}^{\infty} na\right) - a$ whence $a = 0$. □

The following notion is a direct generalization of a σ-algebra of crisp subsets.

A *tribe* of fuzzy sets on a set $\Omega \neq \emptyset$ is a nonvoid system $\mathscr{F} \subseteq [0,1]^{\Omega}$ such that

(i) $1_\Omega \in \mathscr{F}$;
(ii) if $a \in \mathscr{F}$, then $1 - a \in \mathscr{F}$;
(iii) if $\{a_n\}_{n=1}^{\infty}$ is a sequence of elements of $\mathscr{F}$, then

$$\min\left\{\sum_{n=1}^{\infty} a_n, 1\right\} \in \mathscr{F}.$$

(We recall that all above operations with fuzzy sets are defined pointwisely on Ω.)

By [RiNe, Proposition 3.13], if $\mathscr{F}$ is a tribe and if $a, b \in \mathscr{F}$, then (i) $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, (ii) $b - a \in \mathscr{F}$ if $a \leq b$, that is, if $a(\omega) \leq b(\omega)$ for all $\omega \in \Omega$, (iii) if $a_n \in \mathscr{F}$, and $a_n \nearrow a$ (pointwisely), then $a = \lim_n a_n \in \mathscr{F}$. It is simple to verify that $\mathscr{F}$ is a σ-complete MV-algebra of fuzzy sets, where the partial order is determined by the set-theoretical ordering, with the least and greatest elements 0_Ω and 1_Ω, respectively. Moreover, the system $\{s_\omega : \omega \in \Omega\}$, where $s_\omega(f) := f(\omega)$, $f \in \mathscr{F}$, is an order-determining system of σ-additive states on $\mathscr{F}$.

Let $\mathscr{F}$ be a family of fuzzy subsets of Ω. We define $\mathscr{F}_0 := \mathscr{F} \cup \{0_\Omega, 1_\Omega\}$, and for any ordinal number $\alpha > 0$, we define

$$\mathscr{F}_\alpha = \left(\bigcup_{\beta<\alpha} \mathscr{F}_\beta\right)^*,$$

where the family $\mathscr{C}^*$ denotes the set of all functions of the form $\min\left\{\sum_{n=1}^{\infty} f_n, 1\right\}$, where either f_n or $1 - f_n \in \mathscr{C}$ for any $n \geq 1$. Then

(i) $\mathcal{F} \subseteq \mathcal{F}_\beta \subseteq \mathcal{F}_\alpha \subseteq \mathcal{T}(\mathcal{F})$ for $0 < \beta < \alpha$, where $\mathcal{T}(\mathcal{F})$ is a tribe generated by $\mathcal{F}$;

(ii) $\mathcal{T}(\mathcal{F}) = \bigcup_{\beta<\omega_1} \mathcal{F}_\beta$, where ω_1 is the first uncountable ordinal number.

The following result characterizes the tribe generated by $C(X)$, the space of all continuous fuzzy functions on X, where X is a compact Hausdorff space. We recall that by $\mathcal{B}(X)$ we mean the Baire σ-algebra generated by compact G_δ sets on X, or equivalently, by $\{f^{-1}([a,\infty)) : f \in C(X), a \in \mathbb{R}\}$.

PROPOSITION 3.3. *Let X be a compact Hausdorff space. Let $\mathcal{F}(X)$ be the tribe generated by $C(X)$, and let $\mathcal{M}(X)$ be the set of all Baire measurable fuzzy sets on X. Then $\mathcal{F}(X) = \mathcal{M}(X)$.*

PROOF. It is obvious that $\mathcal{M}(X)$ is a tribe containing $C(X)$ hence, $\mathcal{F}(X) \subseteq \mathcal{M}(X)$. Let $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{S}_{\mathcal{M}}$ be the systems of all crisp subsets $A \subseteq X$ such that $\chi_A \in \mathcal{F}(X)$ and $\chi_A \in \mathcal{M}(X)$, respectively. Then $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{S}_{\mathcal{M}}$ are σ-algebras of subsets of X, and $\mathcal{S}_{\mathcal{F}} \subseteq \mathcal{S}_{\mathcal{M}} \subseteq \mathcal{B}(X)$.

Since each $f \in \mathcal{F}(X)$ is $\mathcal{S}_{\mathcal{F}}$-measurable, then each $f \in C(X)$ is $\mathcal{S}_{\mathcal{F}}$-measurable, so that $\mathcal{B}(X) \subseteq \mathcal{S}_{\mathcal{F}}$.

On the other hand, because $\mathcal{F}(X)$ contains $C(X)$, $\mathcal{F}(X)$ contains all constant functions taking values in the interval [0, 1]. By [RiNe, Theorem 8.1.4], this is a necessary and sufficient condition in order $\mathcal{F}(X)$ consists of all $\mathcal{S}_{\mathcal{F}}$-measurable fuzzy subsets of X. Consequently, $\mathcal{F}(X) = \mathcal{M}(X)$. □

4. Loomis-Sikorski Theorem

In the present section, we give a generalization of the Loomis-Sikorski theorem for σ-complete MV-algebras. Before it we present partial results.

For any $a \in M$, we put

$$M(a) := \{A \in \mathcal{M}(M) : a \notin A\}. \tag{4.1}$$

Then $\{M(a) : a \in M\}$ is a base of $\mathcal{T}_{\mathcal{M}}$ of the compact Hausdorff space $\mathcal{M}(M)$, and for $a, b \in M$,

(i) $M(0) = \emptyset$;

(ii) $M(a) \subseteq M(b)$ whenever $a \leq b$;

(iii) $M(a \wedge b) = M(a) \cap M(b)$, $M(a \vee b) = M(a) \cup M(b)$.

An element $a \in M$ is *idempotent* if and only if $a \vee a^* = 1$. It is possible to show that a is idempotent if and only if $a \oplus a = a$ if and only if $a \odot a = a$ if and only if $a \wedge a^* = 0$. Denote by $B(M)$ the set of all idempotent elements of M.

PROPOSITION 4.1. *$B(M)$ is a Boolean algebra with the least and greatest elements 0 and 1 and with unary operation * taken from M. In addition, if $\bigvee_i a_i$ of elements $\{a_i\}$ of $B(M)$ exists in M, then $\bigvee_i a_i \in B(M)$. Consequently, if M is σ-complete or complete, so is $B(M)$.*

PROOF. Let $\{a_i\}$ be a family of elements of $B(M)$ such $a = \bigvee_i a_i \in M$. Then

$$a \wedge a^* = \bigvee_i (a_i \wedge a^*) \leq \bigvee_i (a_i \wedge a_i^*) = 0. \qquad \square$$

PROPOSITION 4.2. *For any $a \in M$,*

$$M(a)^c \subseteq M(a^*).$$

If a is idempotent, then $M(a)^c = M(a^*)$. In general, if $a \leq b$, then

$$M(b) \setminus M(a) \subseteq M(b * a).$$

If a and b are idempotent, $a \leq b$, then $b * a^* = b \wedge a^*$ and $M(b) \setminus M(a) = M(b * a)$.

If M is semisimple, then a is idempotent if and only if $M(a^*) = M(a)^c$.

PROOF. Let $a \leq b$. Take $A \in M(b) \setminus M(a)$. Then $b \notin A$ and $a \in A$. We claim that $b * a \notin A$. If not, then $b * a \in A$ so that $b = a + (b * a) \in A$ which is a contradiction.

Let a and b be two idempotent elements of M and take $A \in M(b * a)$. Then $b * a \notin A$ and $b \notin A$. Since A is a prime ideal of M, then $0 = (b * a) \wedge a \in A$ entails $a \in A$ so that $A \in M(b) \setminus M(a)$. The first condition follows from the above.

Let M be semisimple and $M(a^*) = M(a)^c$. Then for any maximal ideal I of M either $a \in I$ or $a^* \in I$ which entails $a \wedge a^* \in I$, and the semisimplicity of M gives $a \wedge a^* = 0$. $\square$

REMARK 4.1. We recall that if $M(a)$ is clopen, then not necessary $M(a^*) = M(a)^c$. Indeed, take $M = [0, 1]$. Then $\mathscr{M}(M) = \{\{0\}\}$, and for any nonzero $a \in M$, $M(a) = \mathscr{M}(M)$ is clopen, and $M(a)^c = \emptyset$.

Similarly, if M is not semisimple, then the equality $M(a^*) = M(a)^c$ does not entail that a is idempotent. Indeed, take the Chang MV-algebra [Cha] $M = \{0, 1, 2, \dots, n, \dots, \dot{n}, \dots, \dot{2}, \dot{1}, \dot{0}\}$. Then $\mathscr{M}(M) = \{\{0, 1, \dots\}\}$, and $M(1) = \emptyset = M(\dot{1})^c$ and 1 is not idempotent.

A topological space Ω is said to be

(i) *totally disconnected* if every two different points are separated by a clopen subset of Ω;

(ii) *basically disconnected* if the closure of every open F_σ subset of Ω is open;

(iii) *extremally disconnected* if the closure of every open set is open.

PROPOSITION 4.3. *If M is a σ-complete or complete* MV*-algebra, then the space $\mathscr{M}(M)$ is basically disconnected or extremally disconnected, respectively.*

PROOF. Since M is σ-complete (complete) if and only if its representation ℓ-group G is Dedekind σ-complete (complete), according to [Goo, Theorem 8.14], the space $\mathrm{Ext}(\mathscr{S}(M))$ is homeomorphic with the set $\mathscr{M}(B(M))$ of all maximal ideals of the Boolean algebra $B(M)$. Hence, by [Sik, Theorem 22.4], a Boolean algebra is σ-complete (complete) if and only if $\mathscr{M}(B(M))$ is basically disconnected (extremally disconnected). □

PROPOSITION 4.4. *Let M be a semisimple* MV*-algebra. If $a = \bigvee_t a_t \in M$, then*

$$M(a) \setminus \bigcup_t M(a_t),$$

where $M(a)$ is defined by (3.1), *is a nowhere dense subset of $\mathscr{M}(M)$.*

PROOF. Let $a = \bigvee_t a_t$ and suppose $M(a) \setminus \bigcup_t M(a_t)$ is not nowhere dense. Since $\{M(a) : a \in M\}$ is a base of the topological space $\mathscr{M}(M)$, there exists a nonzero element $b \in M$ such that $\emptyset \neq M(b) \subseteq M(a) \setminus \bigcup_t M(a_t)$. Due to $M(b) = M(b) \cap M(a) = M(b \wedge a)$, we take $b_0 := b \wedge a$ which is a nonzero element of M. Then $M(b_0) \cap M(a_t) = \emptyset$ for any t, so that $M(b_0 \wedge a_t) = \emptyset$ and the semisimplicity of M yields $b_0 \wedge a_t = 0$ for any t.

Then

$$b_0 = b_0 \wedge a = b_0 \wedge \bigvee_t a_t = \bigvee_t (b_0 \wedge a_t) = 0,$$

which gives $M(b) = \emptyset$, a contradiction, so that our assumption was false, and consequently, $M(a) \setminus \bigcup_t M(a_t)$ is a nowhere dense set. □

We recall that the converse to Proposition 4.4 is not true for any semisimple MV-algebra. Take $M = [0, 1]$, then $M(0.3) \setminus (M(0.1) \cup M(0.2)) = \emptyset$ but $0.3 \neq 0.1 \vee 0.2$.

PROPOSITION 4.5. *Let M be a semisimple* MV*-algebra and let $a_t \leq a$ for any t. If $\bigcap_t M(a * a_t)$ is a nowhere dense subset of $\mathscr{M}(M)$, then $a = \bigvee_t a_t$.*

PROOF. It is clear that in order to prove $a = \bigvee_t a_t$ it is sufficient to verify that $a_t \leq b \leq a$ for any t implies $b = a$.

So let $\bigcap_t M(a * a_t)$ be nowhere dense, and let $b \neq a$ for some $b \geq a_t$, $b \leq a$. Then $a * b \neq 0$ and $M(b * a)$ is a nonempty open subsets of $\mathscr{M}(M)$. By assumptions, there exists a nonzero open subset $O \subseteq M(b * a)$ such that $O \cap \bigcap_t M(a * a_t) = \emptyset$. Consequently, there is a nonzero element $c \in M$ such that $M(c) \subseteq O$. Hence, for any $I \in M(c) \subseteq M(a * b)$, we have $c \notin I$, $a * b \notin I$ and $I \notin \bigcap_t M(a * a_t)$. This entails that there is an index t such that $a * a_t \in I$. Since $a_t \leq b$, we have $a * b \leq a * a_t \in I$ which implies $a * b \in I$, and this is a contradiction with $a * b \notin I$. Finally, our assumption $b < a$ was false, and whence $b = a$, and $a = \bigvee_t a_t$. □

The converse statement holds, for example, if $\{a_t\}$ is a system of idempotents, Proposition 4.2 and Proposition 4.4. We recall that the converse to Proposition 4.5 is not true for any semisimple MV-algebra. Take $M = [0, 1]$, then $\mathscr{M}(M) = \{\{0\}\}$. If $a_n = 0.5 - 1/n$, then $\bigvee_n a_n = a := 0.5$, and $M(a * a_n) = M(1/n) = \mathscr{M}(M)$ for any $n \geq 1$, so that $\bigcap_n M(a * a_n)$ is not nowhere dense.

It is worth recalling that if M is a Boolean algebra, then $a = \bigvee_t a_t$ if and only if $\bigcap_t M(a * a_t) = M(a) \setminus \bigcup_t M(a_t)$ is a nowhere dense set, and this observation is a corner stone for the Loomis-Sikorski theorem. As we have seen, for semisimple MV-algebras, the analogical statement is not true, in general. Hence, for the validity of the Loomis-Sikorski theorem on σ-complete MV-algebras we have to develop below a more detailed analysis of σ-complete MV-algebras.

Let $a \in M$ and $k \geq 1$. We define

$$k \odot a = a_1 \oplus \cdots \oplus a_k,$$

where $a_1 = \cdots = a_k = a$.

PROPOSITION 4.6. *For any $a \in M$,*

$$M(a) = \bigcup_{k=1}^{\infty} M((k \odot a)^*)^c.$$

If, in addition, M is σ-complete, then the closure of any $M(a)$ is open.

PROOF. If $a = 0$, the statement is evident. Let now $a \neq 0$. Let $I \in M(a)$. Then $a \notin I$, and consequently, there is an integer $k \geq 1$ such that $(k \odot a)^* \in I$, and $I \in M((k \odot a)^*)^c$.

Conversely, let $I \in \bigcup_{k=1}^{\infty} M((k \odot a)^*)^c$. There exists an integer $k \geq 1$ such that $I \in M((k \odot a)^*)^c$. Hence, $(k \odot a)^* \in I$. Since I is a maximal ideal, we conclude that $a \notin I$.

The second part of the assertion follows from Proposition 4.3 and from the fact that $M(a)$ is an open F_σ set. □

REMARK 4.2. We recall that a Boolean algebra M is σ-complete (complete) if and only if $\mathscr{M}(M)$ is basically disconnected (extremally disconnected) [Sik, Theorem 22.4]. For MV-algebras such assertion is not true, in general. Indeed, take the

Chang MV-algebra [Cha] $M = \{0, 1, 2, \dots, n, \dots, \dot{n}, \dots, \dot{2}, \dot{1}, \dot{0}\}$. Then $\mathscr{M}(M) = \{\{0, 1, \dots\}\}$, and it is basically disconnected (extremally disconnected) but M is not σ-complete.

PROPOSITION 4.7. *An element a of a semisimple* MV*-algebra M is idempotent if and only if $\bar{a}$ is a characteristic function.*

PROOF. Let $a \in B(M)$. For any maximal ideal A of M, from $0 = a \wedge a^* \in A$ we conclude that either $a \in A$ or $a^* \in A$ which implies that $\bar{a}(A) \in \{0, 1\}$. Conversely, let $\bar{a}$ be a characteristic function. Then $\bar{a}(A) = a/A$ is either 0 or 1, or equivalently, either $a \in A$ or $a^* \in A$. Hence $a \wedge a^* \in A$ for any $A \in \mathscr{M}(M)$, and the semisimplicity of M entails $a \wedge a^* = 0$. □

For a bounded function $g : X \to \mathbb{R}$ on a topological space X we define

$$\tilde{g}(x) = \inf_{U \in \mathscr{N}(x)} \sup\{g(y) : y \in U\}, \tag{4.2}$$

where $\mathscr{N}(x)$ is the system of open neighbourhoods for $x \in X$. Then

(i) $g(x) \leq \tilde{g}(x)$ for any $x \in X$;
(ii) $\tilde{g}(x) = g(x)$ if g is continuous in x.

If $D(g)$ is the set of discontinuity points for g, then

$$\{x \in X : g(x) \neq \tilde{g}(x)\} \subseteq D(g) = \bigcup_n \{g^{-1}(R_n) - Int(g^{-1}(R_n)\}, \tag{4.3}$$

where $\{R_n\}$ is an open basis in $\mathbb{R}$.

Let f be a real-valued function on $\Omega \neq \emptyset$. We define

$$N(f) := \{\omega \in \Omega : |f(\omega)| > 0\}.$$

Suppose that $\mathscr{F}$ is a Bold algebra of fuzzy sets on Ω. Then for all $f, g \in \mathscr{F}$ we have

(i) $N(f \oplus g) = N(f) \cup N(g)$;
(ii) $N(f * g) = \{\omega \in \Omega : f(\omega > g(\omega)\}$;
(iii) $(f * g) \oplus (g * f) = (f * g) + (g * f)$;
(iv) $N((f * g) \oplus (g * f)) = N(f - g)$;
(v) $N(f) \subseteq N(g)$ if $f \leq g$.

We recall that if M is a semisimple algebra, then

$$M(a) = \{I \in \mathscr{M}(M) : a \notin I\} = N(\bar{a}).$$

Now we are ready to formulate the main result, the representation Loomis-Sikorski theorem for σ-complete MV-algebras.

https://doi.org/10.1017/S1446788700001993 Published online by Cambridge University Press

THEOREM 4.1 (Loomis-Sikorski Theorem). *For every* σ*-complete* MV*-algebra* M *there exist a tribe* $\mathcal{F}$ *of fuzzy sets and an* MV-σ*-homomorphism* h *from* $\mathcal{F}$ *onto* M.

PROOF. Let $\mathcal{F}$ be the tribe of fuzzy sets on $\mathrm{Ext}(\mathcal{S}(M))$ generated by the Bold algebra $\{\hat{a} : a \in M\}$. Consider by $\mathcal{F}'$ the class of all functions $f \in \mathcal{F}$ with the property that for some $b \in M$, $N((f * \hat{b}) \oplus (\hat{b} * f))$ is a meager set.

If b_1 and b_2 are two elements of M such that $N((f * \hat{b}_i) \oplus (\hat{b}_i * f))$ is a meager set for $i = 1, 2$. Then

$$N((\hat{b}_1 * \hat{b}_2) \oplus (\hat{b}_2 * \hat{b}_1)) = N(\hat{b}_1 - \hat{b}_2) \subseteq N(\hat{b}_1 - f) \cup N(f - \hat{b}_2)$$

is a meager set. Due to the Baire theorem saying that any non-empty open set of a compact Hausdorff space cannot be a meager set, we conclude that $\hat{b}_1 = \hat{b}_2$, that is, $b_1 = b_2$.

It is clear that $\mathcal{F}'$ is closed under the formation of complement $f \mapsto 1 - f$ and it is a Bold algebra containing $\{\hat{a} : a \in M\}$.

To show that $\mathcal{F}'$ is a tribe is necessary to verify that $\mathcal{F}'$ is closed under limits of non-decreasing sequences from $\mathcal{F}'$.

Let $\{f_n\}_n$ be a sequence of non-decreasing elements from $\mathcal{F}'$. Choose $b_n \in M$ such that $N(f_n - \hat{b}_n)$ is a meager set. Without loss of generality we can assume that $b_n \leq b_{n+1}$. Denote $f = \lim_n f_n$, $b = \bigvee_{n=1}^{\infty} b_n$, $b_0 = \lim_n \hat{b}_n$. Then $f, b_0 \in \mathcal{F}$ and $b \in M$. We have

$$N(f - \hat{b}) \subseteq N(f - b_0) \cup N(\hat{b} - b_0)$$

and $N(f - b_0) = \{m : f(m) < b_0(m)\} \cup \{m : b_0(m) < f(m)\}$.

If $m \in \{m : f(m) < b_0(m)\}$, then there is an integer $n \geq 1$ such that $f(m) < \hat{b}_n(m) \leq b_0(m)$. Hence $f_n(m) \leq f(m) < \hat{b}_n(m) \leq b_0(m)$ so that $m \in \{m : f_n(m) < \hat{b}_n(m)\}$.

Similarly, we can prove that if $m \in \{m : b_0(m) < f(m)\}$, then there is an integer $n \geq 1$ such that $m \in \{m : \hat{b}_n(m) < f_n(m)\}$.

The last two cases imply

$$N(f - b_0) \subseteq \bigcup_{n=1}^{\infty} N(\hat{b}_n - f_n)$$

which is a meager set.

Apply now (4.1) to the function b_0 to obtain $\tilde{b}_0$, that is

$$\tilde{b}_0(m) := \inf_{U \in \mathcal{N}(m)} \sup\{b_0(y) : y \in U\}.$$

Since $\mathrm{Ext}(\mathcal{S}(M))$ is basically disconnected, compact, Hausdorff, and $b_0^{-1}(\alpha, \infty) = \bigcup_n \hat{b}_n^{-1}(\alpha, \infty)$ for any $\alpha \in \mathbb{R}$, $b_0^{-1}(\alpha, \infty)$ is an open F_σ set, by [Goo, Lemma 9.1],

$\tilde{b}_0$ is continuous. Since b_0 is a point limit of a sequence of continuous functions, by [Kur, pp. 86, 405–6], $D(b_0) \supseteq N(\tilde{b}_0 - b_0)$ is a meager set,

$$b_0 \leq \tilde{b}_0 \leq \hat{b},$$

and $N(\hat{b} - b_0) \subseteq N(\hat{b} - \tilde{b}_0) \cup D(b_0)$.

Finally we show $\tilde{b}_0 = \hat{b}$. Define $\mathscr{C}(X)$, where $X = \mathrm{Ext}(\mathscr{S}(M))$, as the set of all continuous real valued functions defined on X. Since X is a basically disconnected, compact, Hausdorff space, $\mathscr{C}(X)$ is a Dedekind σ-complete ℓ-group with a strong unit 1_X, [Goo, Corollary 9.3]. Using Mundici's functor Γ, we see that the system of all continuous fuzzy functions on X, $C(X) = \Gamma(\mathscr{C}(X), 1_X)$ is a σ-complete MV-algebra. Applying [Goo, Lemma 9.12], we conclude the mapping $a \mapsto \hat{a}$, $a \in M$ preserves countable suprema and infima. Consequently, $\tilde{b}_0 = \hat{b}$, which proves that $\mathscr{F}'$ is a tribe, and whence, $\mathscr{F}' = \mathscr{F}$.

Due to definition of $\mathscr{F}'$, for any $f \in \mathscr{F}$ there is a unique element $h(f) := b \in M$ such that $N(f - \hat{b})$ is meager, which proves that $h : \mathscr{F} \to M$ is a surjective MV-σ-homomorphism in question. □

We recall that if M is an atomic σ-complete MV-algebra with the countable set of atoms, then M is σ-isomorphic with some tribe [DCR].

5. Loomis-Sikorski theorem for Dedekind σ-complete ℓ-groups

In the present section, we apply the Loomis-Sikorski theorem for MV-algebras to Dedekind σ-complete ℓ-groups.

A *g-tribe* is a nonvoid system $\mathscr{T}$ of bounded functions on $\Omega \neq \emptyset$ such that

(i) $0_\Omega, 1_\Omega \in \mathscr{T}$;

(ii) $f \pm g \in \mathscr{T}$ whenever $f, g \in \mathscr{T}$;

(iii) if $\{f_n\}_n$ is a sequence of elements from $\mathscr{T}$ for which there exists $f \in \mathscr{T}$ with $f_n \leq f$ (pointwisely), $n \geq 1$, then $\sup_n f_n \in \mathscr{T}$.

It is evident that $\mathscr{T}$ with respect to the pointwise ordering $f \leq g$ if and only if $f(\omega) \leq g(\omega)$ for every $\omega \in \Omega$ is a Dedekind σ-complete ℓ-group with strong unit 1_Ω. In addition, $\Gamma(\mathscr{T}, 1_\Omega) := \{f \in \mathscr{T} : 0_\Omega \leq f \leq 1_\Omega\}$ is a tribe of fuzzy functions.

Let (G, u) be an ℓ-group with strong unit u. A *state* on G is any mapping $s : G \to \mathbb{R}$ such that $s(g_1 + g_2) = s(g_1) + s(g_2)$, $s(u) = 1$, $s(g) \geq 0$ for $g \in G^+$.

If m is a state on $M = \Gamma(G, u)$, then it can be uniquely extended to a state $\hat{m}$ on G and conversely, any restriction of a state s on G to M gives a state on M.

Let $\mathscr{S}(G, u)$ denote the set of all states on (G, u). Then there exists a one-to-one correspondence $m \leftrightarrow \hat{m}$ between $\mathscr{S}(M)$ and $\mathscr{S}(G, u)$, and extremal points of $\mathscr{S}(M)$ are mapped onto extremal points of $\mathscr{S}(G, u)$ and vice-versa. In addition, this mapping

https://doi.org/10.1017/S1446788700001993 Published online by Cambridge University Press

is a homeomorphism. Since an ℓ-group is Dedekind σ-complete if and only if M is a σ-complete MV-algebra, we conclude that if G is Dedekind σ-complete, then the set $X = \mathrm{Ext}(\mathcal{S}(G, u))$, the set of all extremal points of $\mathcal{S}(G, u)$, is a compact Hausdorff basically disconnected space.

A state s on (G, u) is *discrete* if $s(G) = 1/n\mathbb{Z}$ for some integer $n \geq 1$, where $\mathbb{Z}$ is the set of all integers.

We recall that an ℓ-group σ-homomorphism $\hat{h}$ from $\mathcal{T}$ onto G means: (i) $\hat{h}(f \pm g) = \hat{h}(f) \pm \hat{h}(g)$; (ii) $\hat{h}(\sup_n f_n) = \bigvee_n \hat{h}(f_n)$; (iii) $\hat{h}(1_X) = u$.

Now we present an analogue of the Loomis-Sikorski theorem for Dedekind σ-complete ℓ-group.

THEOREM 5.1. *For any Dedekind σ-complete ℓ-group G with strong unit u there exists a g-tribe $\mathcal{T}$ of bounded functions on a compact Hausdorff space X and an ℓ-group σ-homomorphism $\hat{h}$ from $\mathcal{T}$ onto G with $h(1_X) = u$.*

PROOF. Put $X = \mathrm{Ext}(\mathcal{S}(G, u))$ and let $\mathcal{C}(X)$ be the set of all continuous functions on the compact space X. Denote by

$$B = \{f \in \mathcal{C}(X) : f(s) \in s(G) \text{ for all discrete states } s \in X\}. \tag{5.1}$$

Then B is an ℓ-group with strong unit 1_X. According to [Goo, Corollary 9.14], the mapping $\psi : G \to \mathcal{C}(X)$, defined by $\psi(g)(s) := s(g)$, $g \in G$ $(s \in X)$, defines an isomorphism of (G, u) onto $(B, 1_X)$ (as ordered groups with ordered unit).

Let $\mathcal{T}(B)$ be the g-tribe generated by B. We assert that for the tribe $\mathcal{F}(M)$ of fuzzy sets generated by the set $\hat{M} := \{\psi(g) : g \in M\}$, where $M = \Gamma(G, u)$, we have

$$\mathcal{F}(M) = \Gamma(\mathcal{T}(B), 1_X). \tag{5.2}$$

Step 1. Since for any $g \in G$, $\psi(g) \in \mathcal{F}(M)$ and $\psi(g) \in \Gamma(\mathcal{T}(B), 1_X)$, we conclude from the fact that $\Gamma(\mathcal{T}(B), 1_X)$ is a tribe containing $C(X)$ that

$$\mathcal{F}(M) \subseteq \Gamma(\mathcal{T}(B), 1_X). \tag{5.3}$$

Let now $G(\mathcal{F}(M))$ be an ℓ-group generated by $\mathcal{F}(M)$. In view of (5.3), we have

$$G(\mathcal{F}(M)) \subseteq \mathcal{T}(B). \tag{5.4}$$

Step 2. Since 1_X is a strong unit for $G(\mathcal{F}(M))$, to prove that $G(\mathcal{F}(M))$ is closed under $\sup_n f_n$ for every sequence $\{f_n\}_n$ of element from $G(\mathcal{F}(M))$ with an upper bound in $G(\mathcal{F}(M))$, it is sufficient to verify that for each sequence $\{f_n\}_n$ of functions belonging to $[0, k1_X]$ for $k \geq 1$.

We proceed by induction on k. The case $k = 1$ is trivial. Suppose now that $k > 1$ and that the interval $[0, (k-1)1_X]$ is closed under $\sup_n$. Since $[(k-1)1_X, k1_X]$ is isomorphic to $[0, 1_X]$, we conclude that $[(k-1)1_X, k1_X]$ is closed under $\sup_n$.

Let $\{f_n\}_n$ be a sequence of functions of $[0, k1_X]$. For any $k \geq 1$, we put

$$f_n^1 = \min\{f_n, a\}, \quad f_n^2 = \max\{f_n, a\},$$

where $a = (n-1)1_X$. Then $f_n = f_n^1 + (f_n^2 - a)$ for each $n \geq 1$. Define $f^1 = \sup_n f_n^1$ and $f^2 = \sup_n f_n^2$ and $f = f^1 + f^2$. Then $f^1 \in [0, (k-1)1_X], f^2 \in [(k-1)1_X, k1_X]$ and $f \geq f_n$ for each $n \geq 1$. Let now $s \in X$ and $\epsilon > 0$. Then there exists an integer $n \geq 1$ such that $f^i(s) - f_n^i(s) \leq \epsilon/2$ for $i = 1, 2$. Hence $f(s) - f_n(s) = f^1(s) - f_n^1(s) + f^2(s) - f_n^2(s) < \epsilon$ which proves that $f = \sup_n f_n$. Similarly we prove that $\inf_n f_n$ exists in $[0, k1_X]$.

Step 3. In fact we have verified that $G(\mathcal{F}(M))$ is a g-tribe containing B which entails

$$\mathcal{T}(B) \subseteq G(\mathcal{F}(M)), \tag{5.5}$$

which by (5.4) yields $\mathcal{T}(B) = G(\mathcal{F}(M))$, consequently, $\mathcal{F}(M) = \Gamma(\mathcal{T}(B), 1_X)$.

Step 4. By the Loomis-Sikorski Theorem 4.1, there exists an MV-σ-homomorphism h from $\mathcal{F}(M)$ onto M. Using arguments of Mundici's categorical equivalence of ℓ-groups with strong unit with MV-algebras, given by $M \leftrightarrow \Gamma(G, u)$, see (2.2), we conclude that there exists an extension $\hat{h}$ of h from $\mathcal{T}(B)$ onto G such that $\hat{h}$ is an ℓ-group homomorphism, that is, for $\hat{h}$ we have (i) $\hat{h}(f \pm g) = \hat{h}(f) \pm \hat{h}(g)$, (ii) $\hat{h}(\max\{f, g\}) = \hat{h}(f) \vee \hat{h}(g)$, and (iii) $\hat{1}_X = u$.

To prove that $\hat{h}(\sup_n f_n) = \sup_n \hat{h}(f_n)$ whenever $\sup_n f_n \in \mathcal{T}(B)$, we proceed in the same was is in the Step 2. □

Comparing Proposition 3.4, we have the following statement.

PROPOSITION 5.1. *Let X be a compact Hausdorff space and let $\mathcal{T}_g(X)$ be a g-tribe generated by the set $\mathcal{C}(X)$ of all continuous functions on X. Then $\mathcal{T}_g(X) = \mathcal{M}_b(X)$, where $\mathcal{M}_b(X)$ denotes the set of all bounded Baire measurable functions on X. In addition, $\mathcal{M}(X) = \Gamma(\mathcal{T}_g(X), 1_X)$.*

PROOF. Since $\mathcal{M}_b(X)$ is a g-tribe containing $\mathcal{C}(X)$, we have $\mathcal{T}_g(X) \subseteq \mathcal{M}_b(X)$. $\mathcal{T}_g(X)$ is an ℓ-group with strong unit 1_X, whence $\Gamma(\mathcal{T}_g(X), 1_X)$ is a tribe containing the set $C(X)$ of all continuous fuzzy sets on X, which by Proposition 3.4 entails

$$\mathcal{M}(X) \subseteq \Gamma(\mathcal{T}_g(X), 1_X) \subseteq \mathcal{M}_b(X). \tag{5.6}$$

If is a non-negative function from $\mathcal{M}_b(X)$, then there is an integer $k \geq 1$ such that $f \leq k1_X$, and by Proposition 3.4 it means that $f/k \in \mathcal{M}(X)$. Consequently,

f belongs to the ℓ-group generated by $\mathcal{M}(X)$. This is also true for any function $f \in \mathcal{M}_b(X)$. Consequently, if $f \in \Gamma(\mathcal{T}_g(X), 1_X)$, then $f \in \mathcal{M}(X)$. Hence, $\mathcal{M}_b(X) = \mathcal{T}_g(X)$ and $\mathcal{M}(X) = \Gamma(\mathcal{T}_g(X), 1_X)$. □

Acknowledgement

The author is indebted to the referee for his valuable suggestions which improved readability of the paper.

This research was supported by the grant VEGA No 2/4033/98 of the Slovak Academy of Sciences, Slovakia.

References

[Bel] L. P. Belluce, 'Semisimple algebras of infinite valued logic', *Canad. J. Math.* **38** (1986), 1356–1379.

[Bir] G. Birkhoff, *Lattice theory*, Coll. Publ. 25 (Amer. Math. Soc., Providence, 1997).

[Cha] C. C. Chang, 'Algebraic analysis of many-valued logics', *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

[Cig] R. Cignoli, 'Complete and atomic algebras of the infinite valued Łukasiewicz logic', *Studia Logica* **50** (1991), 375–384.

[CDM] R. Cignoli, I. M. L. D'Ottaviano and D. Mundici, *Algebraic foundations of many-valued reasoning*, (Kluwer Acad. Publ., Dordrecht, to appear).

[DCR] A. Dvurečenskij, F. Chovanec and E. Rybáriková, 'D-homomorphisms and atomic σ-complete Boolean D-posets', *Soft Comput.*, to appear.

[DvGr] A. Dvurečenskij and M. G. Graziano, 'Commutative BCK-algebras and lattice ordered groups with universal property', *Math. Japonica* **49** (1999), 159–174.

[Goo] K. R. Goodearl, *Partially ordered Abelian groups with interpolation*, Math. Surveys Monographs 20 (Amer. Math. Soc., Providence, 1986).

[Jak] J. Jakubík, 'On complete MV-algebras', *Czechoslovak Math. J.* **45 (120)** (1995), 473–480.

[Kur] K. Kuratowski, *Topology I* (Mir, Moskva, 1966).

[Mun] D. Mundici, 'Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus', *J. Funct. Anal.* **65** (1986), 15–63.

[Mun1] ———, 'Averaging the truth-value in Łukasiewicz logic', *Studia Logica* **55** (1995), 113–127.

[RiNe] B. Riečan and T. Neubrunn, *Integral, measures, and ordering* (Kluwer Acad. Publ., Dordrecht, 1997).

[Sik] R. Sikorski, *Boolean algebras* (Springer, Berlin, 1964).

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
Slovakia
e-mail: dvurecen@mat.savba.sk