K. Miyake
Nagoya Math. J.
Vol. 80 (1980), 117–127

ON THE STRUCTURE OF THE IDELE GROUP OF AN ALGEBRAIC NUMBER FIELD

KATSUYA MIYAKE

The purpose of this paper is to present the results of E. Artin and Furtwängler, with which they proved the principal ideal theorem, as a structure theorem of the idele group of an algebraic number field. Such treatment may be helpful to clarify the Arithmetic nature these results possess.

§ 1.

Let F be an algebraic number field (of finite degree over $\boldsymbol{Q}$), and let K/F and L/K be both finite abelian extensions. Suppose that L is a Galois extension of F, and that K is the maximal abelian extension of F contained in L. Then $G = \mathrm{Gal}(L/F)$ is metabelian, and $G' = \mathrm{Gal}(L/K)$ is the commutator subgroup of G.

Let us denote the Artin maps of K/F and L/K by $[\cdot, K/F]$ and $[\cdot, L/K]$ respectively. That is, for a prime ideal $\mathfrak{p}$ of F which is unramified in K/F, $[\mathfrak{p}, K/F]$ is the Frobenius automorphism of $\mathfrak{p}$ in $\mathrm{Gal}(K/F)$.

Let $\mathfrak{a}$ be an ideal of F. Then the extension of $\mathfrak{a}$ to an ideal of K is $\mathfrak{a} \cdot O_K$ where O_K is the maximal order of K.

THEOREM (Artin-Furtwängler). *Let L be a Galois extension of F, and suppose that $G = \mathrm{Gal}(L/F)$ is metabelian. Let K be the maximal abelian extension of F contained in L, and O_K the maximal order of K. Then, if an ideal $\mathfrak{a}$ of F is unramified in K/F, $[\mathfrak{a} \cdot O_K, L/K]$ is trivial.*

E. Artin showed that the map of $G/G' = \mathrm{Gal}(K/F)$ to $G' = \mathrm{Gal}(L/K)$ which gives

$$[\mathfrak{a}, K/F] \longmapsto [\mathfrak{a} \cdot O_K, L/K]$$

is the transfer (Verlagerung) $V_{G \to G'}$ of G/G' to G'. Then Furtwängler proved that $V_{G \to G'}$ is the trivial homomorphism of G/G' to G'. (See [1] and [3].)

Received May 8, 1979.

https://doi.org/10.1017/S0027763000019085 Published online by Cambridge University Press

It may be worth to point out that this theorem is proved without using class field theory.

§ 2.

For an algebraic number field F, the ring of adeles of F is denoted by F_A, and the idele group of F by $F_A^\times$. Let F_{ab} be the maximal abelian extension in the algebraic closure $\bar{F}$ of F, and put $\mathfrak{A}_F = \mathrm{Gal}\,(F_{ab}/F)$ and $\mathfrak{G}_F = \mathrm{Gal}\,(\bar{F}/F)$. Let $F_A^\times = F_f^\times \cdot F_\infty^\times$ be the decomposition of $F_A^\times$ into the product of its non-Archimedean part $F_f^\times$ and its Archimedean part $F_\infty^\times$. Let $F_{\infty+}^\times$ be the connected component of the unity of $F_\infty^\times$, and $F^\#$ the topological closure of $F^\times \cdot F_{\infty+}^\times$ in $F_A^\times$. Here and after, F and $F^\times$ are considered to be diagonally embedded in F_A and $F_A^\times$ respectively.

By class field theory, Artin map or canonical morphism

$$[\cdot, F]\colon F_A^\times \longrightarrow \mathfrak{A}_F$$

is an open, continuous and surjective homomorphism whose kernel is $F^\#$. Our basic reference on class field theory is Weil's book [8] though the notation slightly differs.

Let K be a finite Galois extension of F. Then $\mathrm{Gal}\,(K/F) = \mathfrak{G}_F/\mathfrak{G}_K$ where $\mathfrak{G}_K = \mathrm{Gal}\,(\bar{F}/K)$. The ring of adeles of K is naturally identified with the tensor product $K \otimes_F F_A = K_A$. Then the natural action of $\mathfrak{G}_F$ on K_A is the one defined through the K-factor of the product.

Let $\mathfrak{G}'_K$ be the commutator subgroup of $\mathfrak{G}_K$. Then $\mathfrak{A}_K = \mathrm{Gal}\,(K_{ab}/K) = \mathfrak{G}_K/\mathfrak{G}'_K$. Since $\mathfrak{G}_K$ is a normal subgroup of $\mathfrak{G}_F$, this $\mathfrak{G}_F$ acts on $\mathfrak{G}_K$ through inner automorphisms of $\mathfrak{G}_F$, and also on $\mathfrak{A}_K = \mathfrak{G}_K/\mathfrak{G}'_K$. More precisely, let ξ be an element of $\mathfrak{G}_K$. Then for $\lambda \in \mathfrak{G}_F$, the action of λ on $\xi \bmod \mathfrak{G}'_K$ is defined by

$$(\xi \bmod \mathfrak{G}'_K)^\lambda = \lambda^{-1}\cdot\xi\cdot\lambda \mod \mathfrak{G}'_K\,.$$

THEOREM 1. *For $x \in K_A^\times$ and $\lambda \in \mathfrak{G}_F$,*

$$[x^\lambda, K] = [x, K]^\lambda$$

where $[\cdot, K]\colon K_A^\times \to \mathfrak{A}_K = \mathrm{Gal}\,(K_{ab}/K)$ is Artin map for K.

This theorem is well known. But a proof will be given in § 6 for the completeness.

§ 3.

Now our intended result is ready to be shown. Generalization will

https://doi.org/10.1017/S0027763000019085 Published online by Cambridge University Press

be done in the next section. Note that K does not have to be an abelian extension of F in this theorem.

THEOREM 2. *Let F be an algebraic number field and K a finite Galois extension of F. If an open subgroup U of $K_A^\times$ satisfies*

(i) $U \supset K^\#$

(ii) $U^\sigma = U$ *for any* $\sigma \in \mathrm{Gal}(K/F)$

(iii) $U \cdot N_{K/F}^{-1}(F^\#) = K_A^\times$

then $U \supset F_A^\times$.

Here $N_{K/F}\colon K_A^\times \to F_A^\times$ is the norm map of K over F.

Proof. First we reduce the theorem to the case that K is an abelian extension of F. Let M be the maximal abelian extension of F contained in K. Then

$$F^\times \cdot N_{M/F}(M_A^\times) = F^\times \cdot N_{K/F}(K_A^\times) .$$

Put $V = M^\times \cdot N_{K/M}(U)$. Then V is an open subgroup of $M_A^\times$, and contains $M^\#$. It is obvious that $V^\tau = V$ for $\tau \in \mathrm{Gal}(M/F)$. Since

$$F^\times \cdot N_{M/F}(V) = F^\times \cdot N_{K/F}(U) = F^\times \cdot N_{K/F}(K_A^\times) = F^\times \cdot N_{M/F}(M_A^\times)$$

it is easy to see that

$$V \cdot N_{M/F}^{-1}(F^\#) = V \cdot N_{M/F}^{-1}(F^\times) = M_A^\times .$$

It follows, moreover, from (i) and (ii) that U contains V as a subgroup. Hence it is sufficient to show that V contains $F_A^\times$. Therefore we may assume that K itself is an abelian extension of F.

Now let L be the class field of K corresponding to U. Then

$$U = K^\times \cdot N_{L/K}(L_A^\times) .$$

By Theorem 1, condition (ii) implies that L is a Galois extension of F. From (iii), it follows that K is the maximal abelian extension of F contained in L.

For a prime ideal $\mathfrak{P}$ of K, let $O_{K,\mathfrak{P}}$ be the $\mathfrak{P}$-adic completion of O_K, and $O_{K,\mathfrak{P}}^\times$ the group of units of $O_{K,\mathfrak{P}}$. Then $O_{K,\mathfrak{P}}^\times$ is canonically regarded as a subgroup of $K_A^\times$. Since U is open, the number of such prime ideals $\mathfrak{P}$ that $O_{K,\mathfrak{P}}^\times \not\subset U$ is finite. Let S be the set of all such prime ideals of K. For each $\mathfrak{P} \in S$, fix an integer $e(\mathfrak{P})$ such that

$$1 + \mathfrak{P}^{e(\mathfrak{P})} \cdot O_{K,\mathfrak{P}} \subset U$$

and

$$U_S = \prod_{\mathfrak{P} \notin S} O_{K,\mathfrak{P}}^\times \times \prod_{\mathfrak{P} \in S} (1 + \mathfrak{P}^{e(\mathfrak{P})} \cdot O_{K,\mathfrak{P}}) \times K_{\infty+}^\times$$

$K_{A(S)}^\times$ = the subgroup of $K_A^\times$ generated by U_S and all $K_\mathfrak{P}^\times$ for $\mathfrak{P} \notin S$

$K_S^\times = K^\times \cap K_{A(S)}^\times$

$\mathfrak{M} = \prod_{\mathfrak{P} \in S} \mathfrak{P}^{e(\mathfrak{P})} \times$ product of all infinite places of K

$I_L(S)$ = the group of ideals of L prime to $\mathfrak{M}$

$I_K(S)$ = the group of ideals of K prime to $\mathfrak{M}$

$\mathfrak{S}_K(M)$ = the Strahl ideal class group modulo $\mathfrak{M}$.

Here $K_\mathfrak{P}$ is the $\mathfrak{P}$-adic completion of K, and $K_\mathfrak{P}^\times$ is its multiplicative group. For prime $\boldsymbol{P}$ of L, let $L_{\boldsymbol{P}}$ be the $\boldsymbol{P}$-adic completion, and $L_{\boldsymbol{P}}^\times$ the multiplicative group of $L_{\boldsymbol{P}}$. Put

$L_{A(S)}^\times$ = the subgroup of $L_A^\times$ generated by $\prod_{\boldsymbol{P} \cap K \in S} O_{L,\boldsymbol{P}}^\times$ and all $L_{\boldsymbol{P}}^\times$ for $\boldsymbol{P} \cap K \notin S$.

For idele x of K (resp. of L, of F), denote the corresponding ideal of K (resp. of L, of F) by $\mathscr{I}_K(x)$ (resp. $\mathscr{I}_L(x)$, $\mathscr{I}_F(x)$). Then we have exact sequences

$$1 \longrightarrow U_S \longrightarrow K_{A(S)}^\times \xrightarrow{\mathscr{I}_K} I_K(S) \longrightarrow 1$$
$$1 \longrightarrow K_S^\times \cdot U_S \longrightarrow K_{A(S)}^\times \longrightarrow \mathfrak{S}_K(S) \longrightarrow 1$$
$$L_{A(S)}^\times \cap N_{L/K}^{-1}(K_{A(S)}^\times) \xrightarrow{\mathscr{I}_L} I_L(S) \longrightarrow 1 .$$

Furthermore, for $x \in L_{A(S)}^\times \cap N_{L/K}^{-1}(K_{A(S)}^\times)$,

$$\mathscr{I}_K(N_{L/K}(x)) = N_{L/K}(\mathscr{I}_L(x))$$

and, for $x \in F_A^\times \cap K_{A(S)}^\times$,

$$\mathscr{I}_K(x) = \mathscr{I}_F(x) \cdot O_K .$$

Now apply Artin-Furtwängler theorem to this case. Then, (by Hilbert theory), one can easily conclude that, for $x \in F_A^\times \cap K_{A(S)}^\times$, there exist $a \in K_S^\times$ and $y \in L_{A(S)}^\times \cap N_{L/K}^{-1}(K_{A(S)}^\times)$ such that

$$\mathscr{I}_K(x) = \mathscr{I}_K(a) \cdot N_{L/K}(\mathscr{I}_L(y)) .$$

Therefore

$$x = a \cdot N_{L/K}(y) \cdot u$$

with some $u \in U_S$. Since U contains all of $K_S^\times$, $N_{L/K}(L_A^\times)$ and U_S, it has

https://doi.org/10.1017/S0027763000019085 Published online by Cambridge University Press

been shown that

$$F_A^\times \cap K_{A(S)}^\times \subset U .$$

Because S is a finite set of prime ideals of K, one can easily see by Chinese remainder theorem that $(F_A^\times \cap K_{A(S)}^\times)\cdot F^\times = F_A^\times$. Since U contains $F^\times$,

$$F_A^\times = (F_A^\times \cap K_{A(S)}^\times)\cdot F^\times \subset U\cdot F^\times = U .$$

The proof is done.

§ 4. Generalization

THEOREM 3. *Let F be an algebraic number field, and K a finite Galois extension of F. For an open subgroup U of $K_A^\times$ satisfying*

(i) $U \supset K^\#$

(ii) $U^\sigma = U$ *for any* $\sigma \in \mathrm{Gal}\,(K/F)$

put $m = [K_A^\times \colon U\cdot N_{K/F}^{-1}(F^\#)]$. *Then*

$$(F_A^\times)^m = \{a^m \mid a \in F_A^\times\} \subset U .$$

Proof. Let L be the abelian extension of K corresponding to $U\cdot N_{K/F}^{-1}(F^\#)$. Then $m = [L\colon K]$, and

$$K^\times\cdot N_{L/K}(L_A^\times) = U\cdot N_{K/F}^{-1}(F^\#) .$$

Put $V = N_{L/K}^{-1}(U)$. Then

$$L_A^\times = V\cdot N_{L/F}^{-1}(F^\#)$$

since

$$\begin{aligned} F^\times\cdot N_{L/F}(L_A^\times) &= F^\times\cdot N_{K/F}(K^\times\cdot N_{L/K}(L_A^\times)) \\ &= F^\times\cdot N_{K/F}(U\cdot N_{K/F}^{-1}(F^\#)) \\ &= F^\times\cdot N_{K/F}(U) \\ &= F^\times\cdot N_{L/F}(V) . \end{aligned}$$

Obviously L is a Galois extension of F. Theorem 2, therefore, is applicable to L/F and V, and implies that $V \supset F_A^\times$. Hence for any $a \in F_A^\times$

$$a^m = N_{L/K}(a) \in U .$$

The proof is completed.

COROLLARY. *The notation and the assumptions being as in the theorem, let n be the largest common divisor of m and the degree $[K\colon F]$. Then*

https://doi.org/10.1017/S0027763000019085 Published online by Cambridge University Press

$$(U\cdot N_{K/F}^{-1}(F^{\#}))\cap F_A^{\times}=(U\cdot X)\cap F_A^{\times}$$

where $X=\{x\in N_{K/F}^{-1}(F^{\#})\,|\,x^n\in U\}$.
Therefore especially

$$(U\cdot N_{K/F}^{-1}(F^{\#}))\cap F_A^{\times}=U\cap F_A^{\times}$$

if n *is prime to the index* $[U\cdot N_{K/F}^{-1}(F^{\#})\colon U]$.

Proof. Put $d=[K\colon F]$. For $a\in(U\cdot N_{K/F}^{-1}(F^{\#}))\cap F_A^{\times}$, choose $u\in U$ and $v\in N_{K/F}^{-1}(F^{\#})$ so that $a=u\cdot v$. Then $a^d=N_{K/F}(a)=N_{K/F}(u)\cdot N_{K/F}(v)$. Condition (ii) implies that $N_{K/F}(u)\in U$. Since $N_{K/F}(v)\in F^{\#}$, we conclude that $a^d\in U\cap F_A^{\times}$. It follows from the theorem that a^m belongs to $U\cap F_A^{\times}$. Therefore a^n belongs to $U\cap F_A^{\times}$ where $n=(m,d)$. Since $a^n=u^n\cdot v^n$, we see that $v\in X$. The proof is done.

§ 5. Remarks on $F^{\#}$

Let F be an algebraic number field of finite degree d over $\boldsymbol{Q}$, and $d=r_1+2\cdot r_2$ where r_1 is the number of real Archimedean primes of F. Put $r=r_1+r_2-1$. Let E_+ be the multiplicative group of all the totally positive units of F. (We exclude the roots of 1 in F from E_+ when $r_1=0$.) Then E_+ is a free $\boldsymbol{Z}$-module of rank r.

Let E_{+f} be the projection of E_+ to the non-Archimedean part $F_f^{\times}$ of $F_A^{\times}$, and $\overline{E_{+f}}$ the topological closure of E_{+f} in $F_f^{\times}$.

PROPOSITION 1. *The closure* $F^{\#}$ *of* $F^{\times}\cdot F_{\infty+}^{\times}$ *in* $F_A^{\times}$ *is equal to* $\overline{E_{+f}}\cdot F^{\times}\cdot F_{\infty+}^{\times}$. *Moreover, for every positive integer* n,

$$\overline{E_{+f}}=E_{+f}\cdot\{x^n\,|\,x\in\overline{E_{+f}}\}$$
$$F^{\#}=F^{\times}\cdot\{x^n\,|\,x\in F^{\#}\}\,.$$

(*See Shimura* [7], 2.2.)

PROPOSITION 2. (1) $F^{\times}\cap\{x^n\,|\,x\in F^{\#}\}=\{a^n\,|\,a\in F^{\times}\}$.
(2) *For* $x\in F^{\#}$, $x^n=1\Rightarrow x\in F^{\times}\cdot F_{\infty+}^{\times}$.
(See [6], 3.1.)

PROPOSITION 3. *As topological groups,* $\overline{E_{+f}}$ *is isomorphic to the direct product of r copies of* $\tilde{Z}=\prod_{p,\,\mathrm{prime}}Z_p$ *where* Z_p *is the ring of p-adic integers.*

Proof. By Chevalley [2], the topology induced on free $\boldsymbol{Z}$-module E_{+f} of rank r is the one defined by taking all the subgroups of finite index as the basis of the neighbourhood of 0. Therefore $\overline{E_{+f}}$ is isomorphic to the completion $\tilde{Z}^r$.

PROPOSITION 4. *Let K be a finite extension of F (not necessarily Galois). Then*

$$N_{K/F}^{-1}(F^\#)/K^\# \cdot N_{K/F}^{-1}(1) \cong N_{K/F}(K_A^\times) \cap F^\times/N_{K/F}(K^\times) .$$

Proof. Put $N = N_{K/F}$, and $d = [K: F]$. First we see $N^{-1}(F^\#) = N^{-1}(F^\times) \cdot F^\#$. For $x \in N^{-1}(F^\#)$, choose $a \in F^\times$ and $b \in F^\#$ by Prop. 1 so that $N(x) = a \cdot b^d$. Put $y = x \cdot b^{-1}$. Then $N(y) = a \in F^\times$, and $x = y \cdot b$.

Next we show $N^{-1}(F^\times) \cap K^\# = K^\times \cdot (N^{-1}(1) \cap K^\#)$. Obviously the right is contained by the left. For $z \in K^\#$, suppose that $N(z) \in F^\times$. By Prop. 1 for K, choose $u \in K^\times$ and $v \in K^\#$ so that $z = u \cdot v^d$. Then $N(v)^d = N(z) \cdot N(u)^{-1} \in F^\times$. Therefore by Prop. 2, (1), we can find $a \in F^\times$ such that $N(v)^d = a^d$. Then $z = (u \cdot a) \cdot (a^{-1} \cdot v^d)$ with $u \cdot a \in K^\times$ and $N(a^{-1} v^d) = 1$. Now

$$\begin{aligned} N^{-1}(F^\#)/K^\# \cdot N^{-1}(1) &= N^{-1}(F^\times) \cdot F^\#/K^\# \cdot N^{-1}(1) \\ &\cong N^{-1}(F^\times)/N^{-1}(F^\times) \cap (K^\# \cdot N^{-1}(1)) \\ &= N^{-1}(F^\times)/(N^{-1}(F^\times) \cap K^\#) \cdot N^{-1}(1) \\ &= N^{-1}(F^\times)/K^\times \cdot N^{-1}(1) \\ &\cong N(K_A^\times) \cap F^\times/N(K^\times) . \end{aligned}$$

The proof is done.

§ 6. Proof of Theorem 1

Let K be a finite Galois extension of an algebraic number field F. Let the notation and the situation be as in § 2. We have to prove that canonical homomorphism $[\cdot, K]: K_A^\times \to \mathfrak{A}_K = \mathrm{Gal}(K_{ab}/K)$ of class field theory is compatible with the action of $\mathfrak{G}_F = \mathrm{Gal}(\bar{F}/F)$ (modulo $\mathfrak{G}_K$).

Let $\mathfrak{p}$ be a prime divisor of F, $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$, and $\bar{F}_\mathfrak{p}$ the algebraic closure of $F_\mathfrak{p}$. Fix an isomorphism ι of $\bar{F}$ into a subfield $\iota(\bar{F})$ of $\bar{F}_\mathfrak{p}$, which is identical on F. Put $\tilde{K} = \iota(K) \cdot F_\mathfrak{p}$. This is a Galois extension of $F_\mathfrak{p}$. Put $\mathfrak{G}_\mathfrak{p} = \mathrm{Gal}(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$ and $\tilde{\mathfrak{G}} = \mathrm{Gal}(\bar{F}_\mathfrak{p}/\tilde{K})$. The latter is a normal subgroup of the former. Note that $\bar{F}_\mathfrak{p} = \iota(\bar{F}) \cdot F_\mathfrak{p}$, $F_{\mathfrak{p},ab} = \iota(F_{ab}) \cdot F_\mathfrak{p}$, and $\tilde{K}_{ab} = \iota(K_{ab}) \cdot \tilde{K}$ where $F_{\mathfrak{p},ab}$ and $\tilde{K}_{ab}$ are the maximal abelian extension of $F_\mathfrak{p}$ and $\tilde{K}$ in $\bar{F}_\mathfrak{p}$ respectively. Hence the restriction of the action of $\mathfrak{G}_\mathfrak{p}$ on $\iota(\bar{F})$ gives an isomorphic embedding of $\mathfrak{G}_\mathfrak{p}$ into $\iota \circ \mathfrak{G}_F \circ \iota^{-1}$. Let $\mathfrak{Z}_\mathfrak{p}$ be the subgroup of $\mathfrak{G}_F$ corresponding to $\mathfrak{G}_\mathfrak{p}$. That is, $\iota \circ \mathfrak{Z}_\mathfrak{p} \circ \iota^{-1} = \mathfrak{G}_\mathfrak{p}$. We also have

https://doi.org/10.1017/S0027763000019085 Published online by Cambridge University Press

$$\mathfrak{G}'_{\mathfrak{p}} = \mathfrak{G}_{\mathfrak{p}} \cap (\iota \circ \mathfrak{G}'_F \circ \iota^{-1})$$
$$\tilde{\mathfrak{G}}' = \tilde{\mathfrak{G}} \cap (\iota \circ \mathfrak{G}'_K \circ \iota^{-1})$$

where $\mathfrak{G}'_{\mathfrak{p}}$ and $\tilde{\mathfrak{G}}'$ are the commutator subgroups of $\mathfrak{G}_{\mathfrak{p}}$ and $\tilde{\mathfrak{G}}$ respectively.

Fix a set of representatives $S = \{\sigma_1, \cdots, \sigma_g\}$ of the left cosets of $\mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K$ in $\mathfrak{G}_F$. (Remember that $\mathfrak{G}_F$ acts on both of K_A and $\mathfrak{A}_K$ from the right.) For $\sigma \in \mathfrak{G}_F$, the representative in S of $\mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K \cdot \sigma$ is denoted by $[\sigma]$. Put

$$\iota(\sigma) = \iota \circ [\sigma]^{-1} \qquad (\sigma \in \mathfrak{G}_F)\,.$$

Then $\iota(\sigma)$ depends only on the coset $\mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K \cdot \sigma$. The family of pairs $\{(\iota(\sigma), K) | \sigma \in S\}$ is a set of all non-equivalent proper embeddings of K above $F_{\mathfrak{p}}$. That is, for any proper embedding (λ, L) of K above $F_{\mathfrak{p}}$, there are $\sigma \in S$ and isomorphism ρ of L over $F_{\mathfrak{p}}$ into $\tilde{K}$ such that $\iota(\sigma) = \rho \circ \lambda$. (See Weil [8], p. 51, Cor. 2.) Fix a set of representatives $R = \{\rho_1, \cdots, \rho_f\}$ of $\mathfrak{G}_{\mathfrak{p}}/\tilde{\mathfrak{G}} = \mathrm{Gal}\,(\tilde{K}/F_{\mathfrak{p}})$ where $\rho_i \in \mathfrak{G}_{\mathfrak{p}}$. Then for any two elements σ, τ of $\mathfrak{G}_F$, there is a unique element $\rho(\sigma, \tau)$ of R such that, restricted to K,

$$\iota(\sigma) \circ \tau|_K = \rho(\sigma, \tau) \circ \iota(\sigma\tau^{-1})|_K\,.$$

For σ and $\tau \in \mathfrak{G}_F$, define $\zeta(\sigma, \tau) \in \mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K$ by

$$[\sigma] \cdot \tau^{-1} = \zeta(\sigma, \tau) \cdot [\sigma\tau^{-1}]\,.$$

Then

$$\rho(\sigma, \tau) \equiv \iota \circ \zeta(\sigma, \tau)^{-1} \circ \iota^{-1} \qquad \text{modulo } \tilde{\mathfrak{G}}\,.$$

For each $\sigma \in S$, put

$$\tilde{\mathfrak{G}}_\sigma = \sigma \circ \iota^{-1} \circ \tilde{\mathfrak{G}} \circ \iota \circ \sigma^{-1} = \iota^{-1} \circ [(\iota \circ \sigma^{-1} \circ \iota^{-1}) \cdot \tilde{\mathfrak{G}} \cdot (\iota \circ \sigma \circ \iota^{-1})] \circ \iota\,.$$

Then $\tilde{\mathfrak{G}}_\sigma$ is a subgroup of $\mathfrak{G}_K$ and is a conjugate of $\mathfrak{Z}_{\mathfrak{p}} \cap \mathfrak{G}_K$ in $\mathfrak{G}_F$. It is easy to see that the commutator subgroup $\tilde{\mathfrak{G}}'_\sigma$ of $\tilde{\mathfrak{G}}_\sigma$ coincides with $\tilde{\mathfrak{G}}_\sigma \cap \mathfrak{G}'_K$. Put

$$\mathfrak{A}_{\tilde{K},\sigma} = \tilde{\mathfrak{G}}_\sigma/\tilde{\mathfrak{G}}'_\sigma\,.$$

This is considered as a subgroup of $\mathfrak{A}_K = \mathfrak{G}_K/\mathfrak{G}'_K$. The action of $\mathfrak{G}_F$ on $\mathfrak{A}_K$ maps the family $\{\mathfrak{A}_{\tilde{K},\sigma} | \sigma \in S\}$ onto itself. Each $\mathfrak{A}_{\tilde{K},\sigma}$ is isomorphic to $\mathfrak{A}_{\tilde{K}} = \tilde{\mathfrak{G}}/\tilde{\mathfrak{G}}'$.

Let us now consider the $\mathfrak{p}$-part of K_A. It is naturally identified with $K \otimes_F F_{\mathfrak{p}}$. Take copies of $\tilde{K}$ indexed by S. That is, put $\tilde{K}_\sigma = \tilde{K}$ for each $\sigma \in S$. Then the map $\iota(\sigma)\colon K \to \tilde{K}_\sigma$ for $\sigma \in S$ gives an $F_{\mathfrak{p}}$-linear isomorphism

https://doi.org/10.1017/S0027763000019085 Published online by Cambridge University Press

$\eta_{\mathfrak{p}}$ of $K \otimes_F F_{\mathfrak{p}}$ onto the direct product $\prod_{\sigma \in S} \tilde{K}_\sigma$.

For $\sigma, \tau \in \mathfrak{G}_F$, and for $a \in K$,

$$\begin{aligned} \iota(\sigma)(a^\tau) &= (\iota(\sigma) \circ \tau)(a) = (\rho(\sigma, \tau) \circ \iota(\sigma\tau^{-1}))(a) \\ &= (\iota(\sigma\tau^{-1})(a))^{\rho(\sigma,\tau)} . \end{aligned}$$

Therefore it is easy to see the following:

For $x \in K \otimes_F F_{\mathfrak{p}}$, *let* $\eta_{\mathfrak{p}}(x) = (x_\sigma)_{\sigma \in S} \in \prod_\sigma \tilde{K}_\sigma$.

Then for $\tau \in \mathfrak{G}_F$,

$$\eta_{\mathfrak{p}}(x^\tau) = (y_\sigma)_{\sigma \in S} \in \prod_\sigma \tilde{K}_\sigma$$
$$y_\sigma = (x_{[\sigma\tau^{-1}]})^{\rho(\sigma,\tau)} .$$

Let χ be a (linear) character of $\mathfrak{G}_K$. It is automatically considered as a character of $\mathfrak{A}_K = \mathfrak{G}_K/\mathfrak{G}_K' = \mathrm{Gal}\,(K_{ab}/K)$. For $\lambda \in \mathfrak{G}_F$, define a character χ^λ of $\mathfrak{G}_K$ by

$$\chi^\lambda(\tau) = \chi(\lambda\tau\lambda^{-1}) \qquad (\tau \in \mathfrak{G}_K) .$$

Since $\mathfrak{G}_K$ is normal in $\mathfrak{G}_F$, this is well defined. Note that χ^λ depends only on λ modulo $\mathfrak{G}_K$.

For χ, we can associate characters $\chi_\sigma(\sigma \in S)$ of $\mathfrak{A}_{\tilde{K}} = \tilde{\mathfrak{G}}/\tilde{\mathfrak{G}}' = \mathrm{Gal}\,(\tilde{K}_{ab}/\tilde{K})$ through the isomorphisms of $\mathfrak{A}_{\tilde{K}}$ onto $\mathfrak{A}_{\tilde{K},\sigma}$ established above. Namely for $\mu \in \tilde{\mathfrak{G}}$,

$$\begin{aligned} \chi_\sigma(\mu) &= \chi(\sigma \circ \iota^{-1} \circ \mu \circ \iota \circ \sigma^{-1}) \\ &= \chi(\sigma^{-1} \cdot (\iota^{-1} \circ \mu \circ \iota) \cdot \sigma) \\ &= \chi^{\sigma^{-1}}(\iota^{-1} \circ \mu \circ \iota) . \end{aligned}$$

For a character χ of $\mathfrak{G}_K$, and for $x \in K \otimes_F F_{\mathfrak{p}}$ with $\eta_{\mathfrak{p}}(x) = (x_\sigma)_{\sigma \in S} \in \prod_\sigma \tilde{K}_\sigma$, the canonical pairing $(\chi, x)_{K,\mathfrak{p}}$ is defined by

$$(\chi, x)_{K,\mathfrak{p}} = \prod_{\sigma \in S} (\chi_\sigma, x_\sigma)_{\tilde{K}} ,$$

where each $(\chi_\sigma, x_\sigma)_{\tilde{K}}$ is the canonical pairing of local class field theory for $\tilde{K}_\sigma = \tilde{K}$.

Let λ be an element of $\mathfrak{G}_F$. For $x \in K \otimes_F F_{\mathfrak{p}}$ with $\eta_{\mathfrak{p}}(x) = (x_\sigma)_{\sigma \in S}$, we had $\eta_{\mathfrak{p}}(x^\lambda) = (y_\sigma)$ with $y_\sigma = (x_{[\sigma\lambda^{-1}]})^{\rho(\sigma,\lambda)}$. On the other hand, $(\chi^\lambda)_\sigma(\mu) = \chi^{\lambda\sigma^{-1}}(\iota^{-1} \circ \mu \circ \iota)$ for $\mu \in \tilde{\mathfrak{G}}$. Since $\sigma\lambda^{-1} = \zeta(\sigma, \lambda)[\sigma\lambda^{-1}]$,

$$(\chi^\lambda)_\sigma(\mu) = \chi^{[\sigma\lambda^{-1}]^{-1}\zeta(\sigma,\lambda)^{-1}}(\iota^{-1} \circ \mu \circ \iota)$$

$$
\begin{aligned}
&= \chi^{[\sigma\lambda-1]^{-1}}(\zeta(\sigma,\lambda)^{-1}\cdot(\iota^{-1}\circ\mu\circ\iota)\cdot\zeta(\sigma,\lambda)) \\
&= \chi^{[\sigma\lambda-1]^{-1}}(\zeta(\sigma,\lambda)\circ\iota^{-1}\circ\mu\circ\iota\circ\zeta(\sigma,\lambda)^{-1}) \\
&= \chi^{[\sigma\lambda-1]^{-1}}(\iota^{-1}\circ\rho(\sigma,\lambda)^{-1}\circ\mu\circ\rho(\sigma,\lambda)\circ\iota) \\
&= \chi_{[\sigma\lambda-1]}(\rho(\sigma,\lambda)^{-1}\circ\mu\circ\rho(\sigma,\lambda)) \\
&= \chi_{[\sigma\lambda-1]}(\rho(\sigma,\lambda)\cdot\mu\cdot\rho(\sigma,\lambda)^{-1}) \\
&= (\chi_{[\sigma\lambda-1]})^{\rho(\sigma,\lambda)}(\mu)\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\chi^{\lambda}, x^{\lambda})_{K,\mathfrak{p}} &= \prod_{\sigma\in S}((\chi^{\lambda})_{\sigma}, y_{\sigma})_{\tilde{K}} \\
&= \prod_{\sigma\in S}((\chi_{[\sigma\lambda-1]})^{\rho(\sigma,\lambda)}, (x_{[\sigma\lambda-1]})^{\rho(\sigma,\lambda)})_{\tilde{K}}\,.
\end{aligned}
$$

Since $\rho(\sigma,\lambda)\in\mathfrak{G}_{\mathfrak{p}}=\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$, and since $\tilde{K}$ is a Galois extension of $F_{\mathfrak{p}}$, we have $\tilde{K}^{\rho(\sigma,\lambda)}=\tilde{K}$.
Therefore

$$
((\chi_{[\sigma\lambda-1]})^{\rho(\sigma,\lambda)}, (x_{[\sigma\lambda-1]})^{\rho(\sigma,\lambda)})_{\tilde{K}} = (\chi_{[\sigma\lambda-1]}, x_{[\sigma\lambda-1]})_{\tilde{K}}\,.
$$

(See Weil [8], p 223, Cor. 5.) This shows that

$$
(\chi^{\lambda}, x^{\lambda})_{K,\mathfrak{p}} = (\chi, x)_{K,\mathfrak{p}}\,.
$$

Since this is true for any prime divisor of F,

$$
(\chi^{\lambda}, x^{\lambda})_{K} = (\chi, x)_{K}
$$

for $x\in K_A^{\times}$, $\lambda\in\mathfrak{G}_F$ and a character χ of $\mathfrak{G}_K$. Here $(\chi, x)_K$ is the canonical pairing of K.

The canonical morphism

$$
[\cdot, K]:\ K_A^{\times}\longrightarrow\mathfrak{A}_K=\mathfrak{G}_K/\mathfrak{G}_K'=\mathrm{Gal}(K_{ab}/K)
$$

is defined so that

$$
(\chi, x)_K = \chi([x, K])
$$

for any $x\in K_A^{\times}$ and any χ. For each $[x, K]\in\mathfrak{A}_K$, choose $[x, K]^*\in\mathfrak{G}_K$ so that $[x, K]^*$ modulo $\mathfrak{G}_K'$ is $[x, K]$. Then for $\lambda\in\mathfrak{G}_F$,

$$
(\chi^{\lambda}, x^{\lambda})_K = \chi^{\lambda}([x^{\lambda}, K]^*) = \chi(\lambda\cdot[x^{\lambda}, K]^*\cdot\lambda^{-1})\,.
$$

Therefore

$$
\chi([x, K]) = \chi(\lambda\cdot[x^{\lambda}, K]^*\cdot\lambda^{-1})
$$

for any χ. This implies that

https://doi.org/10.1017/S0027763000019085 Published online by Cambridge University Press

$$[x, K] = \lambda \cdot [x^{\lambda}, K]^{*} \cdot \lambda^{-1} \text{ modulo } \mathfrak{G}'_K\,.$$

Equivalent to say,

$$\lambda^{-1} \cdot [x, K]^{*} \cdot \lambda \equiv [x^{\lambda}, K]^{*} \qquad \text{modulo } \mathfrak{G}'_K\,.$$

This is what Theorem 1 claims. The proof is done.

Reference

[1] E. Artin, Idealklassen in Oberkörpern und allgemeine Reziprozitätsgesetze, Abh. Math. Sem. Hamburg **7** (1930).
[2] C. Chevalley, Deux théorèmes d'arithmétique, J. Math. Soc. Japan **3** (1951).
[3] Ph. Furtwängler, Beweis des Hauptidealsatzes für Klassenkörper algebraischer Zahlkörper, Abh. Math. Sem. Hamburg **7** (1930).
[4] J. Herbrand, Sur les théorèmes du genre principal et des ideaux principaux, Abh. Math. Sem. Hamburg **9** (1933).
[5] S. Iyanaga, Über den allgemeinen Hauptidealsatz, Jap. J. Math. **7** (1930).
[6] K. Miyake, Models of certain automorphic function fields, Acta Math. **126** (1971).
[7] G. Shimura, On canonical models of arithmetic quotient of bounded symmetric domains II, Ann. of Math. **92** (1970).
[8] A. Weil, Basic Number Theory, Springer-Verlag, Berlin (1967).

Nagoya University