Appendix A

Functional integration

In this appendix we outline the basis of functional methods which are employed in the text. Path-integral techniques appear at first sight to be rather formal and abstract. However, it is remarkable how easy it is to obtain practical information from them. Very often they add insight or new results, which are difficult to obtain from canonical quantization.

A–1 Quantum-mechanical formalism

Before attempting to address the full field-theoretic formalism we first review the application of such techniques within the more familiar setting of nonrelativistic quantum mechanics in one spatial dimension. Unless otherwise specified we hereafter set $\hbar = 1$.

Path-integral propagator

Simply stated, the functional integral is an alternative way of evaluating the quantity

$$D(x_f, t_f; x_i, t_i) = \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle \equiv \langle x_f, t_f | x_i, t_i \rangle. \tag{1.1}$$

This matrix element, usually called the *propagator*, is the amplitude for a particle located at position x_i and time t_i to be found at position x_f and subsequent time t_f. The propagator can also be written as a functional integral

$$D(x_f, t_f; x_i, t_i) = \int \mathcal{D}[x(t)]\, e^{iS[x(t)]}, \tag{1.2}$$

where the integration is over all histories (i.e. paths) of the system which begin at spacetime point x_i, t_i and end at x_f, t_f. The paths are identified by specifying the coordinate x at each intermediate time t, so that the symbol $\int \mathcal{D}[x(t)]$ represents

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a sum over all such trajectories. The contribution of each path to the integral is weighted by the exponential involving the classical action

$$S[x(t)] = \int_{t_i}^{t_f} dt \left(\frac{m}{2} \dot{x}^2(t) - V(x(t)) \right), \tag{1.3}$$

which, since it depends on the detailed shape of $x(t)$, is a functional of the trajectory.[1] Although the validity of the path-integral representation, Eq. (1.2), may not be obvious, its correctness can be verified by beginning with Eq. (1.1) and breaking the time interval $t_f - t_i$ into N discrete steps of size $\epsilon = (t_f - t_i)/N$. Using the completeness relation

$$\mathbf{1} = \int_{-\infty}^{\infty} dx_n \, |x_n\rangle\langle x_n|,$$

one can write Eq. (1.1) as

$$D(x_f, t_f; x_i, t_i) = \int_{-\infty}^{\infty} dx_{N-1} \cdots \int_{-\infty}^{\infty} dx_1 \\ \langle x_N | e^{-i\epsilon H} | x_{N-1} \rangle \langle x_{N-1} | e^{-i\epsilon H} | x_{N-2} \rangle \cdots \langle x_1 | e^{-i\epsilon H} | x_0 \rangle, \tag{1.4}$$

where $x_0 \equiv x_i$, $x_N \equiv x_f$. In the limit of large N the time slices become infinitesimal, implying

$$\begin{aligned} \langle x_\ell | e^{-iH\epsilon} | x_{\ell-1} \rangle &= \langle x_\ell | e^{-i\epsilon\left(\frac{p^2}{2m} + V(x)\right)} | x_{\ell-1} \rangle \\ &= e^{-i\epsilon V(x_\ell)} \langle x_\ell | e^{-i\epsilon \frac{p^2}{2m}} | x_{\ell-1} \rangle + \mathcal{O}(\epsilon^2). \end{aligned} \tag{1.5}$$

Inserting a complete set of momentum states and introducing a convergence factor $e^{-\kappa p^2}$ for the resulting integral over momentum, we have

$$\begin{aligned} \langle x_\ell | e^{-i\epsilon \frac{p^2}{2m}} | x_{\ell-1} \rangle &= \lim_{\kappa \to 0} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \, e^{ip\left(x_\ell - x_{\ell-1}\right) - i\epsilon p^2/2m - \kappa p^2} \\ &= \sqrt{\frac{m}{2\pi i \epsilon}} \, e^{i \frac{m}{2\epsilon} \left(x_\ell - x_{\ell-1}\right)^2}. \end{aligned} \tag{1.6}$$

[1] It is important to understand the difference between the concept of a function and that of a functional. A real-valued function involves the mapping from the space of real numbers onto themselves

$$\text{reals} \longleftarrow [f : \text{reals}].$$

On the other hand, a real-valued functional such as $S[x(t)]$ is a mapping from the space of functions $x(t)$ onto real numbers

$$\text{reals} \longleftarrow [S : x(t)].$$

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Upon taking the continuum limit we obtain

$$D\left(x_f, t_f; x_i, t_i\right) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}}\left[\prod_{n=1}^{N-1}\int_{-\infty}^{\infty} dx_n\right] e^{i\sum_{\ell=1}^{N}\left(\frac{m}{2}\frac{(x_\ell - x_{\ell-1})^2}{\epsilon} - \epsilon V(x_\ell)\right)}. \tag{1.7}$$

It is clear then that we can make connection with Eq. (1.2) by identifying each path with the sequence of locations $(x_1, \ldots, x_{N-1})$ at times $\epsilon, 2\epsilon, \ldots, (N-1)\epsilon$. Integration over these intermediate positions is what is meant by the symbol $\int \mathcal{D}[x(t)]$, viz.

$$\int \mathcal{D}[x(t)] \equiv \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{N/2}\prod_{n=1}^{N-1}\int_{-\infty}^{\infty} dx_n. \tag{1.8}$$

Each trajectory has an associated exponential factor $e^{iS[x(t)]}$, where the quantity

$$S[x(t)] = \sum_{\ell=1}^{N}\epsilon\left(\frac{m}{2}\frac{(x_\ell - x_{\ell-1})^2}{\epsilon^2} - V(x_\ell)\right) \tag{1.9}$$

becomes the classical action in the limit $N \to \infty$. We have thus demonstrated the equivalence of the operator (Eq. (1.1)) and path-integral (Eq. (1.2)) representations of the propagator.[2] It is important to realize that in the latter all quantities are *classical* – no operators are involved.

The path-integral propagator contains a great deal of information, and there are a variety of techniques for extracting it. For example, the spatial wavefunctions and energies are all present, as can be seen by inserting a complete set of energy eigenstates $\{|\, n\rangle\}$ into the definition of the propagator given in Eq. (1.1),

$$D(x_f, t_f; x_i, t_i) = \sum_{n=0}^{\infty}\psi_n(x_f)\psi_n^*(x_i)e^{-iE_n(t_f - t_i)}. \tag{1.10}$$

[2] For completeness, we note that by combining Eqs. (1.5)–(1.8), one can also write the propagator in a corresponding hamiltonian path-integral representation

$$\begin{aligned} D(x_f, t_f; x_i, t_i) &= \lim_{N\to\infty}\int \frac{dp_0}{2\pi}dx_1\frac{dp_1}{2\pi}dx_2\cdots dx_{N-1}\frac{dp_{N-1}}{2\pi} \\ &\quad\times e^{i\sum_{\ell=1}^{N}\left(p_\ell(x_\ell - x_{\ell-1}) - \left(\frac{p_\ell^2}{2m} + V(x_\ell)\right)\epsilon\right)} \\ &\equiv \int \mathcal{D}[x(t)]\,\mathcal{D}[p(t)]\,e^{i\int dt\,(p\dot{x} - H(p,x))}. \end{aligned}$$

This form is useful when one is dealing with non-cartesian variables or with constrained systems.

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In addition, other quantum-mechanical amplitudes can be found by use of the identity[3]

$$\begin{aligned}&\langle x_f, t_f | T\left(x(t_1)\cdots x(t_n)\right)|x_i, t_i\rangle \\ &\quad = \int \mathcal{D}[x(t)]\; x(t_1)\cdots x(t_n)\; e^{i\int_{t_i}^{t_f} dt\left(\frac{m}{2}\dot{x}^2(t) - V(x(t))\right)},\end{aligned} \tag{1.11}$$

where 'T' is the time-ordered product.

External sources

An important technique involves the addition of an external source. In the quantum-mechanical case this is added like an arbitrary external 'force' $j(t)$,

$$\langle x_f, t_f | x_i, t_i\rangle_{j(t)} = \int \mathcal{D}[x(t)]\, e^{i\int_{t_i}^{t_f} dt\left[\frac{m}{2}\dot{x}^2(t) - V(x(t)) + j(t)x(t)\right]}. \tag{1.12}$$

The amplitude is now a functional of the source $j(t)$. From this quantity one can obtain all matrix elements using *functional differentiation*, which can be defined by means of the relation

$$j(t) = \int dt'\delta(t-t')j(t') \quad \Rightarrow \quad \frac{\delta j(t)}{\delta j(t')} = \delta(t-t') \tag{1.13}$$

and yields the result we seek,

$$\begin{aligned}&\langle x_f, t_f | T\left(x(t_1)\cdots x(t_n)\right)|x_i, t_i\rangle \\ &\quad = (-i)^n \frac{\delta^n}{\delta j(t_1)\ldots\delta j(t_n)}\langle x_f, t_f | x_i, t_i\rangle_{j(t)}\Big|_{j=0}.\end{aligned} \tag{1.14}$$

For many applications it is necessary only to consider matrix elements between the lowest energy states (*vacuum*) of the quantum system. This can be

[3] One can prove this relation by choosing a particular ordering, say

$$t_i < t_1 < t_2 < \cdots < t_f,$$

and noting that

$$\begin{aligned}&\langle x_f, t_f | T\left(x(t_1)\cdots x(t_n)\right)|x_i, t_i\rangle = \langle x_f, t_f | x(t_n)x(t_{n-1})\cdots x(t_1)|x_i, t_i\rangle \\ &\quad = \prod_{k=1}^{n}\int_{-\infty}^{\infty} dx_k \langle x_f, t_f | x_n, t_n\rangle\, x_n \langle x_n, t_n | x_{n-1}, t_{n-1}\rangle x_{n-1}\cdots x_1\langle x_1, t_1|, x_i, t_i\rangle,\end{aligned}$$

where we have used completeness and have defined $x_k = x(t_k)$ $(k = 1, 2, \ldots, n)$. The amplitudes $\langle x_k, t_k | x_{k-1}, t_{k-1}\rangle$ are simply free propagators as in Eq. (1.1), and can be evaluated by means of the time-slice methods outlined above. Thus, the above expression is identical to the right-hand side of Eq. (1.11). In the case of a different time ordering the same result goes through provided one always places the times such that the later time always appears to the left of an earlier counterpart. However, this is simply the definition of the time-ordered product and hence the proof holds in general.

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accomplished in either of two ways. First, it is possible to explicitly project out this amplitude using the ground-state wavefunction

$$\langle x, t|0\rangle = \psi_0(x)e^{-iE_0 t}, \tag{1.15}$$

which implies

$$\langle 0|T\,(x(t_1)\cdots x(t_n))\,|0\rangle \equiv \int_{-\infty}^{\infty} dx_f \int_{-\infty}^{\infty} dx_i\; \psi_0^*(x_f)\, e^{iE_0 t_f} \langle x_f, t_f|T\,(x(t_1)\cdots x(t_n))\,|x_i, t_i\rangle \psi_0(x_i)\, e^{-iE_0 t_i}. \tag{1.16}$$

However, this amplitude can be isolated in a simpler fashion. If we consider the amplitude $\langle x_f, t_f|x_i, t_i\rangle$ in the unphysical limit $t_f \to -i\tau_f$, $t_i \to +i\tau_i$ we find for large $\tau_f + \tau_i$,

$$\langle x_f, t_f|x_i, t_i\rangle \to \sum_n \psi_n(x_f)\psi_n^*(x_i)\, e^{-E_n(\tau_f+\tau_i)} \underset{\tau_f+\tau_i\to\infty}{\longrightarrow} \psi_0(x_f)\psi_0^*(x_i)\, e^{-E_0(\tau_f+\tau_i)}. \tag{1.17}$$

Generalizing, we have

$$\lim_{\substack{t_f\to -i\infty \\ t_i\to i\infty}} \frac{e^{iE_0(t_f-t_i)}}{\psi_0(x_f)\psi_0^*(x_i)} \langle x_f, t_f|T\,(x(t_1)\cdots x(t_n))\,|x_i, t_i\rangle = \langle 0|T\,(x(t_1)\cdots x(t_n))\,|0\rangle \tag{1.18}$$

which is operationally a much simpler procedure than Eq. (1.16).

The generating functional

We may combine all these techniques in the so-called *generating functional*, defined by

$$Z[j] = \lim_{\substack{t_f\to -i\infty \\ t_i\to +i\infty}} \langle x_f, t_f|x_i, t_i\rangle_{j(t)}. \tag{1.19}$$

This has the path-integral representation

$$Z[j] = \lim_{\substack{t_f\to -i\infty \\ t_i\to i\infty}} \int \mathcal{D}[x(t)]\, e^{i\int_{t_i}^{t_f} dt\, \left(\frac{m}{2}\dot{x}^2(t) - V(x(t)) + x(t)j(t)\right)}. \tag{1.20}$$

Noting that for $t_i = i\tau_i$ and $t_f = -i\tau_f$,

$$\langle x_f, t_f|x_i, t_i\rangle \to \psi_0(x_f)\psi_0^*(x_i)\, e^{-E_0(\tau_f+\tau_i)} \underset{\tau_i,\tau_f\to\infty}{\longrightarrow} Z[0], \tag{1.21}$$

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we find that ground-state matrix elements as in Eq. (1.16) can be given in terms of the generating functional $Z[j]$,

$$\langle 0|T\left(x(t_1)\cdots x(t_n)\right)|0\rangle = (-i)^n \frac{1}{Z[0]}\,\frac{\delta^n}{\delta j(t_1)\cdots\delta j(t_n)} Z[j]\Big|_{j=0}. \tag{1.22}$$

It often happens with path integrals that formal procedures are best defined, as above, by using the imaginary-time limits $t \to \pm i\infty$. However, in practice it is common instead to express the theory in terms of Minkowski spacetime. Thus, the generating functional will involve the real-time limits $t \to \pm\infty$. Does the dominance of the ground-state contribution, as in Eq. (1.21), continue to hold? The answer is 'yes'. At an intuitive level, one understands this as a consequence of the rapid variation of the phase $e^{-iE_n t}$ in the limit $t \to \infty$. The more rapid phase variation accompanying the increased energy E_n of any excited state washes out its contribution relative to that of the ground state. In a more formal sense, the real-time limit is defined by an analytic continuation from imaginary time. To properly define the continuation, one must introduce appropriate '$i\epsilon$' factors into the Green's functions in order to deal with various singularities. Beginning with the next section, we shall often employ the Minkowski formulation and thus explicitly display the '$i\epsilon$' terms in our formulae.

The prescription given in Eq. (1.22) represents a powerful but formal procedure for the generation of matrix elements in the presence of an arbitrary potential $V(x)$. Unfortunately, an explicit evaluation is no more generally accessible via this route than is an exact solution of the Schrödinger equation. In practice, aside from an occasional special case, the only path integrals which can be performed exactly are those in quadratic form. However, approximation procedures are generally available.

One of the most common of these is perturbation theory. Suppose that the full potential $V(x)$ is the sum of two parts $V_1(x)$ and $V_2(x)$, where $V_1(x)$ is such that the generating functional can be evaluated exactly while $V_2(x)$ is in some sense small. Then we can write

$$\begin{aligned}
Z[j] &= \lim_{\substack{t_f\to -i\infty \\ t_i \to i\infty}} \int \mathcal{D}[x(t)]e^{i\int_{t_i}^{t_f} dt\left[\frac{m}{2}\dot{x}^2(t) - V_1(x(t)) - V_2(x(t)) + x(t)j(t)\right]} \\
&= \lim_{\substack{t_f\to -i\infty \\ t_i \to i\infty}} e^{-i\int_{t_i}^{t_f} dt\; V_2\left(-i\frac{\delta}{\delta j(t)}\right)} Z^{(0)}[j] \\
&= \lim_{\substack{t_f\to -i\infty \\ t_i \to i\infty}} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\left[\int_{t_i}^{t_f} dt\; V_2\left(-i\frac{\delta}{\delta j(t)}\right)\right]^n Z^{(0)}[j],
\end{aligned} \tag{1.23}$$

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where

$$Z^{(0)}[j] = \lim_{\substack{t_f \to -i\infty \\ t_i \to i\infty}} \int \mathcal{D}[x(t)]\, e^{i \int_{t_i}^{t_f} dt \left[\frac{m}{2}\dot{x}^2(t) - V_1(x(t)) + x(t)j(t)\right]} \tag{1.24}$$

is the generating functional for $V_1(x)$ alone. Obviously, Eq. (1.23) defines an expansion for $Z[j]$ in powers of the perturbing potential $V_2(x)$.

A–2 The harmonic oscillator

It is useful to interrupt our formal development by considering the harmonic oscillator as an example of these methods. This treatment turns out to reproduce known oscillator properties with the use of functional methods, which are very similar to corresponding field-theory techniques.

It is most convenient to address the problem by employing Fourier transforms,

$$x(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi}\, e^{-iEt} \tilde{x}(E), \tag{2.1}$$

whereby for $t_i = -\infty$ and $t_f = +\infty$,

$$\begin{aligned} S^j[x(t)] &= \int_{-\infty}^{\infty} dt \left(\frac{m}{2}\dot{x}^2(t) - \frac{m\omega^2}{2} x^2(t) + x(t)j(t) \right) \\ &= \int_{-\infty}^{\infty} \frac{dE}{2\pi} \left[\frac{m}{2}(E^2 - \omega^2)\tilde{x}(E)\tilde{x}(-E) + \frac{1}{2}\tilde{j}(E)\tilde{x}(-E) + \frac{1}{2}\tilde{j}(-E)\tilde{x}(E) \right] \\ &= \int_{-\infty}^{\infty} \frac{dE}{2\pi} \left\{ \frac{m}{2}(E^2 - \omega^2)\tilde{x}'(E)\tilde{x}'(-E) - \frac{1}{2m}\tilde{j}(E) \frac{1}{E^2 - \omega^2 + i\epsilon} \tilde{j}(-E) \right\}, \end{aligned} \tag{2.2}$$

with the definition $\tilde{x}'(E) \equiv \tilde{x}(E) + \tilde{j}(E)/(mE^2 - m\omega^2 + i\epsilon)$. An infintesimal imaginary part $i\epsilon$ has been introduced to make the integration precise. Upon taking the inverse Fourier transform

$$x'(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{x}'(E) = x(t) + \frac{1}{m} \int_{-\infty}^{\infty} dt'\, D(t - t') j(t'), \tag{2.3}$$

where

$$D(t - t') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iE(t-t')} \frac{1}{E^2 - \omega^2 + i\epsilon} = -\frac{i}{2\omega} e^{-i\omega|t'-t|}, \tag{2.4}$$

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we have

$$S^j[x(t)] = \int_{-\infty}^{\infty} dt \left(\frac{m}{2}\dot{x}'^2(t) - \frac{m\omega^2}{2} x'^2(t) \right) \\ - \frac{1}{2m} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, j(t) D(t-t') j(t'). \tag{2.5}$$

Finally, changing variables from $x(t)$ to $x'(t)$ we obtain the generating functional

$$Z[j] = \int \mathcal{D}\left[x'(t)\right] e^{i \int_{-\infty}^{\infty} dt \left(\frac{m}{2}\dot{x}'^2(t) - \frac{m\omega^2}{2} x'^2(t) \right)} \\ \times e^{-\frac{i}{2m} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, j(t) D(t-t') j(t')} \\ = Z[0] e^{-\frac{i}{2m} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, j(t) D(t-t') j(t')}. \tag{2.6}$$

Note that the above change of variables has left the measure invariant ($\int \mathcal{D}[x(t)] = \int \mathcal{D}\left[x'(t)\right]$).

We can use this result to calculate arbitrary oscillator matrix elements. Thus for $t_2 > t_1$, we have for the ground state

$$\langle 0| T\left(x(t_2)x(t_1)\right) |0\rangle = (-i)^2 \frac{1}{Z[0]} \left. \frac{\delta^2 Z[j]}{\delta j(t_2) \delta j(t_1)} \right|_{j=0} \\ = \frac{i}{m} D(t_2 - t_1) = \frac{e^{-i\omega(t_2 - t_1)}}{2m\omega}, \tag{2.7}$$

which, in the limit $t_2 \to t_1$, reproduces the familiar result

$$\langle 0|x^2|0\rangle = \frac{1}{2m\omega}. \tag{2.8}$$

Although only ground-state expectation values have been treated thus far, it is also possible to deal with arbitrary oscillator matrix elements with this formalism by generalizing the operator relation

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle, \tag{2.9}$$

where

$$a^\dagger = \sqrt{\frac{m\omega}{2}} \left(x - \frac{i}{m\omega} p \right) \tag{2.10}$$

is the usual creation operator. First, however, it is convenient to use the classical relation $p = m\dot{x}$ to rewrite the operator $a^\dagger$ as

$$a^\dagger = \sqrt{\frac{m\omega}{2}} \left(1 - \frac{i}{\omega} \frac{d}{dt} \right) x(t). \tag{2.11}$$

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In a simple application, we calculate that

$$\begin{aligned}
\langle 0|x|1\rangle &= \lim_{t_2\to t_1^+}\sqrt{\frac{m\omega}{2}}\left(1-\frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\langle 0|x(t_2)x(t_1)|0\rangle \\
&= \lim_{t_2\to t_1^+}(-i)^2\sqrt{\frac{m\omega}{2}}\left(1-\frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\frac{1}{Z[0]}\frac{\delta^2}{\delta j(t_2)\delta j(t_1)}Z[j] \\
&= \lim_{t_2\to t_1^+}\sqrt{\frac{m\omega}{2}}\left(1-\frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\frac{i}{m}D(t_2-t_1) \\
&= \frac{1}{\sqrt{2m\omega}},
\end{aligned} \tag{2.12}$$

which agrees with the result obtained by more conventional means,

$$\langle 0|x|1\rangle = \sqrt{\frac{1}{2m\omega}}\langle 0|\left(a+a^\dagger\right)|1\rangle = \frac{1}{\sqrt{2m\omega}}. \tag{2.13}$$

More complicated matrix elements can also be found, as with

$$\begin{aligned}
\langle 1|x^2|1\rangle &= \frac{m\omega}{2}\lim_{\substack{t_2\to t^-\\ t_1\to t^+}}\left(1+\frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\left(1-\frac{i}{\omega}\frac{\partial}{\partial t_2}\right)\langle 0|x(t_1)x^2(t)x(t_2)|0\rangle \\
&= \frac{(-i)^4}{Z[0]}\lim_{\substack{t_2\to t'^-,t_1\to t^+\\ t'\to t^-}}\left(\frac{m\omega}{2}\right)\left(1+\frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\left(1-\frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \\
&\qquad \frac{\delta^4}{\delta j(t_1)\delta j(t_2)\delta j(t)\delta j(t')}Z[j]\bigg|_{j=0} \\
&= \frac{m\omega}{2}\left(\frac{i}{m}\right)^2\lim_{\substack{t_2\to t^-\\ t_1\to t^+}}\left(1+\frac{i}{\omega}\frac{\partial}{\partial t_1}\right)\left(1-\frac{i}{\omega}\frac{\partial}{\partial t_2}\right) \\
&\quad\times\left[D(t_1-t_2)D(0)+2D(t_1-t)D(t-t_2)\right] = \frac{3}{2m\omega},
\end{aligned} \tag{2.14}$$

which agrees with

$$\langle 1|x^2|1\rangle = \frac{1}{2m\omega}\langle 1|\left(a+a^\dagger\right)\left(a+a^\dagger\right)|1\rangle = \frac{3}{2m\omega}. \tag{2.15}$$

In this manner, arbitrary oscillator matrix elements can be reduced to ground-state expectation values, which in turn can be determined from the generating functional $Z[j]$. The ground-state amplitude in the presence of an arbitrary source $j(t)$ contains *all* the information about the harmonic oscillator.

One should note the analogy of the above methods to those of quantum field theory. The 'one-particle' matrix elements involving $|1\rangle$ have been reduced to vacuum matrix elements by use of Eq. (2.9). This is similar to the LSZ reduction of fields

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(see App. B–3). As a result, all that one needs to deal with are the vacuum Green's functions. The generating functional is ideal for this purpose, as we shall see in our development of functional techniques in field theory.

A–3 Field-theoretic formalism

One of the advantages of the functional approach to quantum mechanics is that it can be taken over with little difficulty to quantum field theory. An important difference is that instead of trajectories $x(t)$, which pick out a particular point in space at a given time, one must deal with fields $\varphi(x,t)$ which are defined at *all* points in space at a given time t. Also, instead of a sum $\int \mathcal{D}[x(t)]$ over trajectories one has instead a sum $\int [d\varphi(x)]$ over all possible field configurations. Nevertheless, the analogy is rather direct.

Path integrals with fields

The formal transition from quantum mechanics to field theory can be accomplished by dividing spacetime, both time and space, into a set of tiny four-dimensional cubes of volume $\delta t\, \delta x\, \delta y\, \delta z$. Within each cube one takes the field

$$\varphi\left(x_i, y_j, z_k, t_\ell\right) \tag{3.1}$$

as a constant. Derivatives are defined in terms of differences between fields in neighboring blocks, e.g.,

$$\partial_t \varphi\big|_{x_i,y_j,x_k,t_l} \simeq \frac{1}{\delta t}\left(\varphi\left(x_i, y_j, z_k, t_l + \delta t\right) - \varphi\left(x_i, y_j, z_k, t_l\right)\right). \tag{3.2}$$

The lagrangian is easily found,

$$\mathcal{L}(\varphi, \partial_\mu \varphi)\big|_{x_i,y_j,z_k,t_l} \simeq \mathcal{L}\left(\varphi\left(x_i, y_j, z_k, t_l\right), \partial_\mu \varphi\left(x_i, y_j, z_k, t_l\right)\right), \tag{3.3}$$

and the action is written as

$$S \simeq \sum_{i,j,k,l} \delta x \delta y \delta z \delta t\ \mathcal{L}\left(\varphi\left(x_i, y_j, z_k, t_l\right), \partial_\mu \varphi\left(x_i, y_j, z_k, t_l\right)\right). \tag{3.4}$$

The field-theory analog of the path integral can then be constructed by summing over all possible field values in each cell

$$D \sim \prod_{i,j,k,l} \int_{-\infty}^{\infty} d\varphi\left(x_i, y_j, z_k, t_l\right) e^{iS\left[\varphi(x_i,y_k,z_k,t_l),\ \partial_\mu \varphi\left(x_i,y_j,z_k,t_l\right)\right]}. \tag{3.5}$$

Formally, in the limit in which the cell size is taken to zero, this is written as

$$\int [d\varphi(x)]\, e^{iS[\varphi(x),\ \partial_\mu \varphi(x)]}. \tag{3.6}$$

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By analogy with the quantum mechanical case (cf. Eq. (1.18)), it is clear that, since the time integration for S in Eq. (3.4) is from $-\infty$ to $+\infty$, this amplitude is to be identified with the vacuum-to-vacuum amplitude of the field theory,

$$\langle 0|0\rangle = N \int [d\varphi(x)]\, e^{iS[\varphi(x),\partial_\mu\varphi(x)]}. \tag{3.7}$$

Generally, quantum field theory is formulated in terms of vacuum expectation values of time-ordered products of the fields

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0|T\,(\varphi(x_1)\cdots\varphi(x_n))\,|0\rangle \tag{3.8}$$

i.e., the *Green's functions* of the theory. By analogy with the quantum-mechanical case, one is naturally led to the path-integral definition

$$G^{(n)}(x_1, \ldots, x_n) = N \int [d\varphi(x)]\, \varphi(x_1)\cdots\varphi(x_n) e^{iS[\varphi(x),\partial_\mu\varphi(x)]}, \tag{3.9}$$

where N is a normalization factor. Again we emphasize that all quantities here are c numbers and no operators are involved. In terms of a functional representation, we then have from Eqs. (3.7), (3.9),

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\int [d\varphi(x)]\;\varphi(x_1)\cdots\varphi(x_n) e^{iS[\varphi(x),\partial_\mu\varphi(x)]}}{\int [d\varphi(x)]\, e^{iS[\varphi(x),\partial_\mu\varphi(x)]}}. \tag{3.10}$$

Generating functional with fields

These Green's functions can most easily be evaluated by use of the generating functional

$$Z[j] = N \int [d\varphi(x)]\, e^{\left(iS[\varphi(x),\partial_\mu\varphi(x)]+i\int d^4x\, j(x)\varphi(x)\right)} \tag{3.11}$$

Functional differentiation for fields is defined by

$$\frac{\delta\varphi(y)}{\delta\varphi(x)} = \delta^{(4)}(x-y), \tag{3.12}$$

which lets us obtain (cf. Eq. (3.9))

$$G^{(n)}(x_1, \ldots, x_n) = (-i)^n \frac{1}{Z[0]}\, \frac{\delta^n}{\delta j(x_1)\cdots\delta j(x_n)} Z[j]\bigg|_{j=0}. \tag{3.13}$$

As an example of this formalism consider the free scalar field theory

$$\mathcal{L}^{(0)}(x) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2. \tag{3.14}$$

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In general, we have

$$Z^{(0)}[j] = Z^{(0)}[0] \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\prod_{k=1}^{n} \int_{-\infty}^{\infty} dx_k \; j(x_k) \right] G^{(n)}(x_1, x_2, \ldots, x_n), \tag{3.15}$$

where the generating functional $Z^{(0)}[j]$ is given by

$$Z^{(0)}[j] = N \int [d\varphi(x)]\, e^{i \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + j\varphi \right)}. \tag{3.16}$$

There exist two common ways in which to handle the issue of convergence for such functional integrals, i.e., to ensure acceptable behavior for large φ^2. One is to give the mass an infinitesimal negative imaginary part, $m^2 \to m^2 - i\epsilon$. This is the approach we shall employ in the discussion to follow. The second involves a continuation to euclidean space by means of $t \to -i\tau$ wherein the functional integral becomes

$$\langle 0|0\rangle = N \int [d\varphi(x)] e^{-\int d^4x_E \left(\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{m^2}{2}\varphi^2 \right)}, \tag{3.17}$$

and is now convergent due to the negative argument of the exponential. Continuation back to Minkowski space then yields the desired result.

Integrating by parts, we have from Eq. (3.16)

$$\begin{aligned} Z^{(0)}[j] &= N \int [d\varphi]\, e^{-i \int d^4x \left[\frac{1}{2}\varphi(x) O_x \varphi(x) - \varphi(x) j(x) \right]} \\ &= N \int \left[d\varphi' \right] e^{-\frac{i}{2}\left[\int d^4x \; \varphi'(x) O_x \varphi'(x) + \int d^4x \int d^4y \; j(x) \Delta_F(x-y) j(y) \right]} \end{aligned} \tag{3.18}$$

where $O_x = \Box_x + m^2 - i\epsilon$ and

$$\begin{aligned} \varphi'(x) &= \varphi(x) + \int d^4y \; \Delta_F(x-y) j(y), \\ i\Delta_F(x-y) &= \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}, \\ (\Box_x + m^2)\Delta_F(x-y) &= -\delta^{(4)}(x-y). \end{aligned} \tag{3.19}$$

Note that we have used invariance of the measure $\left(\int [d\varphi] = \int [d\varphi'] \right)$. Finally, we recognize a factor of $Z^{(0)}[0]$ in Eq. (3.18), thus leading to the expression

$$Z^{(0)}[j] = Z^{(0)}[0] e^{-\frac{i}{2} \int d^4x \int d^4y \; j(x) \Delta_F(x-y) j(y)}. \tag{3.20}$$

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We can now determine the Green's functions for the free field theory, e.g.,

$$G^{(2)}(x_1, x_2) = \frac{(-i)^2}{Z^{(0)}[0]} \frac{\delta^2}{\delta j(x_1)\delta j(x_2)} Z^{(0)}[j]\bigg|_{j=0} = i\Delta_F(x_1 - x_2),$$

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) &= \frac{(-i)^4}{Z^{(0)}[0]} \frac{\delta^4}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)} Z^{(0)}[j]\bigg|_{j=0} \\ &= G^{(2)}(x_1, x_2)G^{(2)}(x_3, x_4) + G^{(2)}(x_1, x_3)G^{(2)}(x_2, x_4) \\ &\quad + G^{(2)}(x_1, x_4)G^{(2)}(x_2, x_3). \end{aligned} \tag{3.21}$$

More interesting is the case of a self-interacting field theory for which the lagrangian becomes

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\text{int}}(\varphi) \equiv \mathcal{L}^{(0)}(\varphi) + \mathcal{L}_{\text{int}}(\varphi). \tag{3.22}$$

The theory is no longer exactly soluble, but one can find a perturbative solution by use of the generating functional

$$\begin{aligned} Z[j] &= N \int [d\varphi(x)]\, e^{i\int d^4x\ \left(\mathcal{L}^{(0)}(\varphi) + \mathcal{L}_{\text{int}}(\varphi) + j(x)\varphi(x)\right)} \\ &= N e^{i\int d^4x\ \mathcal{L}_{\text{int}}\left(-i\frac{\delta}{\delta j(x)}\right)} Z^{(0)}[j]. \end{aligned} \tag{3.23}$$

As before, the Green's functions of the theory are given by

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{Z[0]} \left[\prod_{k=1}^{n} \frac{-i\delta}{\delta j(x_k)}\right] e^{i\int d^4x\, \mathcal{L}_{\text{int}}\left(-i\frac{\delta}{\delta j(x)}\right)} Z^{(0)}[j]\bigg|_{j=0}. \tag{3.24}$$

For most purposes one requires only the *connected* portions of the Green's function, i.e., those diagrams which cannot be broken into two or more disjoint pieces. This is illustrated in Fig. A–1 which can be found by dividing the full Green's function

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0|T(\varphi(x_1)\cdots\varphi(x_n))|0\rangle \tag{3.25}$$

into products of connected particle sectors and dividing by the vacuum-to-vacuum amplitude $\langle 0|0\rangle$ in each sector.

Mathematically, one eliminates the disconnected diagrams by defining

$$Z[j] = e^{iW[j]}. \tag{3.26}$$

Then one can show that $W[j]$ is the generating functional for connected Green's functions,

$$iW[j] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n\ j(x_1)\cdots j(x_n) G^{(n)}_{\text{conn}}(x_1, \ldots, x_n), \tag{3.27}$$

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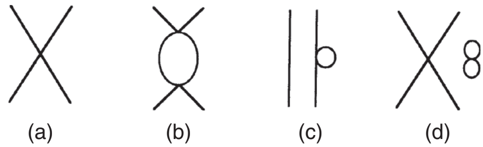

Fig. A–1 Contributions to the four-point Green's function in φ^4 theory: (a)–(b) connected, (c)–(d) disconnected.

where

$$G^{(n)}_{\rm conn}(x_1, \ldots, x_n) = (-i)^{n-1} \frac{\delta^n}{\delta j(x_1) \cdots \delta j(x_n)} W[j]\bigg|_{j=0}. \tag{3.28}$$

A–4 Quadratic forms

The most important example of a soluble path integral is one that is quadratic in the fields because, at least formally, it can be solved exactly.

Let us consider an action quadratic in the fields,

$$S = -\int d^4x\ \varphi(x) O \varphi(x), \tag{4.1}$$

where O is some differential operator which may contain fields distinct from φ within it. The general result for the quadratic path integral is given by

$$I_{\rm quad} = \int [d\varphi(x)] e^{-i\int d^4x\ \varphi(x) O \varphi(x)} = N[\det\, O]^{-1/2}, \tag{4.2}$$

where $\det\, O$ is the determinant of the operator O. In order to prove this, one can expand $\varphi(x)$ in terms of eigenfunctions of O,

$$\varphi(x) = \sum_n a_n \varphi_n(x), \tag{4.3}$$

where $\varphi_n(x)$ satisfies

$$O\varphi_n(x) = \lambda_n \varphi_n(x) \quad \text{and} \quad \int d^4x\ \varphi_n(x)\varphi_m(x) = \delta_{nm}. \tag{4.4}$$

The sum over all field values can then be performed by summing over all values of the expansion coefficients a_n,

$$\begin{aligned} I_{\rm quad} &= N \left[\prod_n \int_{-\infty}^{\infty} da_n \right] e^{-i\int d^4x \sum_{k=1}^{\infty} a_k \varphi_k(x) \sum_{l=1}^{\infty} a_l \varphi_\ell(x) \lambda_l} \\ &= N \prod_n \int_{-\infty}^{\infty} da_n\ e^{-i\lambda_n a_n^2} = N' \, (\det\ O)^{-1/2}, \end{aligned} \tag{4.5}$$

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where N, N' are normalization constants and

$$\det\ O = \prod_{n=1}^{\infty} \lambda_n \tag{4.6}$$

denotes, as usual, the product of operator eigenvalues.

In general, some effort is required to evaluate the determinant of an operator. One valuable relation, easily proven for finite dimensional matrices and generalizable to infinite dimensional ones is[4]

$$\det\ O = \exp(\operatorname{tr}\ \ln O). \tag{4.7}$$

This trace now denotes a summation over spacetime points, i.e.,

$$\operatorname{tr}\ \ln O = \int d^4x\ \langle x \mid \ln O \mid x \rangle, \tag{4.8}$$

which is the most commonly used form in practice.

Background field method to one loop

We can illustrate one use of this result by constructing an expansion about a background field configuration (which satisfies the classical equation of motion) and retaining the quantum fluctuations up to quadratic order. Consider a scalar field theory with interaction $\mathcal{L}_{\text{int}}(\varphi(x))$. We define $\bar{\varphi}$ as a solution to

$$\left(\Box + m^2\right)\bar{\varphi}(x) - \mathcal{L}'_{\text{int}}(\bar{\varphi}(x)) = j(x). \tag{4.9}$$

Writing

$$\varphi(x) = \bar{\varphi}(x) + \delta\varphi(x), \tag{4.10}$$

leads to the generating functional

$$Z[j] = e^{\left(iS[\bar{\varphi}(x)] + i\int d^4x\, j(x)\bar{\varphi}(x)\right)} \int [d\delta\varphi]\, e^{i\int d^4x\left(\frac{1}{2}\partial_\mu\delta\varphi\partial^\mu\delta\varphi - \frac{1}{2}\left(m^2 - \mathcal{L}''_{\text{int}}(\bar{\varphi}(x))\right)\delta\varphi^2\right)} + \cdots, \tag{4.11}$$

where

$$S[\bar{\varphi}(x)] = \int d^4x \left(\frac{1}{2}\partial_\mu\bar{\varphi}(x)\partial^\mu\bar{\varphi}(x) - \frac{m^2}{2}\bar{\varphi}^2(x) + \mathcal{L}_{\text{int}}(\bar{\varphi}(x))\right). \tag{4.12}$$

[4] For a discrete basis, this follows from the result

$$\exp(\operatorname{tr}\ \ln O) = \exp\sum_n \ln\lambda_n = \prod_n \exp(\ln\lambda_n) = \prod_n \lambda_n = \det O,$$

where λ_n are the eigenvalues of the operator O.

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Integration by parts gives

$$Z[j] = e^{\left(iS[\bar{\varphi}(x)]+i\int d^4x\ \bar{\varphi}(x)j(x)\right)} \int [d\delta\varphi]\, e^{-\frac{i}{2}\int d^4x\, \delta\varphi(x) O_x \delta\varphi(x)}, \tag{4.13}$$

where

$$O_x \equiv \Box_x + m^2 - \mathcal{L}''_{\text{int}}\left(\bar{\varphi}(x)\right). \tag{4.14}$$

The functional integration can then be performed (cf. Eq. (4.5)) and we obtain

$$Z[j] = \text{const.}\ (\det O_x)^{-1/2}\, e^{\left(iS[\bar{\varphi}(x)]+i\int d^4x\ j(x)\bar{\varphi}(x)\right)}. \tag{4.15}$$

It is convenient to normalize the determinant somewhat differently by defining

$$O_{0x} \equiv \Box_x + m^2. \tag{4.16}$$

Then, suppressing the x subscript, we write

$$(\det O)^{-1/2} = \text{const.}\ \left(\det O_0^{-1} O\right)^{-1/2}, \tag{4.17}$$

where

$$\text{const.} = (\det O_0)^{-1/2}, \tag{4.18}$$

and

$$O_0^{-1} O = 1 + \Delta_F \mathcal{L}''_{\text{int}}(\varphi). \tag{4.19}$$

Using Eq. (4.2) we have

$$Z[j] = N e^{\left[iS[\bar{\varphi}(x)]+i\int d^4x\ j(x)\bar{\varphi}(x) - \frac{1}{2}\text{Tr}\ \ln\left(1+\Delta_F \mathcal{L}''(\bar{\varphi})\right)\right]}. \tag{4.20}$$

The generating functional for connected diagrams can now be identified immediately as

$$\begin{aligned} W[j] &= S[\bar{\varphi}] + \int d^4x\ j(x)\bar{\varphi}(x) + \frac{i}{2}\text{Tr}\ \ln\left(1 + \Delta_F \mathcal{L}''_{\text{int}}(\bar{\varphi})\right) \\ &= \int d^4x \left[\frac{1}{2} j(x)\bar{\varphi}(x) + \mathcal{L}_{\text{int}}(\bar{\varphi}) - \frac{1}{2}\bar{\varphi}(x)\mathcal{L}'_{\text{int}}(\bar{\varphi})\right] \\ &\quad + \frac{i}{2}\text{Tr}\ \ln\left(1 + \Delta_F \mathcal{L}''_{\text{int}}(\bar{\varphi})\right). \end{aligned} \tag{4.21}$$

The trace ‘Tr’ includes the integration over spacetime variables and can be interpreted as follows,

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$$\text{Tr}\ \ln\left[1 + \Delta_F \mathcal{L}''_{\text{int}}(\bar{\varphi})\right] = \text{Tr}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\left(\Delta_F \mathcal{L}''_{\text{int}}(\bar{\varphi})\right)^n ,$$

$$\text{Tr}\left[\Delta_F \mathcal{L}''_{\text{int}}(\bar{\varphi})\right] = \int d^4x\ \Delta_F(x-x)\mathcal{L}''_{\text{int}}(\bar{\varphi}) ,$$

$$\text{Tr}\left[\Delta_F \mathcal{L}''_{\text{int}}(\bar{\varphi})\ \Delta_F \mathcal{L}''_{\text{int}}(\bar{\varphi})\right] = \int d^4x \int d^4y\ \Delta_F(x-y)\mathcal{L}''_{\text{int}}(\bar{\varphi}(y)) \times \Delta_F(y-x)\mathcal{L}''_{\text{int}}(\bar{\varphi}(x)) . \qquad (4.22)$$

In this manner, one-loop diagrams containing arbitrary numbers of $\mathcal{L}''_{\text{int}}(\bar{\varphi})$ factors are generated. The physics associated with this approximation can be gleaned from counting arguments. The overall power of $\hbar$ attached to a particular diagram can be found by noting that associated with a propagator and a vertex are the powers $\hbar$ and $\hbar^{-1}$, respectively. There is also an overall factor of $\hbar$ for each diagram. Then with the relation

$$\text{no. internal lines} - \text{no. internal vertices} = \text{no. loops} - 1$$

we see that this approximation corresponds to an expansion to one loop. The classical phase generates the tree diagram $\left(\mathcal{O}(\hbar^0)\right)$ contribution and the determinant yields the one-loop $\left(\mathcal{O}(\hbar^1)\right)$ correction to a given amplitude.

A–5 Fermion field theory

Thus far, our development has been performed within the simple context of scalar fields. It is important also to consider the case of fermion fields where the requirements of antisymmetry impose interesting modifications on functional integration techniques. The key to the treatment of anticommuting fields is the use of Grassmann variables. Thus, while ordinary c-number quantities (hereafter denoted by roman letters $a, b, \ldots$) commute with one another,

$$[a, a] = [a, b] = [a, c] = \cdots = 0, \qquad (5.1)$$

the Grassmann numbers (hereafter denoted by Greek letters $\alpha, \beta, \ldots$) anticommute,

$$\{\alpha, \alpha\} = \{\alpha, \beta\} = \{\alpha, \gamma\} = \cdots = 0. \qquad (5.2)$$

It follows that the square of a Grassmann quantity must vanish,

$$\alpha^2 = \beta^2 = \gamma^2 = \cdots = 0, \qquad (5.3)$$

and that any function must have the general expansion

$$f(\alpha) = f_0 + f_1\alpha , \qquad g(\alpha, \beta) = g_0 + g_1\alpha + g_2\beta + g_3\alpha\beta. \qquad (5.4)$$

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Differentiation is defined correspondingly via

$$\frac{d\alpha}{d\alpha} = \frac{d\beta}{d\beta} = \cdots = 1\,, \qquad \frac{d\beta}{d\alpha} = \frac{d\alpha}{d\beta} = \cdots = 0, \tag{5.5}$$

so that in the notation of Eq. (5.4) we have

$$\frac{df}{d\alpha}(\alpha) = f_1, \qquad \frac{dg}{d\beta}(\alpha, \beta) = g_2 - g_3\alpha. \tag{5.6}$$

Second derivatives then have the property

$$\frac{d^2}{d\alpha d\alpha} = 0. \tag{5.7}$$

We must also define the concept of Grassmann integration. If we demand that integration have the property of translation invariance

$$\int d\alpha\, f(\alpha) = \int d\alpha\, f(\alpha + \beta), \tag{5.8}$$

it follows that

$$\int d\alpha\, f_1\beta = 0 \quad \text{or} \quad \int d\alpha = 0. \tag{5.9}$$

The normalization in the diagonal integral can be chosen for convenience,

$$\int d\alpha\, \alpha = 1, \qquad \int d\alpha\, f(\alpha) = f_1. \tag{5.10}$$

Let us extend this formalism to a matrix notation by considering the discrete sets $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\bar{\alpha} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ of Grassmann variables. A class of integrals which commonly arises in a functional framework is

$$Z[M] = \int d\bar{\alpha}_n \cdots d\bar{\alpha}_1\, d\alpha_n \cdots d\alpha_1 e^{i\bar{\alpha}M\alpha}. \tag{5.11}$$

As an example, the simple 2×2 case is calculated to be

$$\begin{aligned} Z[M] = \int d\bar{\alpha}_2\, d\bar{\alpha}_1\, d\alpha_2\, d\alpha_1\, \big[1 + i\bar{\alpha}_i M_{ij}\alpha_j \\ + \bar{\alpha}_2\bar{\alpha}_1\alpha_2\alpha_1\,(M_{11}M_{22} - M_{12}M_{21})\big]. \end{aligned} \tag{5.12}$$

Only the final term survives the integration, and we obtain

$$Z[M] = \det M. \tag{5.13}$$

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This result generalizes to the $n \times n$ system [Le 82] yielding essentially the inverse of the result found for Bose fields,

$$Z[M]_{\text{Fermi}} = \int d\bar{\alpha}_n \cdots d\bar{\alpha}_1 \, d\alpha_n \cdots d\alpha_1 e^{i\bar{\alpha}M\alpha} = \det M,$$
$$Z[M]_{\text{Bose}} = \int da_n^* \cdots da_1^* \, da_n \cdots da_1 e^{-a^* M a} \propto (\det M)^{-1} \,. \tag{5.14}$$

We can now extend this formalism to the case of fermion *fields* $\psi(x)$ and $\bar{\psi}(x)$. Since such quantities always enter the lagrangian quadratically, the functional integral can be performed exactly to yield

$$Z[O] = \int [d\psi] \left[d\bar{\psi}\right] e^{i \int d^4x \, \bar{\psi}(x) O \psi(x)} = N \det O. \tag{5.15}$$

The remaining development proceeds parallel to that given for scalar fields. Given the free field lagrangian

$$\mathcal{L}_0 \left(\bar{\psi}, \psi\right) = \bar{\psi}(x) \, (i\partial\!\!\!/ - m) \, \psi(x), \tag{5.16}$$

the generating functional for the noninteracting spin one-half field becomes

$$Z[\eta, \bar{\eta}] = \int [d\psi] \left[d\bar{\psi}\right] e^{i \int d^4x \left[\bar{\psi}(x) O_x \psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right]}, \tag{5.17}$$

where $O_x \equiv i\partial\!\!\!/_x - m + i\epsilon$ and $\bar{\eta}(x), \eta(x)$ are Grassmann fields. Introducing the change of variables

$$\psi'(x) = \psi(x) - \int d^4y \; S_F(x, y)\eta(y),$$
$$\bar{\psi}'(x) = \bar{\psi}(x) - \int d^4y \; \bar{\eta}(y) S_F(y, x),$$
$$i S_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x-y)} \frac{i}{k\!\!\!/ - m + i\epsilon}$$
$$(i\partial\!\!\!/_x - m) S_F(x - y) = \delta^{(4)}(x - y), \tag{5.18}$$

we find that an alternative form for the generating functional is

$$Z[\eta, \bar{\eta}] = \int \left[d\psi'\right] \left[d\bar{\psi}'\right] e^{i \int d^4x \, \bar{\psi}'(x) O_x \psi'(x) - i \int d^4x \int d^4y \, \bar{\eta}(x) S_F(x,y) \eta(x)},$$
$$= Z[0, 0] e^{-i \int d^4x \int d^4y \, \bar{\eta}(x) S_F(x,y) \eta(x)}. \tag{5.19}$$

Thus, the generating functional for connected diagrams is

$$W[\eta, \bar{\eta}] = -\int d^4x \int d^4y \, \bar{\eta}(x) S_F(x, y) \eta(y), \tag{5.20}$$

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and the only nonvanishing connected Green's function is

$$G^{(2)}_{\text{conn}}(x_1, x_2) = (-i)^2 \frac{\delta^2 W}{\delta\eta(x_2)\delta\bar{\eta}(x_1)}$$
$$= S_F(x_1, x_2) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1-x_2)} \frac{i}{\not{k} - m + i\epsilon}, \tag{5.21}$$

which is the usual Feynman propagator.

A–6 Gauge theories

For our final topic, we examine gauge theories within a functional framework. We shall employ *QED* as the archetypical example, for which the action is

$$S[A_\mu] = -\frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x\ (A_\mu \Box_x A^\mu - A_\mu \partial^\mu_x \partial^\nu_x A_\nu)$$
$$\equiv \frac{1}{2}\int d^4x\ A_\mu O^{\mu\nu}_x A_\nu, \tag{6.1}$$

where the second line follows from the first by an integration by parts and

$$O^{\mu\nu}_x \equiv g^{\mu\nu}\Box_x - \partial^\mu_x \partial^\nu_x. \tag{6.2}$$

In the presence of a source j_μ, the generating functional is then

$$Z[j_\mu] = N \int [dA_\mu] e^{iS[A_\mu] + i\int d^4x\ j_\mu A^\mu}. \tag{6.3}$$

Due to the bilinear form of Eq. (6.1), it would appear that one could perform the functional integration as usual, resulting in

$$Z[j_\mu] = Z[0] e^{-\frac{i}{2}\int d^4x \int d^4y\ j^\mu(x) D_{F\mu\nu}(x,y) j^\nu(y)}, \tag{6.4}$$

where the inverse operator $D_{F\mu\nu}(x, y)$ is defined as

$$O^{\lambda\mu}_x D_{F\mu\nu}(x, y) \equiv \delta^\lambda_\nu \delta^{(4)}(x - y). \tag{6.5}$$

However, this is illusory since the inverse does not exist. That is, acting on Eq. (6.5) from the left with the derivative ∂^x_λ yields

$$0 \times D_{F\mu\nu}(x, y) = \partial^x_\nu \delta^{(4)}(x - y), \tag{6.6}$$

implying that $D_{F\mu\nu}$ must be infinite. An alternative way to demonstrate that $O^{\mu\nu}_x$ is a singular operator is to observe that

$$O^{\mu\nu}_x \partial^x_\nu \alpha = 0. \tag{6.7}$$

Thus, any four-gradient $\partial^x_\nu \alpha$ is an eigenfunction of $O^{\mu\nu}_x$ having eigenvalue zero, and an operator having zero eigenvalues does not possess an inverse.

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Gauge fixing

The occurrence of such a divergence in the generating functional of a gauge theory can be traced to gauge invariance. For *QED*, any gauge transformation of vector potentials (cf. Eq. (II–1.3)),

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x), \tag{6.8}$$

leaves the action invariant,

$$S[A'_\mu(x)] = S[A_\mu(x)]. \tag{6.9}$$

If we partition the full field integration $[dA_\mu]$ into a component $[d\bar{A}_\mu]$ that includes only those configurations which are *not* related by a gauge transformation and a component $[d\alpha]$ that denotes all possible gauge transformations, then we have

$$\int [dA_\mu]\, e^{iS[A_\mu]} = \int [d\bar{A}_\mu]\, e^{iS[\bar{A}_\mu]} \times \int [d\alpha]. \tag{6.10}$$

But $\int [d\alpha]$ is clearly infinite and this is the origin of the problem. The solution, first given by Faddeev and Popov [FaP 67], involves finding a procedure which somehow isolates the integration over the distinctly different vector potentials $\bar{A}_\mu(x)$. In order to understand this technique, we shall first examine a finite-dimensional analog [Ra 89].

Consider the functional

$$Z[\mathbf{A}] = \left[\prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i\right] e^{-\sum_{k,l} x_k A_{kl} x_l}, \tag{6.11}$$

where $\mathbf{A}$ is an $N \times N$ matrix. Suppose that $\mathbf{A}$ is brought into diagonal form $\mathbf{A^D}$ by linear transformation $\mathbf{R}$,

$$\mathbf{A^D} \equiv \mathbf{RAR}^{-1}. \tag{6.12}$$

Letting $\vec{y} = \mathbf{R}\vec{x}$ denote the coordinates in the diagonal basis, we have

$$\begin{aligned} Z[\mathbf{A}] &= \left[\prod_{i=1}^{N} \int dy_i\right] e^{-\sum_{k,l} y_k A^D_{kl} y_l} = \left[\prod_{i=1}^{N} \int dy_i\right] e^{-\sum_k y_k^2 A^D_{kk}} \\ &= \prod_{i=1}^{N} \left(\frac{\pi}{A^D_{ii}}\right)^{1/2} = \pi^{N/2} [\det \mathbf{A}]^{-1/2}. \end{aligned} \tag{6.13}$$

Suppose that the last n of the N eigenvalues belonging to $\mathbf{A}$ vanish. The exponential factor in Eq. (6.13) is then *independent* of the coordinates $y_{N-n+1}, \ldots, y_N$ and the corresponding integrations $\int dy_{N-n+1} \ldots \int dy_N$ diverge. This is reflected in the vanishing of $\det \mathbf{A}$, and causes the quantities in Eq. (6.13) to diverge. The infinity

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is removed if the integration is restricted to only variables associated with nonzero eigenvalues, in which case we obtain the finite result

$$Z^f[\mathbf{A}] = \left[\prod_{i=1}^{N-n} \int dy_i\right] e^{-\sum_{k,l} y_k A^D_{kl} y_l}. \tag{6.14}$$

It is possible to express $Z^f[\mathbf{A}]$ as an integral over the *full* range of indices $1 \leq i \leq N$ by defining variables

$$z_i = \begin{cases} y_i & (1 \leq i \leq N-n), \\ \text{arbitrary} & (N-n+1 \leq i \leq N), \end{cases} \tag{6.15}$$

and writing for the generating functional

$$Z^f[\mathbf{A}] = \left[\prod_{i=1}^{N} \int dz_i\right] \delta(z_{N-n+1}) \cdots \delta(z_N) e^{-\sum_{k,l} z_k(x) A_{kl} z_l(x)}. \tag{6.16}$$

Upon tranforming back to an arbitrary set of coordinates $\{x_i\}$, we obtain the useful expression

$$Z^f[\mathbf{A}] = \left[\prod_{i=1}^{N} \int dx_i\right] \det\left|\frac{\partial \vec{z}}{\partial \vec{x}}\right| \prod_{j=N-n+1}^{N} \delta\left(z_j(\vec{x})\right)\, e^{-\sum_{k,l} x_k A_{kl} x_l}. \tag{6.17}$$

Let us now return to the subject of gauge fields, broadening the scope of our discussion to include even nonabelian gauge theories. By analogy, corresponding to the variables $z_{N-n+1}, \ldots, z_N$ will be the gauge degrees of freedom and the prescription of Faddeev and Popov becomes for generic gauge fields $A^a_\mu(x)$,

$$Z^f = \prod_a \int [dA^a_\mu] \prod_{b=1}^{n} \delta(G_b(A^a_\mu)) \det|\delta G_b/\delta\alpha_a|\, e^{iS[A^a_\mu]}, \tag{6.18}$$

where the $\{\alpha_a\}$ are gauge-transformation parameters (cf. Sect. I–4) and the $\{G_b(A^a_\mu)\}$ are functions which vanish for some value of $A^a_\mu(x)$. Since the $\{G_b\}$ serve to define the gauge, such contributions to the generating functional are referred to as *gauge-fixing* terms. The variation $\delta G_b/\delta\alpha_a$ signifies the response of the gauge-fixing function G_b to a gauge-transformation parameter α_a.

For any gauge theory, there are a variety of choices possible for the gauge-fixing function G. In *QED*, one defines the *axial gauge* by

$$G(A_\mu) = n_\mu A^\mu, \tag{6.19}$$

where n_μ is an arbitrary spacelike four-vector. Due to the presence of the four-vector n^μ, one must forgo manifestly covariant Feynman rules in this approach. Thus, one often employs a *covariant* gauge-fixing condition such as

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$$G(A_\mu) = \partial^\mu A_\mu - F, \tag{6.20}$$

where F is an arbitrary constant. Under the gauge transformation of Eq. (6.8), we find

$$G(A_\mu) \to G(A_\mu) + \Box\alpha, \tag{6.21}$$

so that

$$\delta G/\delta\alpha = \Box. \tag{6.22}$$

Referring back to the general formula of Eq. (6.18), we see in this case that $\det|\delta G/\delta\alpha|$ is independent of the gauge field and thus may be dropped from the functional integral. The *QED* generating functional then becomes

$$\begin{aligned} Z[j_\mu] &= N\int [dA_\mu]\,\delta(\partial^\mu A_\mu - F)\, e^{i\int d^4x\,\left(\frac{1}{2}A_\mu\Box_x A^\mu - \frac{1}{2}A_\mu\partial_x^\mu\partial_x^\nu A_\nu + j_\mu A^\mu\right)} \\ &= N\int [dA_\mu] e^{i\int d^4x\,\left(\frac{1}{2}A_\mu\Box_x A^\mu + j_\mu A^\mu\right)} \\ &= Z[0] e^{-\frac{i}{2}\int d^4x\int d^4y\; j_\lambda(x) D_F^{\lambda\nu}(x,y) j_\nu(y)}. \end{aligned} \tag{6.23}$$

Note that, as promised, this result is finite and leads to a photon propagator in Feynman gauge

$$D_F^{\nu\lambda}(x,y) = \frac{1}{Z[0]}\frac{\delta^2 Z[j_\mu]}{\delta j_\nu(x)\delta j_\lambda(y)}\Big|_{j_\mu=0} = -i\int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)}\frac{g^{\nu\lambda}}{q^2+i\epsilon}. \tag{6.24}$$

The result is independent of the choice of F. Consequently, even if the constant F is evaluated to the status of a field $F(x)$, one can functionally integrate over $F(x)$ with an arbitrary weighting factor since this will only affect the overall normalization of the generating functional. A common choice is

$$\int [dF]\,\delta(\partial^\mu A_\mu - F(x)) e^{-\frac{i}{2\xi}\int d^4x\; F^2(x)} = e^{-\frac{i}{2\xi}\int d^4x\;(\partial^\mu A_\mu)^2}, \tag{6.25}$$

where ξ is a real-valued parameter. In this case, the generating functional becomes

$$Z[j_\mu] = N\int [dA_\mu]\, e^{i\int d^4x\left(\frac{1}{2}A_\mu\left(\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + j_\mu A^\mu\right)}. \tag{6.26}$$

The integrand of the above spacetime integral can be regarded as the effective lagrangian of the theory, and the gauge-fixing term appears as one of its contributions. At this point, the functional integration can be carried out with impunity to obtain

$$Z[j_\mu] = Z[0] e^{-\frac{i}{2}\int d^4x\int d^4y\; j_\mu(x) D_F^{\mu\nu}(x,y) j_\nu(y)}, \tag{6.27}$$

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where $D_F^{\mu\nu}$ is defined as

$$\left(\Box_x g^{\mu\nu} - (1 - \xi^{-1})\partial_x^\mu \partial_x^\nu\right) D_{F\nu\lambda}(x - y) = \delta_\lambda^\mu \delta^{(4)}(x - y). \tag{6.28}$$

We find in this way the form of the photon propagator in an arbitrary gauge, as appearing in Eq. (II–1.17).

Ghost fields

In the path-integral formalism, if the generating functional can be written in purely exponential form, then one can read off the lagrangian of the theory from the exponent. However, the general formula in Eq. (6.18) for a gauge-fixed generating functional contains a seemingly nonexponential factor, the determinant factor det $|\delta G_b/\delta\alpha_a|$. A fruitful procedure, due to Faddeev and Popov, for expressing the determinant as an exponential factor is motivated by the identity (cf. Eq. (5.15)),

$$\det M = N \int [dc][d\bar{c}]\, e^{i\bar{c}Mc}, \tag{6.29}$$

where $c, \bar{c}$ are Grassmann fields. This identity suggests that we replace the determinant factor with an appropriate functional integration over Grassmann variables.

For *QED*, the generating functional can then be written in the concise form

$$Z[j_\mu] = N \int [dA_\mu][dc][d\bar{c}]\, e^{i\int d^4x\, \left(A_\mu(\Box g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \bar{c}\Box_x c + j_\mu A^\mu\right)}. \tag{6.30}$$

As pointed out earlier, for this case the integration over $c, \bar{c}$ yields only an unimportant constant and may be discarded. However, for nonabelian gauge theory Eq. (6.30) generalizes to

$$Z[j_\mu^a] = \int \prod_{a,b,d} [dA_\mu^a][dc^b][d\bar{c}^d]\, e^{i\int d^4x\, \left[\mathcal{L}[A_\mu^a] + j_\mu^a A_a^\mu + \bar{c}^b M_{be} c^e - \frac{1}{2\xi}\sum_b F_b^2(A_\mu^a)\right]}, \tag{6.31}$$

where repeated indices are summed over. The quantities

$$M_{be} \equiv \frac{\delta F_b(A_\mu^a)}{\delta\alpha^e} \tag{6.32}$$

will generally depend upon the fields A_μ^a themselves. Thus, the fields $\{c_a\}, \{\bar{c}_a\}$ will appear as degrees of freedom in the defining lagrangian of the theory. However, although coupled to the gauge fields A_μ^a through $\bar{c}\mathbf{M}c$, they do not interact with any source terms and therefore can only appear in closed loops inside more complex diagrams.[5] Since these Grassmann quantities are unphysical, they are

[5] Such loops must include a multiplicative factor of −1 to account for the anticommuting nature of these variables.

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often called *Faddeev–Popov ghost fields*. They are scalar, anticommuting variables, which transform as members of the regular representation of the gauge group, e.g., for the gauge group $SU(n)$, there are $n^2 - 1$ of the $\{c_a\}$ and $\{\bar{c}_a\}$ fields.

To complete the discussion, let us determine the ghost-field contribution to the *QCD* lagrangian. We choose $F_b = \partial_\mu A_b^\mu$ and note the form of a gauge transformation (cf. Eqs. (I–5.12), (I–5.17) with α_a infinitesimal),

$$A_b^\mu \to A_b'^\mu = A_b^\mu + \frac{1}{g_3}\partial^\mu \alpha_b - f_{bae} A_a^\mu \alpha_e. \tag{6.33}$$

Then we find from a direct evaluation of $\partial F_b / \partial \alpha_e$ followed by the rescaling $-g_3^{-1}\bar{c}c \to \bar{c}c$,

$$\mathcal{L}_{\text{gh}} = -\bar{c}_b \partial_\nu [\delta_{be}\partial^\nu - g_3 f_{bae} A_a^\nu] c_e. \tag{6.34}$$

Upon performing an integration by parts in the first term and relabeling the indices in the second, we obtain the ghost contribution to the *QCD* lagrangian of Eq. (II–2.25).

Problems

(1) **The van Vleck determinant**

The semiclassical approximation to the propagator (valid as $\hbar \to 0$) can be derived by expanding about the classical path. Writing

$$x(t) = x_{\text{cl}}(t) + \delta x(t),$$

we have

$$D(x_f, t_f; x_i, t_i) = e^{iS[x_{\text{cl}}(t)]} \int \mathcal{D}[\delta x(t)] e^{\frac{i}{2}\int dt dt' \, \delta x(t) \frac{\delta^2 S}{\delta x(t) \delta x(t')} \delta x(t')},$$

where

$$\frac{\delta^2 S}{\delta x(t)\delta x(t')} = -\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2 V[x_{\text{cl}}(t)]}{\partial x_{\text{cl}}^2(t)} \right) \delta(t - t')$$

and we have dropped the term linear in $\delta x(t)$ by Hamilton's condition. Performing the path integration we have then

$$D(x_f, t_f; x_i, t_i) = N \left[\det \frac{\delta^2 S}{\delta x(t)\delta x(t')} \right]^{-1/2} e^{iS[x_{\text{cl}}(t)]},$$

where N is a normalization constant and the quantity inside the square root is called the *van Vleck determinant*.

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(a) Show that this can be written in the form

$$N\left[\frac{\delta^2 S}{\delta x(t)\delta x(t')}\right]^{-\frac{1}{2}} = \left[\frac{1}{2\pi i}\frac{\partial^2 S[x_{\rm cl}(t)]}{\partial x_f \partial x_i}\right]^{1/2}.$$

Hint: The following argument is hardly rigorous but leads to the correct answer. Write

$$D(x_f, t_f; x_i, t_i) \equiv A(x_f, x_i; t_f - t_i)e^{iS_{\rm cl}(x_f, x_i; t_f - t_i)}$$

and use completeness to show that at equal times

$$\begin{aligned}&\delta(x_f - x_i)D(x_f, t_i; x_i, t_i)\\ &\quad = \int dx\, A(x_f, x; T)A^*(x_i, x; T)e^{i(S_{\rm cl}(x_f, x; T) - S_{\rm cl}(x_i, x; T))},\end{aligned}$$

where T is an arbitrary positive time. Now define $\rho(x_i, x; T) \equiv \partial S_{\rm cl}(x_i, x; T)/\partial x_i$ so that

$$S_{\rm cl}(x_f, x; T) - S_{\rm cl}(x_i, x; T) \simeq (x_f - x_i)\rho(x_i, x; T).$$

Finally, change variables from x to ρ and compare with the free particle result to obtain

$$A(x_f, x_i; T) = \left[\frac{1}{2\pi i}\frac{\partial^2 S_{\rm cl}}{\partial x_f \partial x_i}\right]^{1/2}.$$

(b) Show that

$$S_{\rm cl}(x_f, x_i; T) = -ET + \int_{x_i}^{x_f} dx\sqrt{2m(E - V(x))}$$

and verify that

$$\left[\frac{1}{2\pi i}\frac{\partial^2 S_{\rm cl}}{\partial x_f \partial x_i}\right]^{\frac{1}{2}} = \left[\frac{m}{2\pi i \dot{x}_{\rm cl}(t_f)\dot{x}_{\rm cl}(t_i)\int_{x_i}^{x_f} dx\, \dot{x}_{\rm cl}^{-3}(x)}\right]^{1/2}.$$

Hint: Recall that t is an independent variable, so that

$$0 = \frac{\partial t}{\partial x_f} = \frac{\partial t}{\partial t_i}.$$

We thus have the result for the semiclassical propagator

$$D(x_f, t_f; x_i, t_i) = \left[\frac{m}{2\pi i \dot{x}_{\rm cl}(t_i)\dot{x}_{\rm cl}(t_f)\int_{x_i}^{x_f} dx\, \dot{x}_{\rm cl}^{-3}(x)}\right]^{1/2} e^{iS_{\rm cl}},$$

which is identical to that found from WKB methods.

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(2) **Propagator for the charged scalar field**
The lagrangian for a charged scalar field φ of mass m and charge e in the presence of an external (c-number) potential A_μ is

$$\mathcal{L} = D^\mu \varphi^* D_\mu \varphi - m^2 \varphi^* \varphi,$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative.

(a) Show that the full Feynman propagator,

$$D_F(x'; x) = \frac{\int [d\varphi][d\varphi^*]\varphi(x')\varphi^*(x)e^{i\int d^4x\ \mathcal{L}(x)}}{\int [d\varphi][d\varphi^*]e^{i\int \mathcal{L}(x)}},$$

can be written as

$$D_F(x'; x) = -i\langle x'|(D^\mu D_\mu + m^2 - i\epsilon)^{-1}|x\rangle.$$

Suggestion: This is a quadratic form. Use the generating functional to integrate it.

(b) By expanding $D_F(x'; x)$ as a power series in $A_\mu(x)$, show that an alternative representation for the propagator is

$$D_F(x'; x) = \langle x'| \int_0^\infty ds\ e^{-is(D^\mu D_\mu + m^2 - i\epsilon)}|x\rangle.$$

(3) **Functional methods and φ^4 theory**
Consider a scalar field theory with the self-interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\varphi^4(x).$$

(a) Show that the generating functional can be written as

$$Z[j] = Ne^{-i\frac{\lambda}{4!}\int d^4z\left(\frac{\delta^4}{\delta j^4(z)}\right)} e^{-\frac{1}{2}\int d^4x \int d^4y\ j(x) i\Delta_F(x,y) j(y)},$$

where the free field Feynman propagator $i\Delta_F(x, y)$ is as in Eq. (C–2.12).

(b) Evaluate the two-point function to $\mathcal{O}(\lambda^2)$. Associate a Feynman diagram with each term of this expansion and separate the connected and disconnected diagrams.

(c) Calculate the connected generating functional via

$$W[j] = W_0[j] - i\ \ln\left[1 + e^{-iW^{(0)}[j]}\left(e^{-i\frac{\lambda}{4!}\int d^4z \frac{\delta^4}{\delta j^4(z)}} - 1\right)e^{iW^{(0)}[j]}\right]$$

where

$$W^{(0)}[j] = \frac{1}{2}\int d^4x \int d^4y\ j(x) i\Delta_F(x, y) j(y).$$

(d) Compare the connected diagrams found in parts (b) and (c).

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