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ON WAVES IN THE PRESENCE OF VERTICAL POROUS BOUNDARIES

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Abstract

In this paper various wave motions in water of infinite depth containing vertical porous boundaries are determined when the water is of infinite extent on one or both sides. Initially surface tension is ignored and simple solutions for incident waves are obtained before going on to harder wave source and wave-maker solutions. A reduction method is developed to obtain solutions for two-sided boundaries from those for one-sided, which are obtained by standard techniques. The effect of surface tension that precludes simple solutions is also considered, although a present lack of information on dynamical edge behaviour for porous boundaries means that the formal mathematical solutions must be left in terms of arbitrary edge constants. In conclusion, some solutions are noted for finite depth.

1. Introduction

In contrast to the extensive traditional theory involving impermeable boundaries, surface wave motion on water in the presence of porous boundaries has received but little attention. Two reasonably recent time-harmonic investigations of Chwang [3] for a vertical wave-maker and Chakrabarti and Sahoo [2] for incident waves against a nearly vertical wall seek to redress this situation. Both are based on the assumption of the boundaries having fine pores, when a linear condition given by Taylor [10] relating normal velocity to pressure jump may be used.

In this paper we continue the investigation and examine several problems for time-harmonic surface waves in the presence of a vertical porous boundary, when the water is on either one side only or both sides and extends to infinity. In the former situation traditional methods can be employed, while for the latter a reduction method can be developed to determine the motion quite easily from the former solutions. This method is similar to one used for two superposed liquids with a horizontal interface in Rhodes-Robinson [8].

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The problems solved are for the scattering of incident waves by, and wave sources outside, fixed boundaries and vertical wave-maker problems for moving boundaries. Linearised potential theory is used and for simplicity two-dimensional motion on infinite depth is taken; the extension to three-dimensional motion and finite constant depth is straightforward, however, and some of the solutions are noted for the latter.

The effect of surface tension is omitted in the initial formulation, but the modifications needed to include this are also discussed in some detail. Solutions are more complicated now and involve arbitrary edge constants as for impermeable boundaries. A dynamical edge condition that would enable these to be evaluated is not known at present. A generalised slope potential for a porous wall is fundamental to the solutions. The effect of surface tension was omitted in [3, 5] but included in two fairly recent investigations by Chakrabarti [1] and Gorgui, Faltas and Ahmed [4] for porous boundaries, although the edge conditions are dubious.

2. Formulation

The infinitesimal wave motion of water considered herein is two-dimensional in the x, y-plane and harmonic in time t with angular frequency σ. Motion takes place in a region $y > 0$ of infinite depth with horizontal mean free surface $y = 0$. A vertical boundary $x = 0$ extends throughout the region and contains water of semi-infinite horizontal extent either on one or both sides. The effect of surface tension is ignored for now and motion is under the influence of gravity alone with acceleration g. No motion occurs at infinite depth.

For a *one-sided* boundary containing water in $x > 0$, the motion may be described by a velocity potential of the form $\mathrm{Re}[\phi(x, y)e^{-i\sigma t}]$ that allows suppression of the time henceforth. Then the complex-valued potential ϕ satisfies the well-known basic requirements

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} = 0, \tag{2.1a}$$

$$K\phi + \phi_y = 0 \quad \text{on} \quad y = 0, \tag{2.2a}$$

$$\phi \to 0 \quad \text{as} \quad y \to \infty \tag{2.3a}$$

in the region $x > 0,\ y > 0$, where $K = \sigma^2/g$. Further

$$\phi_x + ik\phi = 0 \quad \text{on} \quad x = 0 \tag{2.4a}$$

if the porous boundary (a 'wall') is fixed, where $k > 0$ is a porosity constant. This condition was determined by Chakrabarti and Sahoo [2], following the approach of Chwang [3] noted below. The traditional impermeable or 'hard' wall corresponds to

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$k = 0$ and a completely porous or 'soft' wall to $k \to \infty$, when (2.4a) becomes $\phi_x = 0$ and $\phi = 0$ respectively.

For a *two-sided* boundary containing water on both sides, let $\phi = \phi_1$ $(x > 0)$, $\phi = \phi_2$ $(x < 0)$. Then the potential pair ϕ_1, ϕ_2 satisfies the basic requirements

$$\nabla^2\phi_1 = 0 = \nabla^2\phi_2, \tag{2.1b}$$

$$K\phi_1 + \phi_{1y} = 0 = K\phi_2 + \phi_{2y} \quad \text{on} \quad y = 0, \tag{2.2b}$$

$$\phi_1, \phi_2 \to 0 \quad \text{as} \quad y \to \infty \tag{2.3b}$$

in their regions. Now there are two coupling conditions

$$\phi_{1x} = -ik(\phi_1 - \phi_2) = \phi_{2x} \quad \text{on} \quad x = 0 \tag{2.4b}$$

if the porous boundary (a 'barrier') is fixed; this condition was essentially determined by Chwang [3] by reference to experimental work of Taylor [10], assuming that the boundary has fine pores. The cases $k = 0$, $k \to \infty$ correspond to hard and soft barriers that either allow no interaction between the regions or else are removable; these are not significant.

Other conditions apply in specific problems to give single or coupled linearised boundary-value problems. Note that straightforward modifications to (2.4) are needed if the boundary is moving; these are noted later.

Two simple results involving only progressive wave solutions $e^{-Ky\pm iKx}$ of (2.1 – 2.3) having wave number K are now obtained, before going on to some more complicated problems.

3. Incident waves against a wall

The problem for incident progressive waves with potential $e^{-Ky-iKx}$ that are reflected by a vertical porous wall $x = 0$ is considered first. Here the extra condition on the potential is

$$\phi \to e^{-Ky-iKx} + Re^{-Ky+iKx} \quad \text{as} \quad x \to \infty$$

in addition to (2.1 –2.4a), where the reflexion constant R is part of the solution. This problem is easily solved as for $k = 0$ by trying a purely progressive wave solution of this form with no local disturbance, which satisfies the remaining porous-wall condition (2.4a) if

$$K - KR - k(1 + R) = 0$$

so that

$$R = \frac{K - k}{K + k}.$$

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Hence the solution is

$$\phi = e^{-Ky-iKx} + \frac{K-k}{K+k} e^{-Ky+iKx}. \tag{3.1}$$

Note that there is always an energy loss for $k > 0$ as $|R|^2 < 1$, and in fact the incident waves are completely absorbed ($R = 0$) if $k = K$. The known result for $k = 0$ is

$$\phi = e^{-Ky-iKx} + e^{-Ky+iKx} \tag{3.2}$$

and in this case ($R = 1$) energy is conserved as $|R|^2 = 1$; the same is also true for $k \to \infty$ ($R = -1$). The energy graph is shown in Figure 1.

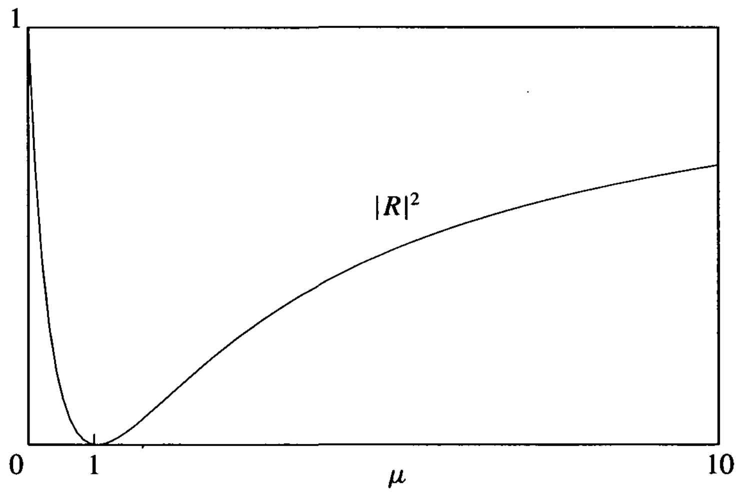

FIGURE 1. Graph of $|R|^2 = (1-\mu)^2/(1+\mu)^2$ vs $\mu = k/K$. This indicates the proportion of incident wave energy that is reflected by a wall.

4. Incident waves against a barrier

The similar problem for incident progressive waves that are partly reflected and partly transmitted by a vertical porous barrier $x = 0$ is also considered. Here the extra conditions on the potentials are

$$\phi_1 \to e^{-Ky-iKx} + Re^{-Ky+iKx} \quad \text{as} \quad x \to \infty,$$
$$\phi_2 \to Te^{-Ky-iKx} \quad \text{as} \quad x \to -\infty$$

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in addition to (2.1 – 2.4b), where the reflexion and transmission constants R, T are part of the solution. These problems are also easily solved by trying progressive wave solutions of these forms, which satisfy the porous-barrier conditions (2.4b) if

$$K - KR = k(1 + R - T) = KT$$

so that

$$R = \frac{K}{K + 2k}, \quad T = \frac{2k}{K + 2k}.$$

Hence the solutions are

$$\phi_1 = e^{-Ky - iKx} + \frac{K}{K + 2k} e^{-Ky + iKx}, \tag{4.1}$$

$$\phi_2 = \frac{2k}{K + 2k} e^{-Ky - iKx}. \tag{4.2}$$

Note that there is an energy loss again for $k > 0$ as $|R|^2 + |T|^2 < 1$, but this cannot now be complete as $|R|^2 + |T|^2 \geq 0.5$ also. The energy graphs are shown in Figure 2.

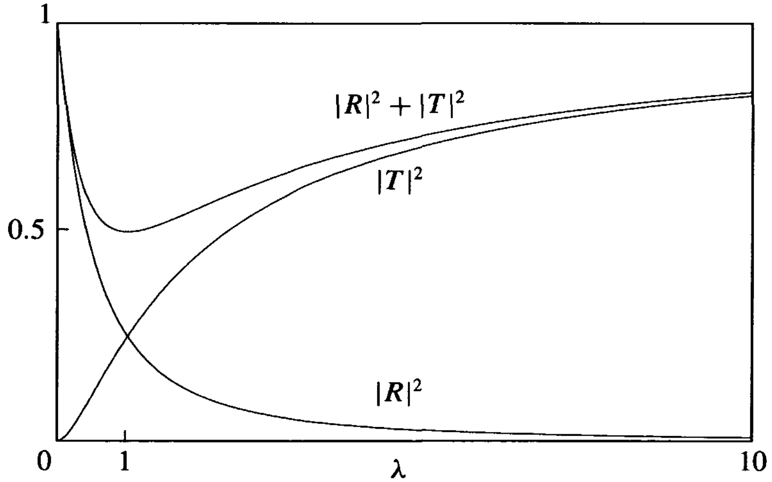

FIGURE 2. Graphs of $|R|^2 = 1/(1+\lambda)^2$, $|T|^2 = \lambda^2/(1+\lambda)^2$ and $|R|^2 + |T|^2 = (1+\lambda^2)/(1+\lambda)^2$ vs $\lambda = 2k/K$. These indicate the proportions of incident wave energy that are reflected, transmitted and scattered by a barrier.

5. Wave source in front of a wall

The problem for a wave source situated at (X, Y) in front of a vertical porous wall $x = 0$ is now considered. Here the extra conditions on the potential are

$$\phi \sim \ln\rho \quad \text{as} \quad \rho \equiv \left[(x-X)^2 + (y-Y)^2\right]^{\frac{1}{2}} \to 0,$$
$$\phi \to \text{a multiple of } e^{-Ky+iKx} \quad \text{as} \quad x \to \infty$$

for $X > 0$, $Y > 0$ in addition to (2.1 – 2.4a), the first of which does not hold at (X, Y). This problem is solved by correcting the solution ψ (say) for $k = 0$, which satisfies $\psi_x = 0$ on $x = 0$ and is easily obtained by taking a source at (X, Y) in a horizontally unbounded region and superposing an equal image source at $(-X, Y)$ in the wall. These well-known fundamental wave sources have the potentials

$$G(x, y; \pm X, Y) = -2\int_0^\infty \frac{e^{-u|x\mp X|}}{u(u^2+K^2)} F(u; y)F(u; Y)\,du - 2\pi i e^{-K(y+Y)+iK|x\mp X|}$$

respectively from Thorne [11], where $F(u; s) = u\cos us - K\sin us$ $(u > 0, s > 0)$, so that

$$\begin{aligned}\psi &= G(x, y; X, Y) + G(x, y; -X, Y)\\ &= -4\int_0^\infty \frac{e^{-uX}\cosh ux}{u(u^2+K^2)} F(u; y)F(u; Y)\,du\\ &\quad - 4\pi i e^{-K(y+Y)+iKX}\cos Kx \quad (0 < x < X)\end{aligned} \tag{5.1}$$

in particular. Now put $\phi = \psi + \phi'$, where the correction potential $\phi'$is regular in $x > 0$ and has the form of an integral superposition of $e^{-ux}F(u; y)$ for $u > 0$ and multiple of $e^{-Ky+iKx}$ in terms of basic solutions of (2.1 – 2.3a). Since $\phi'_x + ik\phi' = -ik\psi$ on $x = 0$, we find that

$$\begin{aligned}\phi' = &-4ik\int_0^\infty \frac{e^{-u(x+X)}}{u(u-ik)(u^2+K^2)} F(u; y)F(u; Y)\,du\\ &+ \frac{4\pi ik}{K+k} e^{-K(y+Y)+iK(x+X)}.\end{aligned} \tag{5.2}$$

Hence on adding (5.1), (5.2),

$$\begin{aligned}\phi = &-4\int_0^\infty \frac{e^{-uX}(u\cosh ux - ik\sinh ux)}{u(u-ik)(u^2+K^2)} F(u; y)F(u; Y)\,du\\ &- 4\pi i \frac{K\cos Kx - ik\sin Kx}{K+k} e^{-K(y+Y)+iKX} \quad (0 < x < X)\end{aligned} \tag{5.3}$$

and likewise

$$\phi = -4\int_0^\infty \frac{e^{-ux}(u\cosh uX - ik\sinh uX)}{u(u-ik)(u^2+K^2)} F(u;y)F(u;Y)\,du$$
$$-4\pi i\frac{K\cos KX - ik\sin KX}{K+k}e^{-K(y+Y)+iKx} \quad (x > X). \tag{5.4}$$

Denote this potential by $\phi = G^{\text{mod}}(x, y; X, Y; k)$ for later use; then also $\psi = G^{\text{mod}}(x, y; X, Y; 0)$. Note from (5.3), (5.4) the evident reciprocity property between source and observation positions as for $k = 0$, which may also be established independently by applying Green's theorem over the region to a pair of equal sources; the contribution from the wall vanishes identically in a similar way to that from the free surface. There is now no obvious identification of the image potential for a porous wall as there is for $k = 0$, except that an equal and opposite source at $(-X, Y)$ is obtained for $k \to \infty$. The potential for a source of double strength on the wall is obtained by formally putting $X = 0$ in (5.4).

6. One-sided wave-maker problem

The problem for the motion with outgoing progressive waves due to the horizontal simple harmonic oscillations of a one-sided vertical porous wave-maker $x = 0$ is considered next. Here the condition (2.4a) on the potential is replaced by

$$\phi_x + ik\phi = U(y) \quad \text{on} \quad x = 0$$

if the wave-maker has velocity $\text{Re}[U(y)e^{-i\sigma t}]$ in the x-direction and the extra condition in addition to (2.1 – 2.3a) is

$$\phi \to \text{ a multiple of } e^{-Ky+iKx} \quad \text{as} \quad x \to \infty$$

if it is assumed that $U(y) \to 0$ as $y \to \infty$. This problem may be solved by applying Green's theorem over the region with G^{mod} in Section 5 to obtain the form

$$\phi(X, Y) = \frac{1}{2\pi}\int_0^\infty G^{\text{mod}}(0, y; X, Y)U(y)\,dy. \tag{6.1}$$

Thus from (5.3)

$$\phi = -\frac{2}{\pi}\int_0^\infty \frac{a(u)e^{-ux}}{(u-ik)(u^2+K^2)}F(u;y)\,du - \frac{2iKA}{K+k}e^{-Ky+iKx} \tag{6.2}$$

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on interchanging the variables, where

$$a(u) = \int_0^\infty U(Y)F(u;Y)\,dY \quad (u > 0),$$
$$A = \int_0^\infty U(Y)e^{-KY}\,dY.$$

The solution (6.2) may also be obtained by simple modification of the known solution of Havelock [5] for $k = 0$ to accommodate the new condition on $x = 0$. Using the source reciprocity property in the form (6.1), the solution can be represented by a distribution of sources on $x = 0$ as

$$\phi(X, Y) = \frac{1}{2\pi}\int_0^\infty G^{\text{mod}}(X, Y; 0, y)U(y)\,dy.$$

Chakrabarti and Sahoo [2] considered incident waves against a nearly vertical porous wall using perturbation techniques, determining the unperturbed and first-order perturbation solutions from first principles with Fourier transforms; these potentials are just the vertical wall and wave-maker solutions (3.1), (6.2) above for a special $U(y)$.

7. Reduction method for two-sided boundaries

To obtain the solutions of the last two problems for two-sided boundaries, a general reduction procedure can be set up to deduce them from the one-sided solutions. This is first developed for fixed boundaries; the modification for moving boundaries is noted later.

The barrier potentials ϕ_1, ϕ_2 satisfy (2.1 – 2.4b) in $x > 0$, $x < 0$ respectively. Define a second potential ϕ_1^* in $x > 0$ by

$$\phi_1^*(x, y) = \phi_2(-x, y), \tag{7.1}$$

which clearly satisfies (2.1 – 2.3b). Further

$$\phi_{1x} = -ik(\phi_1 - \phi_1^*) = -\phi_{1x}^* \quad \text{on} \quad x = 0 \tag{7.2}$$

from (2.4b). Now define two reduction potentials

$$\Phi = \phi_1 - \phi_1^*, \qquad \Psi = \phi_1 + \phi_1^* \tag{7.3}$$

in $x > 0$, which clearly satisfy

$$\nabla^2\Phi = \nabla^2\Psi = 0, \tag{7.4}$$
$$K\Phi + \Phi_y = K\Psi + \Psi_y = 0 \quad \text{on} \quad y = 0, \tag{7.5}$$
$$\Phi, \Psi \to 0 \quad \text{as} \quad y \to \infty. \tag{7.6}$$

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Further

$$\Phi_x + 2ik\Phi = \Psi_x = 0 \quad \text{on} \quad x = 0 \tag{7.7}$$

from (7.2), which are *uncoupled*. Hence by reference to (2.1 – 2.4a) Φ, Ψ are just the potentials for waves in the presence of porous and impermeable *walls* respectively; the porosity constant $k \to 2k$ for the former and $k = 0$ for the latter. The boundary-value problems are completed with other conditions in specific problems. Once the reduction potentials are found, the required solutions are calculated using

$$\phi_1 = \frac{1}{2}(\Psi + \Phi), \quad \phi_1^* = \frac{1}{2}(\Psi - \Phi) \tag{7.8}$$

from (7.3) and then

$$\phi_2(x, y) = \phi_1^*(-x, y) \tag{7.9}$$

from (7.1).

A similar method was developed in Rhodes-Robinson [8] for two superposed liquids in symmetric layers with two coupling conditions at their interface.

To illustrate the method, the problem of incident waves against a barrier in Section 4 is solved again using the result of Section 3 for a wall. Here the extra conditions on the reduction potentials are

$$\Phi, \Psi \to e^{-Ky-iKx} + (R \mp T)e^{-Ky+iKx} \quad \text{as} \quad x \to \infty$$

respectively in addition to (7.4 – 7.7). By reference to Section 3 the potentials Φ, Ψ describe problems for the same incident waves, and the solutions are deduced from (3.1), (3.2) as

$$\Phi = e^{-Ky-iKx} + \frac{K - 2k}{K + 2k} e^{-Ky+iKx},$$
$$\Psi = e^{-Ky-iKx} + e^{-Ky+iKx}.$$

Thus by (7.8), (7.9),

$$\phi_1 = e^{-Ky-iKx} + \frac{K}{K + 2k} e^{-Ky+iKx},$$
$$\phi_2 = \frac{2k}{K + 2k} e^{-Ky-iKx}$$

as in (4.1), (4.2).

8. Wave source in front of a barrier

The problem for a wave source at (X, Y) in front of a vertical porous barrier $x = 0$ is now solved by the reduction method, using the result of Section 5 for a wall. Suppose first that $X > 0$. Then the extra conditions on the potentials are

$$\phi_1 \sim \ln \rho \quad \text{as} \quad \rho \to 0,$$
$$\phi_1, \phi_2 \to \text{ multiples of } e^{-Ky \pm iKx} \quad \text{as} \quad x \to \pm\infty$$

respectively in addition to (2.1 – 2.4b), the first of which does not hold at (X, Y) for ϕ_1. Hence the extra conditions on the reduction potentials are

$$\Phi, \Psi \sim \ln \rho \quad \text{as} \quad \rho \to 0,$$
$$\Phi, \Psi \to \text{ multiples of } e^{-Ky+iKx} \quad \text{as} \quad x \to \infty$$

in addition to (7.4 – 7.7), the first of which does not hold at (X, Y). By reference to Section 5 the potentials Φ, Ψ describe problems for the same source, and the solutions are

$$\Phi = G^{\text{mod}}(x, y; X, Y; 2k), \qquad \Psi = G^{\text{mod}}(x, y; X, Y; 0).$$

Thus by (7.8), (7.9),

$$\phi_1 = \frac{1}{2}\left[G^{\text{mod}}(x, y; X, Y; 0) + G^{\text{mod}}(x, y; X, Y; 2k)\right],$$
$$\phi_2 = \frac{1}{2}\left[G^{\text{mod}}(-x, y; X, Y; 0) - G^{\text{mod}}(-x, y; X, Y; 2k)\right]$$

and it is deduced from (5.3), (5.4) that

$$\begin{aligned}\phi_1 &= -4\int_0^\infty \frac{e^{-uX}[(u-ik)\cosh ux - ik\sinh ux]}{u(u-2ik)(u^2+K^2)} F(u; y)F(u; Y)\,du \\ &\quad - 4\pi i\frac{(K+k)\cos Kx - ik\sin Kx}{K+2k} e^{-K(y+Y)+iKX} \quad (0 < x < X), \\ &= -4\int_0^\infty \frac{e^{-ux}[(u-ik)\cosh uX - ik\sinh uX]}{u(u-2ik)(u^2+K^2)} F(u; y)F(u; Y)\,du \\ &\quad - 4\pi i\frac{(K+k)\cos KX - ik\sin KX}{K+2k} e^{-K(y+Y)+iKx} \quad (x > X) \qquad (8.1)\end{aligned}$$

and

$$\begin{aligned}\phi_2 &= 4ik\int_0^\infty \frac{e^{u(x-X)}}{u(u-2ik)(u^2+K^2)} F(u; y)F(u; Y)\,du \\ &\quad - \frac{4\pi ik}{K+2k} e^{-K(y+Y)-iK(x-X)}. \qquad (8.2)\end{aligned}$$

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For $X < 0$ it is found likewise that

$$\phi_1 = 4ik \int_0^\infty \frac{e^{-u(x-X)}}{u(u-2ik)(u^2+K^2)} F(u;y)F(u;Y)\,du - \frac{4\pi ik}{K+2k} e^{-K(y+Y)+iK(x-X)} \tag{8.3}$$

and

$$\begin{aligned}\phi_2 &= -4\int_0^\infty \frac{e^{uX}[(u-ik)\cosh ux + ik\sinh ux]}{u(u-2ik)(u^2+K^2)} F(u;y)F(u;Y)\,du \\ &\quad - 4\pi i \frac{(K+k)\cos Kx + ik\sin Kx}{K+2k} e^{-K(y+Y)-iKX} \quad (X < x < 0), \\ &= -4\int_0^\infty \frac{e^{ux}[(u-ik)\cosh uX + ik\sinh uX]}{u(u-2ik)(u^2+K^2)} F(u;y)F(u;Y)\,du \\ &\quad - 4\pi i \frac{(K+k)\cos KX + ik\sin KX}{K+2k} e^{-K(y+Y)-iKx} \quad (x < X).\end{aligned} \tag{8.4}$$

These may also be deduced from (8.1), (8.2) by reversing the horizontal coordinate direction.

Again note from (8.1 – 8.4) evident source reciprocity properties on the same side and opposite sides of the barrier, which may also be established independently using the reduction procedure and reciprocity property noted in Section 5.

9. Two-sided wave-maker problem

In conclusion the problem for the antisymmetric motion with outgoing progressive waves due to the oscillations of a two-sided vertical wave-maker $x = 0$ is now solved by the reduction method, using the result of Section 6 for a one-sided wave-maker. Here the conditions (2.4b) on the potentials are replaced by

$$\phi_{1x} = U(y) - ik(\phi_1 - \phi_2) = \phi_{2x} \quad \text{on} \quad x = 0$$

if the wave-maker has the same velocity again and the extra conditions are

$$\phi_1, \phi_2 \to \text{ multiples of } e^{-Ky \pm iKx} \quad \text{as} \quad x \to \pm\infty$$

respectively in addition to (2.1 – 2.3b). Note from (7.1) that

$$\phi_1^*(x,y) = \phi_2(-x,y) = -\phi_1(x,y)$$

due to the antisymmetry, and (7.3) simplifies to $\Phi = 2\phi_1$, $\Psi = 0$. Thus Φ remains to be found and then $\phi_1 = \frac{1}{2}\Phi$ from (7.8), which it is sufficient to find. Hence, to continue, the condition (7.7) on this potential is replaced by

$$\Phi_x + 2ik\Phi = 2U(y) \quad \text{on} \quad x = 0$$

and the extra condition is

$$\Phi \to \text{ a multiple of } e^{-Ky+iKx} \quad \text{as} \quad x \to \infty$$

in addition to (7.4 – 7.6) for Φ. By reference to Section 6 the potential Φ describes a one-sided porous ($k \to 2k$) wave-maker ($U \to 2U$) problem, and the solution is deduced from (6.1). Thus

$$\phi_1 = -\frac{2}{\pi}\int_0^\infty \frac{a(u)e^{-ux}}{(u-2ik)(u^2+K^2)} F(u;y)\,du - \frac{2iKA}{K+2k} e^{-Ky+iKx}, \tag{9.1}$$

where $a(u)$, A are the same again.

10. Modifications for surface tension

If there is surface tension T in addition to gravity the free-surface condition (2.2a) for a *one-sided* boundary becomes

$$K\phi + \phi_y + M\phi_{yyy} = 0 \quad \text{on} \quad y = 0, \tag{10.1a}$$

where $M = T/\rho g$ (ρ is the density of the water), and progressive waves now have wave number κ satisfying the cubic equation $M\kappa^3 + \kappa - K = 0$. Also for uniqueness a condition at the edge $(0,0)$ is needed, which is determined below and generalizes the condition prescribing $\phi_{xy}(0+,0)$ for $k = 0$. Because of this no realistic simple solutions now exist as in the absence of surface tension ($M = 0$), and there are significant changes to our previous results.

Now it is reasonable to suppose that for a porous *wall* there is a potential corresponding to the familiar slope potential G_0 for $k = 0$, which satisfies $G_{0x} = 0$ on $x = 0$, $G_{0xy}(0+,0) = \pi/M$, and represents outgoing waves as $x \to \infty$ (see Rhodes-Robinson [7, 9]). To determine this generalised slope potential we follow the same procedure as for $k = 0$ and construct it from the potential for a wave source in front of a wall (or one form thereof), for the moment leaving the modified edge condition unspecified.

The simplest potential ϕ for a *wave source* at (X, Y) *in front of a wall* is found as for $M = 0$ in Section 5 by superposing the fundamental source potentials

$$G(x, y; \pm X, Y) = -2\int_0^\infty \frac{e^{-u|x\mp X|}}{u[u^2(1-Mu^2)^2+K^2]} F(u; y)F(u; Y)\,du - 2\pi i\frac{1+M\kappa^2}{1+3M\kappa^2}e^{-\kappa(y+Y)+i\kappa|x\mp X|}$$

from Rhodes-Robinson [7] to obtain the solution for $k = 0$, where now $F(u; s) = u(1-Mu^2)\cos us - K\sin us$ $(u > 0, s > 0)$, and then adding on the correction potential

$$-4ik\int_0^\infty \frac{e^{-u(x+X)}}{u(u-ik)[u^2(1-Mu^2)^2+K^2]}F(u; y)F(u; Y)\,du + 4\pi i\frac{1+M\kappa^2}{1+3M\kappa^2}\frac{k}{\kappa+k}e^{-\kappa(y+Y)+i\kappa(x+X)}$$

to satisfy (2.4a). Hence

$$\begin{aligned}\phi &= -4\int_0^\infty \frac{e^{-uX}(u\cosh ux - ik\sinh ux)}{u(u-ik)[u^2(1-Mu^2)^2+K^2]}F(u; y)F(u; Y)\,du \\ &\quad -4\pi i\frac{1+M\kappa^2}{1+3M\kappa^2}\frac{\kappa\cos\kappa x - ik\sin\kappa x}{\kappa+k}e^{-\kappa(y+Y)+i\kappa X} \quad (0 < x < X), \quad (10.2)\\ &= -4\int_0^\infty \frac{e^{-ux}(u\cosh uX - ik\sinh uX)}{u(u-ik)[u^2(1-Mu^2)^2+K^2]}F(u; y)F(u; Y)\,du \\ &\quad -4\pi i\frac{1+M\kappa^2}{1+3M\kappa^2}\frac{\kappa\cos\kappa X - ik\sin\kappa X}{\kappa+k}e^{-\kappa(y+Y)+i\kappa x} \quad (x > X). \quad (10.3)\end{aligned}$$

Denote this potential by $\phi = G^{\text{mod}}(x, y; X, Y; k)$ again. To determine the *generalised slope potential* ϕ now apply Green's theorem with G^{mod} over the region to give

$$\begin{aligned}\phi(X, Y) &= \frac{M}{2\pi K}\left[\phi_{xy}(0+, 0)G_y^{\text{mod}}(0, 0; X, Y) - G_{xy}^{\text{mod}}(0, 0; X, Y)\phi_y(0, 0)\right]\\ &= \frac{M}{2\pi K}\left[\phi_{xy}(0+, 0) + ik\phi_y(0, 0)\right]G_y^{\text{mod}}(0, 0; X, Y),\end{aligned}$$

since $G_{xy}^{\text{mod}}(0, 0; X, Y) = -ikG_y^{\text{mod}}(0, 0; X, Y)$. Details are omitted as they are similar to a calculation for $k = 0$ in Rhodes-Robinson [7], Section 4. Hence for uniqueness

$$\phi_{xy}(0+, 0) + ik\phi_y(0, 0) = \frac{\pi}{M}$$

must be prescribed at the edge, the value for $k = 0$ being taken for consistency. Thus

$$\begin{aligned}\phi(X, Y) &= \frac{1}{2K} G_y^{\text{mod}}(0, 0; X, Y) \\ &= 2 \int_0^\infty \frac{u e^{-uX}}{(u - ik)[u^2(1 - Mu^2)^2 + K^2]} F(u; Y)\, du \\ &\quad + \frac{2\pi i}{1 + 3M\kappa^2} \frac{\kappa}{\kappa + k} e^{-\kappa Y + i\kappa X}\end{aligned} \tag{10.4}$$

from (10.2).

Denote this potential by $\phi = G_0^*(x, y; k)$ in terms of the original variables. Then $G_0 = G_0^*(x, y; 0)$. Note that it is not defined for $M = 0$. It follows in our other problems that

$$\phi_{xy}(0+, 0) + ik\phi_y(0, 0) = \pi\lambda \tag{10.5a}$$

(say) must be prescribed in terms of an edge constant λ for uniqueness; the formal solutions then contain the multiple $M\lambda G_0^*$, since G_0^* has edge constant $1/M$. Note that this is not necessary in selecting source potentials for use in Green's theorem, when the simplest form corresponding to $\lambda = 0$ as in (10.2), (10.3) for G^{mod} is sufficient.

To extend the earlier solutions the easiest method is to first subtract out the solutions corresponding to $\lambda = 0$, which are inferred from the earlier ones. This leaves the same problem each time with solution $M\lambda G_0^*$. Hence, for example, the solution for *incident waves* $e^{-\kappa y - i\kappa x}$ *against a wall* is

$$\phi = e^{-\kappa y - i\kappa x} + \frac{\kappa - k}{\kappa + k} e^{-\kappa y + i\kappa x} + M\lambda G_0^*(x, y; k), \tag{10.6}$$

in which the reflected amplitude constant is

$$R = \frac{1}{\kappa + k}\left[\kappa - k + \frac{2\pi i M\kappa\lambda}{1 + 3M\kappa^2}\right].$$

The known solution for $k = 0$ is

$$\phi = e^{-\kappa y - i\kappa x} + e^{-\kappa y + i\kappa x} + M\lambda G_0^*(x, y; 0) \tag{10.7}$$

as in Rhodes-Robinson [9].

Note finally that a dynamical edge condition is still needed in such problems, presumably relating $\phi_y(0, 0)$ and $\phi_{xy}(0+, 0)$ as for $k = 0$, in order to determine the actual values of λ by applying the condition to formal solutions like (10.6). This has been done for $k = 0$ using the condition from Hocking [6]. However, the appropriate condition needed here is not known at present on the good authority of the last-named author.

For a *two-sided* boundary the free-surface conditions (2.2b) become

$$K\phi_1 + \phi_{1y} + M\phi_{1yyy} = 0 = K\phi_2 + \phi_{2y} + M\phi_{2yyy} \quad \text{on} \quad y = 0 \qquad (10.1\text{b})$$

and the edge conditions taken are

$$\phi_{1xy}(0+, 0) - \pi\lambda_1 = -ik[\phi_{1y}(0, 0) - \phi_{2y}(0, 0)] = \phi_{2xy}(0-, 0) - \pi\lambda_2 \quad (10.5\text{b})$$

(say) in terms of edge constants λ_1, λ_2. The *reduction method* is unchanged, except that the conditions (7.5) on the reduction potentials become

$$K\Phi + \Phi_y + M\Phi_{yyy} = K\Psi + \Psi_y + M\Psi_{yyy} = 0 \quad \text{on} \quad y = 0 \qquad (10.8)$$

and also

$$\Phi_{xy}(0+, 0) + 2ik\Phi_y(0, 0) = \pi(\lambda_1 + \lambda_2), \qquad \Psi_{xy}(0+, 0) = \pi(\lambda_1 - \lambda_2) \quad (10.9)$$

from (10.5b), and so a similar interpretation in terms of one-sided boundaries may again be made. Hence, for example, the solutions for *incident waves against a barrier* are deduced from (10.6), (10.7) as

$$\begin{aligned}
\phi_1 = {} & e^{-\kappa y - i\kappa x} + \frac{\kappa}{\kappa + 2k} e^{-\kappa y + i\kappa x} \\
& + \frac{1}{2} M\lambda_1 [G_0^*(x, y; 0) + G_0^*(x, y; 2k)] \\
& - \frac{1}{2} M\lambda_2 [G_0^*(x, y; 0) - G_0^*(x, y; 2k)], \qquad (10.10) \\
\phi_2 = {} & \frac{2k}{\kappa + 2k} e^{-\kappa y - i\kappa x} \\
& + \frac{1}{2} M\lambda_1 [G_0^*(-x, y; 0) - G_0^*(-x, y; 2k)] \\
& - \frac{1}{2} M\lambda_2 [G_0^*(-x, y; 0) + G_0^*(-x, y; 2k)], \qquad (10.11)
\end{aligned}$$

in which the reflected and transmitted amplitude constants are

$$\begin{aligned}
R &= \frac{1}{\kappa + 2k} \left[\kappa + 2\pi i M \frac{(\kappa + k)\lambda_1 - k\lambda_2}{1 + 3M\kappa^2} \right], \\
T &= \frac{1}{\kappa + 2k} \left[2k + 2\pi i M \frac{k\lambda_1 - (\kappa + k)\lambda_2}{1 + 3M\kappa^2} \right].
\end{aligned}$$

Again the values of λ_1, λ_2 would be determined from appropriate dynamical edge conditions.

Note that there are now two generalised slope potential pairs having edge constants $1/M$, 0 and 0, $1/M$ that can be identified in solutions such as (10.10), (10.11), which contain the multiples $M\lambda_1$, $M\lambda_2$ respectively of these pairs.

Gorgui, Faltas and Ahmed [4] have considered the two-sided wave-maker problem and incident wave problem against a barrier, but their edge conditions appear to be incorrect.

11. Extension to finite depth

The results found above may all be extended to water of finite constant depth h, when the regions are in $0 < y < h$ and the conditions (2.3) are replaced by $\phi_y = 0$ and $\phi_{1y} = 0 = \phi_{2y}$ on the bottom $y = h$. This includes the reduction method. Some selected results for a one-sided boundary are now stated in conclusion, derived as for infinite depth.

In the absence of surface tension the potential for normalised *incident waves against a wall* is

$$\phi = \frac{\cosh k_0(h-y)}{\cosh k_0 h}\left[e^{-ik_0 x} + \frac{k_0 - k}{k_0 + k} e^{ik_0 x}\right] \tag{11.1}$$

and the potential for a *wave source in front of a wall* is

$$\begin{aligned} G^{\mathrm{mod}} = &- 4\pi i \frac{k_0 \cos k_0 x - ik \sin k_0 x}{k_0 + k} \frac{\cosh k_0(h-y)\cosh k_0(h-Y) e^{ik_0 X}}{k_0 h + \sinh k_0 h \cosh k_0 h} \\ &- 4\pi \sum_{n=1}^{\infty} \frac{k_n \cosh k_n x - ik \sinh k_n x}{k_n - ik} \frac{\cos k_n(h-y)\cos k_n(h-Y) e^{-k_n X}}{k_n h + \sin k_n h \cos k_n h} \end{aligned} \tag{11.2}$$

$(0 < x < X)$, etc. $(x > X)$ by reciprocity, where $k_0 \tanh k_0 h = K$, $k_n \tan k_n h + K = 0$ $(n = 1, 2, \dots)$. Chwang [3] has previously solved the two-sided wave-maker problem using the method of Havelock [5].

In the presence of surface tension the *generalised slope potential* with edge constant $1/M$ is

$$\begin{aligned} G_0^* = &\, 2\pi i \frac{\kappa_0}{\kappa_0 + k} \frac{\cosh \kappa_0 h \cosh \kappa_0 (h-y) e^{i\kappa_0 x}}{\kappa_0 h(1 + M\kappa_0^2) + (1 + 3M\kappa_0^2)\sinh \kappa_0 h \cosh \kappa_0 h} \\ &+ 2\pi \sum_{n=1}^{\infty} \frac{\kappa_n}{\kappa_n - ik} \frac{\cos \kappa_n h \cos \kappa_n (h-y) e^{-\kappa_n x}}{\kappa_n h(1 - M\kappa_n^2) + (1 - 3M\kappa_n^2)\sin \kappa_n h \cos \kappa_n h}, \end{aligned} \tag{11.3}$$

where $\kappa_0(1 + M\kappa_0^2)\tanh \kappa_0 h = K$, $\kappa_n(1 - M\kappa_n^2)\tan \kappa_n h + K = 0$ $(n = 1, 2, \dots)$. Chakrabarti [1] has considered the two-sided piston wave-maker problem, but only for $\lambda_1 = 0 = \lambda_2$.

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