ON CHARACTERISTIC MORPHISMS

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1. Introduction

This note is concerned with a translation of some concepts and results about characteristic subgroups of a group into the language of categories. As an example, consider strictly characteristic and hypercharacteristic subgroups of a group: the subgroup H of the group G is called strictly characteristic [1] in G if it admits all ependomorphisms of G; that is all homomorphic mappings of G onto G; and H is called hypercharacteristic [2] in G if it is the least normal subgroup with factor group isomorphic to G/H, that is if H is contained in every normal subgroup K of G with $G/K \cong G/H$.

It is not difficult to translate these definitions into universal algebra: it is clear what it means that a subalgebra B of an arbitrary algebra A admits all ependomorphisms of A; and when it comes to translating 'hypercharacteristic', one notices at once that this will properly appertain not to subalgebras, but to congruences: it is obvious what it means that one congruence on an algebra A is contained in another congruence on A, and also what it means that they have isomorphic factor algebras. It is also obvious how one would define that a congruence on A *admits* an ependomorphism of A, so that such notions as 'strictly characteristic' can be applied to congruences on A, not only to subalgebras of A. Such translations into the language of universal algebra have been carried out and discussed in detail by Jürgen Schmidt [7].

A translation to category language is less immediate; one has to translate such notions as 'ependomorphism' and 'admits' and 'is contained in' and 'have isomorphic factor objects'. Some of the translations are straightforward, some not quite: not that appropriate translations are lacking, but there may be several contenders. This *embarras de richesse* is common in extensions of definitions to more general situations.

It follows at once from the definition that a strictly characteristic

[1] Introduced by Baer [1].

[2] Introduced as 'übercharakteristisch' in [5].

subgroup is characteristic, and it is also easy to see that a hypercharacteristic subgroup is characteristic [5, Satz 2.3]. Even more is true: every hypercharacteristic subgroup is strictly characteristic [7, Satz 16; 4, Theorem 2.1]. By the same token, every hypercharacteristic congruence is, in the natural translation into universal algebra, also strictly characteristic [7, Satz 16, Korollar]. We shall wish to examine whether the corresponding proposition is true in categories, using our translated concepts; and correspondingly for other interrelations between the different kinds of characteristic subgroups, congruences, and morphisms.

2. Preliminaries

As there is no unanimity on category notation, we first fix ours. Elements of a category are called morphisms and denoted by lower case italic letters. A partial multiplication, denoted by juxtaposition, is assumed between morphisms, and this is subject to the associative law:

If two out of xy, yz, $(xy)z$ or out of xy, yz, $x(yz)$ are defined, then so is the third (and thus all four); and then $(xy)z = x(yz)$.

There are identity morphisms, for which we reserve the letters e, f, g: such an identity e is characterized by the property that if ex or xe is defined, then it equals x; and ee is defined (and thus $ee = e$). *To every morphism a there is a left identity,* say e, *and a right identity,* say f, such that $ea = a = af$. We write in this case

$$a : e \to f.$$

The product ab exists if, and only if, a right identity of a is a left identity [3] *of b*, say $a : e \to f$ and $b : f \to g$. It then follows that $ab : e \to g$, and moreover the left and right identities of a morphism are unique. They need not, however, be distinct; if

$$h : e \to e,$$

then h is called an *endo* of e; and we reserve the letter h for endos.

We do not introduce objects as separate entities, but use the identity morphisms as their synonyms.

The morphism a is called *epic* if it is left cancellable [3], that is if

$$ax = ay \text{ implies } x = y;$$

dually it is called *monic* if it is right cancellable, that is if

$$xa = ya \text{ implies } x = y.$$

[3] Note that we use algebraic rather than topological conventions.

If $a : e \to f$, we write $a : e \twoheadrightarrow f$ if a is epic, $a : e \rightarrowtail f$ if a is monic, and $a : e \rightarrowtail\!\!\!\twoheadrightarrow f$ if a is both. We call $a : e \to f$ *left invertible* if there is a morphism $a' : f \to e$ such that $a'a = f$; and *right invertible* if there is a morphism $a'' : f \to e$ such that $aa'' = e$. If a is both left and right invertible, it is called *iso*; then a' and a'' are unique and coincide. A left invertible morphism is an epic, and a right invertible morphism is a monic; but if $a : e \rightarrowtail\!\!\!\twoheadrightarrow f$ is both epic and monic, it need not be an iso.

The classes of endos, epics, monics, left invertibles, right invertibles, isos are relatively closed: if the product of two endos exists, then it is again an endo; and so for the others.

We say that the morphism *a left divides* the morphism b if there is a morphism c such that $ac = b$; and dually for right divisibility. Thus a left invertible morphism is one that right divides its right identity. A morphism that right divides an epic is itself an epic; a morphism that right divides a left invertible morphism is itself left invertible; and dually for monics and right invertibles.

3. Translations

We have already translated 'endomorphisms' by calling h an *endo* if its right and left identities coincide:

$$h : e \to e.$$

If h is, moreover, an epic, we call it an *ependo*:

$$h : e \twoheadrightarrow e;$$

and a *monendo* if it is monic:

$$h : e \rightarrowtail e.$$

If h is both epic and monic, we call it a *weak auto*:

$$h : e \rightarrowtail\!\!\!\twoheadrightarrow e;$$

and h is a *strong auto* if it is an iso. Clearly a strong auto is also a weak auto; but not in general conversely. In the category formed by all homomorphisms of groups, however, both weak and strong autos coincide simply with automorphisms.

One could invent autos that are intermediate between weak and strong, and thus also translate automorphisms of groups: thus one could consider ependos that are right invertible, or left invertible monendos; they might be called right strong and left strong autos, respectively.

We next seek a translation of the partial order of normal subgroups of a group, or of congruences of an algebra. Again several possibilities

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present themselves; we single out two, which we again distinguish as 'weak' and 'strong' and denote by $\leqq_w$ and $\leqq_s$, respectively. We define $\leqq_w$ by

(3.1) $a \leqq_w b$ *if, and only if, for all monics* x, y, $xa = ya$ *implies* $xb = yb$.

This means that if xa and ya are both defined and equal, then xb and yb are both defined and equal; and it follows that a and b must have the same left identity.

We define $\leqq_s$ simply as left divisibility:

(3.2) $a \leqq_s b$ *if, and only if, there is a morphism* c *such that* $ac = b$.

Again this clearly implies that a and b have the same left identity; and it is easy to see that $a \leqq_s b$ implies $a \leqq_w b$. It should be noted that these are in general not orders, but only quasiorders: they are reflexive and transitive, but not necessarily antisymmetric.

Our choice of 'weak' and 'strong' is somewhat arbitrary. One could, for example, make $\leqq_w$ less weak by asking that $xa = ya$ imply $xb = yb$ for all morphisms x, y, not only for monics; or one could strengthen $\leqq_s$ by demanding that c be an epic. In the only case of importance for applications, namely when a and b are epics, this makes no difference, as c then necessarily is also an epic. All these quasiorders, which we denote collectively by $\leqq$, translate the phrase 'the kernel (congruence) of the epimorphism a is contained in that of the epimorphism b'': we omit the verification.

Next we translate the word 'admits' by means of the definition:

(3.3) *The morphism* $a : e \to f$ *admits the endo* $h : e \to e$ *if, and only if,* $a \leqq ha$.

This is, in fact, more than one definition, because it depends on the interpretation of the quasiorder $\leqq$; when necessary, we distinguish between 'weakly admits' and 'strongly admits'.

Let us, as an example, verify this translation, say for groups. Thus we have a group G, an epimorphism $\alpha : G \twoheadrightarrow H$ with $\ker \alpha = K$, and an endomorphism $\eta : G \to G$, and we want to express, in terms of kernels of the various mappings involved, that K admits η, that is to say that

(3.4) $$K\eta \leqq K.$$

Now $\ker (\eta\alpha) = (\ker \alpha)\eta^{-1} = K\eta^{-1}$, and from (3.4) we get

$$(K\eta)\eta^{-1} \leqq K\eta^{-1}.$$

As, moreover, obviously $K \leqq (K\eta)\eta^{-1}$, we have $K \leqq K\eta^{-1}$; thus (3.4) implies

(3.5) $$\ker \alpha \leqq \ker (\eta\alpha).$$

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Conversely, assume (3.5), which is equivalent to $K \leqq K\eta^{-1}$. This gives $K\eta \leqq (K\eta^{-1})\eta$, and as $(K\eta^{-1})\eta = K$, we derive (3.4). Thus (3.4) and (3.5) are equivalent, and our translation of 'admits' is verified.

Finally we have to translate the statement that factor groups of two normal subgroups are isomorphic, or that one is isomorphic to a subgroup of the other; or the corresponding statements about factor algebras of two congruences. Here we simplify slightly, by making the translation of the one statement

$$a : e \twoheadrightarrow f \quad \textit{and} \quad a' : e \twoheadrightarrow f, \tag{3.6}$$

and of the other

$$a : e \twoheadrightarrow f \quad \textit{and} \quad a' : e \rightarrow f. \tag{3.7}$$

A more literal translation of the first would be

$$a : e \twoheadrightarrow f \textit{ and } a' : e \twoheadrightarrow g \textit{ and there is an iso } i : g \rightarrowtail\!\!\!\twoheadrightarrow f,$$

and similarly for the second; but no generality is thereby gained. We omit the verification.

4. Characteristic morphisms

We are now ready to translate also the definitions of various kinds of characteristic subgroups of a group, or characteristic congruences of an algebra. We recall that the subgroup H of the group G is (i) characteristic, (ii) strictly characteristic, (iii) S-characteristic [4], (iv) fully invariant [5] if it admits (i) all automorphisms, (ii) all ependomorphisms, (iii) all monendomorphisms, (iv) all endomorphisms of G. The corresponding definitions are then:

(4.1) *The morphism* [6] $a : e \rightarrow f$ *is*

(i) *characteristic,* (ii) *strictly characteristic,* (iii) *S-characteristic,* (iv) *fully invariant if*

$$a \leqq ha$$

for all (i) *autos,* (ii) *ependos,* (iii) *monendos,* (iv) *endos*

$$h : e \rightarrow e,$$

respectively.

As these definitions involve the quasiorder $\leqq$, each is capable of more than one interpretation, according as $\leqq$ is taken as $\leqq_w$ or $\leqq_s$ (or one of the other possibilities); we may accordingly speak of 'weakly fully invariant'

[4] Introduced by Baer [1].

[5] Introduced as 'vollinvariant' by Levi [3].

[6] These definitions could reasonably be restricted to epics $a : e \twoheadrightarrow f$.

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or 'strongly fully invariant', and so on. In the case of 'characteristic', a further ambiguity is introduced by the autos, which can be interpreted as weak or strong.

The normal subgroup H of G is called (v) hypercharacteristic, (vi) hyperinvariant [7], (vii) ultracharacteristic [8], (viii) ultrainvariant [9] in G if among all normal subgroups whose factor groups are (v) isomorphic to G/H it is the least, (vi) isomorphic to a subgroup of G/H it is the least, (vii) isomorphic to G/H it is the greatest, (viii) isomorphic to a subgroup of G/H it is the greatest. Only the first two of these are of interest: ultracharacteristic subgroups are barely of any use, and ultrainvariant subgroups of none whatever; the only ultrainvariant subgroup of G is G itself, and this is also the only ultracharacteristic subgroup if, for example, G is free (or relatively free) of infinite rank. The translated definitions are as follows:

(4.2) *The epic* [10] $a : e \twoheadrightarrow f$ *is* (v) *hypercharacteristic,* (vi) *hyperinvariant,* (vii) *ultracharacteristic,* (viii) *ultrainvariant if*

(v), (vi) $\qquad a \leqq a'$ *or*

(vii), (viii) $\qquad a' \leqq a$

for all (v), (vii) *epics* $a' : e \twoheadrightarrow f$ *or* (vi), (viii) *morphisms* $a' : e \to f$, *respectively.*

Again each of these definitions splits into several, depending on the interpretation of the quasiorder $\leqq$.

If H is hypercharacteristic or hyperinvariant in the group G, then it is uniquely determined by its factor group G/H. If $a : e \twoheadrightarrow f$ is hypercharacteristic or hyperinvariant, it need not be unique: if also $a_1 : e \twoheadrightarrow f$ is hypercharacteristic or hyperinvariant, then $a \leqq a_1$ and $a_1 \leqq a$, and thus a and a_1 are equivalent under the natural equivalence derived from the quasiorder $\leqq$; but this equivalence will not in general be equality. The same remark applies to ultracharacteristic and ultrainvariant morphisms.

5. Interrelations

We first note some simple examples of characteristic morphisms.

(5.1) *Every monic is weakly characteristic, weakly strictly characteristic, weakly S-characteristic, weakly fully invariant. Every monic epic is weakly hypercharacteristic, weakly hyperinvariant, weakly ultracharacteristic, weakly ultrainvariant.*

[7] Introduced as 'überinvariant', for congruences in universal algebra, by Schmidt [7].
[8] Introduced in [5].
[9] Introduced here in analogy with the two preceding notions.
[10] Nothing appears to be gained by making these definitions for morphisms in general.

Let $a : e \rightarrowtail f$ be monic and $b : e \to g$ be arbitrary. Then if[11] $xa = ya$, then xb and yb are defined; and moreover $x = y$ as a is monic, and thus $xb = yb$. It follows that

$$a \leqq_w b$$

whenever a is monic and has the same left identity as b, and (5.1) is now obvious. Again if $a : e \rightarrowtail f$ is right invertible and $b : e \to g$ arbitrary, then

$$a \leqq_s b;$$

for there is then a morphism $a'' : f \twoheadrightarrow e$ with $aa'' = e$, and thus there is a morphism c, namely $c = a''b$, such that $ac = b$. Hence every right invertible morphism is strongly less than every morphism with the same left identity, and we have:

(5.2) *Every right invertible is strongly characteristic, strongly strictly characteristic, strongly S-characteristic, strongly fully invariant. Every right invertible epic is strongly hypercharacteristic, strongly hyperinvariant, strongly ultracharacteristic, and strongly invariant.*

It makes no difference here whether 'characteristic' is interpreted in terms of weak or strong autos. Equally obvious are the interrelations that flow from the gradual narrowing down of the classes of endos that are to be admitted:

(5.3) *Every fully invariant morphism is both strictly characteristic and S-characteristic; every strictly characteristic morphism, and also every S-characteristic morphism is characteristic in terms of weak autos; and every characteristic morphism in terms of weak autos is characteristic in terms of strong autos.*

Next we show that some of the less obvious interrelations [7, Satz 16; 4, Theorem 2.1] remain true after translation:

(5.4) *Every hypercharacteristic epic is strictly characteristic; every hyperinvariant epic is fully invariant.*

Let $a : e \twoheadrightarrow f$ be hypercharacteristic, and let $h : e \twoheadrightarrow e$ be an arbitrary ependo. Then $a' = ha : e \twoheadrightarrow f$ is also an epic, and by (4.2) then $a \leqq ha$. This is true for all ependos h of e, and a is, by (4.1), strictly characteristic. Similarly if $a : e \twoheadrightarrow f$ is hyperinvariant and $h : e \to e$ an arbitrary endo of e, then $a \leqq ha$, again by (4.2), and a is, again by (4.2), fully invariant. Note that we have not had to declare whether we were thinking of the weak or strong (or any of the other) notions: the argument is precisely the same, whatever quasiorder $\leqq$ it is based on.

[11] This applies to arbitrary x, y such that xa, ya are defined, though it is needed only for monic x, y.

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By contrast we can show the analogue of 'every ultracharacteristic subgroup is hypercharacteristic' [5, Satz 2.2] only for the corresponding strong notions:

(5.5) *Every strongly ultracharacteristic epic is strongly hypercharacteristic; every strongly ultrainvariant epic is strongly hyperinvariant.*

Let $a : e \twoheadrightarrow f$ be strongly ultracharacteristic, and let $a' : e \twoheadrightarrow f$ be another epic; then we are given, by (4.2), that $a' \leqq_s a$, and have to prove that $a \leqq_s a'$. Now $a' \leqq_s a$ means that there is a morphism c such that $a'c = a$. As a is an epic, so is then $c : f \twoheadrightarrow f$. Put $a'' = ac$; then also $a'' : e \twoheadrightarrow f$ is an epic, and by assumption $a'' \leqq_s a$. Thus there is a morphism d such that $a''d = a$, or

$$acd = a = af.$$

As a is an epic, this implies $cd = f$, and further

$$ad = a'cd = a'f = a'.$$

Thus $a \leqq_s a'$, as claimed, and the first part of (5.5) follows. The second part follows similarly, only with $a' : e \to f$ not being assumed an epic: we omit the details. One observes that d, like e, is an epic, and thus a' is an epic, too. Thus we have the following corollary:

(5.6) *If $a : e \twoheadrightarrow f$ is strongly ultrainvariant, then every morphism $a' : e \to f$ is epic.*

To illustrate that ultrainvariant morphisms are uninteresting, we state, without proof, the following corollary:

(5.7) *The epic $a : e \twoheadrightarrow f$ is ultrainvariant in a category with zeros if, and only if, f is a zero.*

This corresponds to the fact that the only ultrainvariant subgroup of a group G is G itself.

6. Discussion

In defining 'hyperinvariant subgroups' in § 4, I have misquoted Schmidt; he defines [7, § 2], not just in groups but at once in universal algebra, hyperinvariant epimorphisms and hyperinvariant congruences. In the special case of groups these congruences, or rather the corresponding normal subgroups, specialize to those I have here called hyperinvariant. There is, however, an important difference between the epimorphisms that Schmidt uses and the epics in categories in general; we shall return to this presently.

Moreover Schmidt uses, implicitly, only what we have here called the 'strong' quasiorder $\leqq_s$. There are interesting categories of algebras, even of groups, in which the weak quasiorder $\leqq_w$ is distinct from $\leqq_s$, and the corresponding notions may not be entirely useless.

An epimorphism of an algebra is defined as a homomorphism that induces an onto mapping of the carrier of the algebra; this notion is not intrinsic in categories, but uses, as it were, the forgetful functor to the category of sets. The definition of an epic in a category is intrinsic, but does not, in general, coincide with that of an epimorphism defined by onto mappings of the carrier. The most noticeable effect of this discrepancy is perhaps in the notion of 'projective'. Peter M. Neumann [6] has recently analysed this situation in some detail, with special reference to categories of groups.

We need a more elaborate language than we have so far used, and we make (or repeat) the following definitions:

An *epic* is, as before (see § 2), a left cancellable morphism. An *onto epic* is defined only in a 'set based' category, that is a category with a natural 'forgetful' functor to the category of sets: it is then a morphism whose image under this functor is an epic (or onto map) in the category of sets. It is easy to see that every onto epic is an epic, but not conversely. In order to approximate in intrinsic category language to the notion of an onto epic, we define the morphism $a : e \twoheadrightarrow f$ to be a *regular epic* [12] if it is an epic and if, moreover,

$$a \leqq_w b \quad \textit{implies} \quad a \leqq_s b.$$

We shall not demonstrate here the proposition that in many important set based categories, in particular those formed of all homomorphisms in a variety of algebras, the regular epics are precisely the onto epics. The first of the following definitions is standard.

(6.1) *The identity* e *is* (i) *projective,* (ii) *onto projective,* (iii) *regularly projective if to every morphism* $c : e \to g$ *and to every* (i) *epic,* (ii) *onto epic,* (iii) *regular epic* $b : f \twoheadrightarrow g$ *there is a morphism* $a : e \to f$ *such that*

$$ab = c.$$

In the case of a category consisting of all homomorphisms in a class $\mathfrak{X}$ of groups, Peter M. Neumann [6] speaks of (i) $\mathfrak{X}$-projectives and (ii) lifting groups for $\mathfrak{X}$; and shows that when $\mathfrak{X}$ is the variety generated by the icosahedral group, only the trivial groups are projective and all finite groups in $\mathfrak{X}$ are lifting groups. Schmidt [7] calls projective what I have called onto projective and remarks that free algebras are, in his sense,

[12] This term was suggested to me by Professor G. M. Kelly. Grothendieck uses 'épimorphisme strict' [2; définition 2.2].

projective. He proves [7, Satz 17] that, in the language of the present paper, an onto epic from an onto projective is hyperinvariant if it is fully invariant: this extends Satz 2.8 of [5]. This can be translated to categories as follows:

(6.2) *Let e be a regularly projective identity, and let $a : e \twoheadrightarrow f$ be a fully invariant regular epic. Then a is hyperinvariant.*

We have to show that if $a' : e \to f$ is arbitrary, then $a \leqq a'$. By the definition of regularly projective, there is a morphism $h : e \to e$ such that $a' = ha$; and as h is an endo of e and a is fully invariant, then $a \leqq ha = a'$ as claimed. Note that again, as with (5.4), we have proved several propositions at once, according to the interpretation of the quasiorder $\leqq$, and the corresponding interpretation of 'fully invariant' and 'hyperinvariant'.

7. Additional remarks

As a bonus for our translation into category language we get duality; and we may then ask what the dual notions become when translated back into the language of groups — or, to be more precise, interpreted in the category of all homomorphisms of all groups, for if we narrow down the class of groups to, say, the variety generated by the icosahedral group, then the answers are quite different.

Thus we shall think of monics, as defining subgroups, instead of epics, defining normal subgroups. The duals of the quasiorders $\leqq$ we have defined translate simply to the reverse order for the subgroups, and the weak and strong notions again coincide. The dual 'co-admits' of 'admits' is the same as 'admits' in the usual sense. Endos and autos are self-dual, ependos and monendos dual to each other. Hence 'fully invariant' and 'characteristic' are self-dual, and 'strictly characteristic' and 'S-characteristic' dual to each other. The subgroup H is 'co-hypercharacteristic' in the group G if it contains every subgroup of G that is isomorphic to H. It is 'co-hyperinvariant' if it contains every subgroup of G isomorphic to a factor group of H. Similarly H is 'co-ultracharacteristic' or 'co-ultrainvariant' in G if it is contained in every isomorphic copy of itself, or every homomorphic image of itself, in G. Only the trivial group can be co-ultrainvariant; and H is co-ultracharacteristic in G if, and only if, it is unique in G in its isomorphism class — thus a co-ultracharacteristic subgroup is clearly co-hypercharacteristic. It is also obvious that a co-hypercharacteristic subgroup is S-characteristic: compare (5.4).

All this is, in fact, almost trivial and of no great interest, and the bonus obtained from translating, dualizing, and translating back is, as far as groups are concerned, negligible.

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