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Description of Entire Solutions of Eiconal Type Equations

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Abstract. The paper describes entire solutions to the eiconal type non-linear partial differential equations, which include the eiconal equations $(X_1(u))^2 + (X_2(u))^2 = 1$ as special cases, where $X_1 = p_1\partial/\partial z_1 + p_2\partial/\partial z_2$, $X_2 = p_3\partial/\partial z_1 + p_4\partial/\partial z_2$ are linearly independent operators with p_j being arbitrary polynomials in $\mathbf{C}^2$.

1 Introduction

In a recent paper [8], we showed that entire solutions to eiconal (eikonal) type nonlinear partial differential equations

$$u_{z1}^2 + u_{z2}^2 = p \tag{1.1}$$

for a polynomial p in $\mathbf{C}^2$ must be either of the form $u = F(z_1, z_2 - iz_1) + f(z_2 - iz_1)$ or of the form $u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$, where F is a two-variable polynomial and f, ϕ_1, ϕ_2 are one-variable polynomials, which are characterized by certain conditions in terms of p. These nonlinear partial differential equations and certain generalizations in real variables arise in wave propagation, geometric optics, quantum mechanics, for example, in describing the wave fronts of light in an inhomogeneous medium with a variable index of refraction p (see [2, 3, 5]).

We note that an equation (1.1) can be turned to a Monge–Ampére equation by a variable change and differentiation [7, 8]. It is known that entire solutions to the eiconal equation $u_{z_1}^2 + u_{z_2}^2 = 1$, the equation (1.1) with $p = 1$, are necessarily linear [7, 10], which is, as mentioned in [7], close in flavor to a result due to Bernstein on linearity of solutions to a minimal surface equation in a whole plane [1, 9], as well as some of the results stemming from [1], such as the theorem of Jörgens [6] stating that a C^2-solution in $\mathbf{R}^2$ of the non-degenerated Monge–Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 1$ must be a quadratic polynomial.

We observe that when p is a certain linear function, entire solutions of (1.1) may be quadratic polynomials; for instance, $u_{z_1}^2 + u_{z_2}^2 = p$ with $p = z_1 + z_2 + 1$ has the entire solution $u = \frac{1}{4}z_1^2 + \frac{i}{2}z_1z_2 - \frac{1}{4}z_2^2 + z_1$. On the other hand, for some linear functions p, the equation (1.1) does not even admit any entire solutions. It is desirable to further understand these situations. It turns out (see Corollary 2.4 below) that all entire solutions of (1.1), if there are any, must be quadratic polynomials when p is linear, and these quadratic polynomials and such p's can be explicitly given; in fact, the

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result holds for arbitrary irreducible polynomials p. This led us to write the present paper, as a continuation of the work [8]. The result will be treated as a consequence of a theorem for more general equations

$$(X_1(u))^2 + (X_2(u))^2 = p, \tag{1.2}$$

where $X_1 = p_1\partial/\partial z_1 + p_2\partial/\partial z_2$, $X_2 = p_3\partial/\partial z_1 + p_4\partial/\partial z_2$ are linearly independent operators in $\mathbf{C}^2$, *i.e.*, $D = p_1p_4 - p_2p_3 \not\equiv 0$, with p_j's being polynomials in $\mathbf{C}^2$ and p an irreducible polynomial in $\mathbf{C}^2$. This contains some well-known equations as special cases. When $p \equiv 1$, the equations of the form (1.2) are often referred to as eiconal equations. For such equations with arbitrary polynomial coefficients, entire solutions in general may be algebraic or transcendental. For example, entire solutions of the complex Grusin equation $u_{z_1}^2 + (z_1^n u_{z_2})^2 = 1$ (n is a positive integer) are linear, while entire solutions of the equation $u_{z_1}^2 + (z_1z_2^3u_{z_1} - (z_2^4/3)u_{z_2})^2 = 1$ are transcendental (Corollary 2.3 and Remark 2.6 below). All entire solutions of these equations can be explicitly found using the theorem in the paper. We will state the detailed results in Section 2 and give the proofs in Section 3.

2 Results

Theorem 2.1 *Let u be an entire solution of the equation*

$$(X_1(u))^2 + (X_2(u))^2 = p, \tag{2.1}$$

where $X_1 = p_1\partial/\partial z_1 + p_2\partial/\partial z_2$, $X_2 = p_3\partial/\partial z_1 + p_4\partial/\partial z_2$ are linearly independent operators in $\mathbf{C}^2$ and p, p_j $(1 \le j \le 4)$ are polynomials in $\mathbf{C}^2$ with $p \not\equiv 0$ irreducible. Then u is given by

$$\begin{aligned} u_{z_1} &= \frac{1}{D}\Big(p_4\frac{e^{ih} + pe^{-ih}}{2} - cp_2\frac{e^{ih} - pe^{-ih}}{2i}\Big),\\ u_{z_2} &= \frac{1}{D}\Big(cp_1\frac{e^{ih} - pe^{-ih}}{2i} - p_3\frac{e^{ih} + pe^{-ih}}{2}\Big), \end{aligned} \tag{2.2}$$

where $c = \pm 1$, $D = p_1p_4 - p_2p_3$, h is a constant or a nonconstant polynomial determined by

$$h_{z_1} = \frac{acp_2 + bp_4}{cD^2}, \quad h_{z_2} = -\frac{acp_1 + bp_3}{cD^2}, \tag{2.3}$$

and

$$\begin{aligned} a &= -\frac{iC}{2p} + cD(p_2)_{z_2} - cp_2D_{z_2} + cD(p_1)_{z_1} - cp_1D_{z_1},\\ b &= -\frac{C}{2p} + D(p_3)_{z_1} - p_3D_{z_1} + D(p_4)_{z_2} - p_4D_{z_2},\\ C &= -p_3p_{z_1} - p_4p_{z_2} - \frac{1}{i}cp_1p_{z_1} - \frac{1}{i}cp_2p_{z_2}. \end{aligned} \tag{2.4}$$

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The expression in (2.2), which depends only on the given data p_1, p_2, p_3, p_4, and p, characterizes the partial derivatives of the solution u and thus the solution u itself. In particular, when these polynomials are explicitly given, one can find all the existing entire solutions u explicitly by integrating (2.2) (see the examples in Remark 2.6 below).

For the eiconal equations with $p \equiv 1$, Theorem 2.1 is specified to the following.

Corollary 2.2 *Let u be an entire solution of the equation*

$$(X_1(u))^2 + (X_2(u))^2 = 1 \tag{2.5}$$

in $\mathbf{C}^2$, *where* $X_1 = p_1(\partial/\partial z_1) + p_2(\partial/\partial z_2)$, $X_2 = p_3(\partial/\partial z_1) + p_4(\partial/\partial z_2)$ *are linearly independent operators with* p_j *being polynomials in* $\mathbf{C}^2$. *Then u is given by*

$$u_{z_1} = \frac{1}{D}(p_4 \cos h - p_2 \sin h), \quad u_{z_2} = \frac{1}{D}(p_1 \sin h - p_3 \cos h), \tag{2.6}$$

where $D = p_1 p_4 - p_2 p_3$, *h is a constant or a nonconstant polynomial given by*

$$h_{z_1} = \frac{a p_2 + b p_4}{D^2}, \quad h_{z_2} = \frac{-a p_1 + b p_3}{D^2} \tag{2.7}$$

and

$$\begin{aligned} a &= D(p_2)_{z_2} - p_2 D_{z_2} + D(p_1)_{z_1} - p_1 D_{z_1}, \\ b &= D(p_3)_{z_1} - p_3 D_{z_1} + D(p_4)_{z_2} - p_4 D_{z_2}. \end{aligned} \tag{2.8}$$

As mentioned in §1, entire solutions to the equations $u_{z_1}^2 + (z_1^n u_{z_2})^2 = 1$ (where n is a positive integer) are linear. This is clearly contained in the following more general characterization.

Corollary 2.3 *Let* $P(z_1; z_2)$ *and* $Q(z_1; z_2)$ *be arbitrary polynomials in* $\mathbf{C}^2$. *Then u is an entire solution of the equation*

$$(P u_{z_1})^2 + (Q u_{z_2})^2 = 1 \tag{2.9}$$

if and only if $u = c_1 z_1 + c_2 z_2 + c_3$ *is a linear function, where* c_j*'s are constants, and exactly one of the following holds:*

(i) $c_1 = 0$ *and Q is a constant satisfying that* $(c_2 Q)^2 = 1$;
(ii) $c_2 = 0$ *and P is a constant satisfying that* $(c_1 P)^2 = 1$;
(iii) $c_1 c_2 \neq 0$ *and P, Q are both constant satisfying that* $(c_1 P)^2 + (c_2 Q)^2 = 1$.

Another special case of Corollary 2.3 is the eiconal equation $u_{z_1}^2 + u_{z_2}^2 = 1$, which is equation (2.9) with $P = Q = 1$, and thus has exactly linear solutions

$$u = c_1 z_1 + c_2 z_2 + c_3$$

with $c_1^2 + c_2^2 = 1$. This was shown in different ways in [7, 10].

As another corollary, we have the following characterization for the equation $u_{z_1}^2 + u_{z_2}^2 = p$, when p is an irreducible polynomial, which was mentioned in the beginning of the introduction.

Corollary 2.4 *Let $p \not\equiv 0$ be an irreducible polynomial in $\mathbf{C}^2$. Then u is an entire solution to the partial differential equation*

$$u_{z_1}^2 + u_{z_2}^2 = p \tag{2.10}$$

in $\mathbf{C}^2$ if and only if p is a linear function of the form $p = c_1(iz_1 - cz_2) + c_2$, where c_1, c_2, c are constants with $c = \pm 1$ and

$$u = \alpha(iz_1 - cz_2)^2 + \beta(iz_1 - cz_2) + \gamma z_1 + \delta \tag{2.11}$$

is a quadratic polynomial, where $\alpha = (c_1\gamma^{-1}/4i), \beta = (1/2i)(\gamma^{-1}c_2 - \gamma)$, and $\gamma \neq 0, \delta$ are constants.

Remark 2.5 The polynomial p in Corollary 2.4 is assumed to be irreducible. When p is not irreducible, the result fails and entire solutions may be polynomials of arbitrarily given degree. For example, consider the equation $u_{z_1}^2 + u_{z_2}^2 = p$ with $p = -2i(z_2 - iz_1)^{n-1} + 1$, where n is any given positive integer. Then the polynomial $u = \frac{1}{n}(z_2 - iz_1)^n + z_1$ is a solution of the equation with the arbitrarily given degree n. When $n \geq 3$, p is not irreducible and the solution is not quadratic.

Remark 2.6 In view of Corollary 2.3, where entire solutions are linear, and Corollary 2.4, where entire solutions are quadratic, it would be tempting to think that entire solutions in Theorem 2.1 or Corollary 2.2 always could be polynomials and something could be said about the degrees. It turns out that entire solutions may be polynomials of arbitrarily given degrees or even transcendental, as shown in the following examples.

(i) Let n be any given positive integer. Consider the equation

$$u_{z_1}^2 + (z_2^{n-1}u_{z_1} - u_{z_2})^2 = 1,$$

which is the equation (2.1) with $p_1 = 1, p_2 = 0, p_3 = z_2^{n-1}$, and $p_4 = -1$. Thus, $D = -1$. We will use Corollary 2.2 to find all entire solutions of the equation. By the corollary, entire solutions are given by (2.6). If h in (2.6) is a polynomial, by (2.8) we see that $a = b = 0$. Then by (2.7), we see that $h_{z_1} = h_{z_2} = 0$. We thus must have that h is a constant. Let $\cos h = c_1$ and $\sin h = c_2$. Then $c_1^2 + c_2^2 = 1$. And by (2.6), $u_{z_1} = c_1$ and $u_{z_2} = -c_2 + c_1 z_2^{n-1}$. Integrating them yields that entire solutions are exactly $u = c_1 z_1 - c_2 z_2 + \frac{c_1}{n} z_2^n$, where c_1, c_2 are any constants satisfying that $c_1^2 + c_2^2 = 1$. They are polynomials of the given degree n with $c_1 \neq 0$.

(ii) Consider the equation $u_{z_1}^2 + (z_1 z_2^3 u_{z_1} - (z_2^4/3)u_{z_2})^2 = 1$, which is equation (2.5) with $p_1 = 1, p_2 = 0 p_3 = z_1 z_2^3$ and $p_4 = -(z_2^4/3)$. Thus, $D = p_4$. We will use Corollary 2.2 to find all entire solutions of the equation. By the corollary, entire solutions are given by (2.6). If h in (2.6) is a constant, then $u_{z_2} = -(3/z_2^4)(\sin h - z_1 z_2^3 \cos h)$ by (2.6). Since $\cos h$ and $\sin h$ cannot both be zero, we see that u_{z_2} has poles when $z_2 = 0$, which is impossible. Thus, h is a nonconstant polynomial by Corollary 2.2. By (2.8), we have that $a = 0$ and $b = -\frac{1}{3}z_2^7$. By (2.7), we obtain that $h_{z_1} = z_2^3$ and $h_{z_2} = 3z_1 z_2^2$, which yields from integration that $h = z_1 z_2^3 + c_1$

for a constant c_1. Then by (2.6) we deduce that $u_{z_1} = \cos(z_1 z_2^3 + c_1)$ and $u_{z_2} = -3z_2^{-4}\sin(z_1 z_2^3 + c_1) + 3z_1 z_2^{-1}\cos(z_1 z_2^3 + c_1)$. Integrating these two equalities yields that $u = (1/z_2^3)\sin(z_1 z_2^3 + c_1) + c$ for a constant c. But we are solving the given equation for entire solutions u. Thus, we must have that $c_1 = k\pi$ for an integer k, since otherwise u would have poles. Thus, entire solutions of the given equation are exactly $u = \pm(1/z_2^3)\sin(z_1 z_2^3) + c$ for a constant c, which are transcendental entire functions in $\mathbf{C}^2$.

Remark 2.7 Consider the eiconal equation on $\mathbf{R}^2$

$$u_x^2 + u_y^2 = 1, \tag{2.12}$$

$(x, y) \in \mathbf{R}^2$. It is apparent that linear functions $u(x, y) = ax + by + c$ with $a^2 + b^2 = 1$ are solutions of (2.12), as in the complex case. If we impose an initial condition, then for any constant λ the linear functions

$$f_\lambda(x, y) = (x - x_0)\cos\lambda + (y - y_0)\sin\lambda$$

give a family of solutions satisfying the initial condition $f_\lambda(x_0, y_0) = 0$. However, besides the linear solutions, the same eiconal equation (2.12) and initial condition are satisfied also by the radical function $f(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. This result in $\mathbf{R}^2$ can be also carried over to the eiconal equation $|\nabla f|_g^2 = 1$ on Riemannian manifolds (M, g) [2, Theorem 7.3.2]. We refer to [2] for related results and applications.

3 Proofs

Proof of Theorem 2.1 Let u be an entire solution of (2.1). Then

$$X_1^2 + X_2^2 = (X_1 + iX_2)(X_1 - iX_2) = p,$$

where $X_1 = p_1 u_{z_1} + p_2 u_{z_2}$, $X_2 = p_3 u_{z_1} + p_4 u_{z_2}$. Since p is irreducible, there exists an entire function h such that $X_1 + iX_2 = e^{ih}$ and then $X_1 - iX_2 = pe^{-ih}$, or $X_1 - iX_2 = e^{ih}$ and then that $X_1 + iX_2 = pe^{-ih}$; each gives a linear system in X_1, X_2. In each case, solving for X_1 and X_2 we obtain that

$$p_1 u_{z_1} + p_2 u_{z_2} = X_1 = \frac{e^{ih} + pe^{-ih}}{2}, \tag{3.1}$$

$$p_3 u_{z_1} + p_4 u_{z_2} = X_2 = c\frac{e^{ih} - pe^{-ih}}{2i}, \tag{3.2}$$

where $c = 1$ in the first case and $c = -1$ in the second case. To simplify the calculation and expressions below, we introduce

$$\sin_p h = \frac{e^{ih} - pe^{-ih}}{2i}, \quad \cos_p h = \frac{e^{ih} + pe^{-ih}}{2}.$$

Then

$$\frac{\partial}{\partial z_j}\sin_p h = h_{z_j}\cos_p h - \frac{1}{2i}p_{z_j}e^{-ih}, \quad \frac{\partial}{\partial z_j}\cos_p h = -h_{z_j}\sin_p h + \frac{1}{2}p_{z_j}e^{-ih}.$$

Solving u_{z_1} and u_{z_2} from the system (3.1) and (3.2), we obtain that

$$u_{z_1} = \frac{1}{D}(p_4\cos_p h - cp_2\sin_p h), \quad u_{z_2} = \frac{1}{D}(cp_1\sin_p h - p_3\cos_p h)$$

as given in (2.2), where $D = p_1p_4 - p_2p_3$ is the determinant of the above system. We next need to show that h is a constant or a nonconstant polynomial given by (2.3). To this end, we differentiate the above two equalities to obtain

$$\begin{aligned}
u_{z_2z_1} = \frac{1}{D^2}\Big\{ & D((p_4)_{z_2}\cos_p h - p_4h_{z_2}\sin_p h + \frac{1}{2}p_4p_{z_2}e^{-ih} - c(p_2)_{z_2}\sin_p h \\
& - cp_2h_{z_2}\cos h + \frac{1}{2i}cp_2p_{z_2}e^{-ih}) - (p_4\cos_p h - cp_2\sin_p h)D_{z_2}\Big\}, \\
u_{z_1z_2} = \frac{1}{D^2}\Big\{ & D(c(p_1)_{z_1}\sin_p h + cp_1h_{z_1}\cos_p h - \frac{1}{2i}cp_1p_{z_1}e^{-ih} - (p_3)_{z_1}\cos_p h \\
& + p_3h_{z_1}\sin_p h - \frac{1}{2}p_3p_{z_1}e^{-ih}) - (cp_1\sin_p h - p_3\cos_p h)D_{z_1}\Big\}.
\end{aligned}$$

Using the fact that $u_{z_1z_2} = u_{z_2z_1}$, we obtain that

$$\begin{aligned}
\sin_p h\Big\{ & -Dp_4h_{z_2} - cD(p_2)_{z_2} + cp_2D_{z_2} - cD(p_1)_{z_1} - Dp_3h_{z_1} + cp_1D_{z_1}\Big\} \\
& = \cos_p h\Big\{cDp_1h_{z_1} - D(p_3)_{z_1} + p_3D_{z_1} - D(p_4)_{z_2} + cDp_2h_{z_2} + p_4D_{z_2}\Big\} \\
& \quad + \frac{1}{2}e^{-ih}\Big\{-\frac{1}{i}cp_1p_{z_1} - p_3p_{z_1} - p_4p_{z_2} - \frac{1}{i}cp_2p_{z_2}\Big\}.
\end{aligned}$$

Write the above equality as $A\sin_p h = B\cos_p h + \frac{1}{2}e^{-ih}C$, where A, B, C are defined from above, *i.e.*,

$$\begin{aligned}
A &= -Dp_4h_{z_2} - cD(p_2)_{z_2} + cp_2D_{z_2} - cD(p_1)_{z_1} - Dp_3h_{z_1} + cp_1D_{z_1}, \\
B &= cDp_1h_{z_1} - D(p_3)_{z_1} + p_3D_{z_1} - D(p_4)_{z_2} + cDp_2h_{z_2} + p_4D_{z_2}, \\
C &= -\frac{1}{i}cp_1p_{z_1} - p_3p_{z_1} - p_4p_{z_2} - \frac{1}{i}cp_2p_{z_2}.
\end{aligned}$$

We then deduce, by the definitions of $\sin_p h$ and $\cos_p h$ that

$$\frac{e^{2ih} - p}{2i}A = \frac{e^{2ih} + p}{2}B + \frac{1}{2}C$$

or $A(e^{2ih} - p) = iB(e^{2ih} + p) + iC$, *i.e.*,

$$(A - iB)e^{2ih} = ipB + pA + iC. \tag{3.3}$$

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If h is a constant, the theorem already holds. Assume in the following that h is not a constant.

We assert that $A - iB \equiv 0$ in (3.3). Otherwise, we would have from (3.3) that

$$e^{2ih} = \frac{ipB + pA + iC}{A - iB}. \tag{3.4}$$

Then h cannot be a nonconstant polynomial, since otherwise the left-hand side of (3.4) is transcendental, while the right-hand side of (3.4) is a polynomial, which is impossible. Thus, h is transcendental. We can then adopt Nevanlinna theory to derive a contradiction. Let $T(r, F)$ denote the Nevanlinna characteristic function of a meromorphic function F in $\mathbf{C}^n$. On one hand, we have that

$$\lim_{r\to\infty} \frac{T(r, e^{2ih})}{T(r, h)} = +\infty$$

(see [4, p. 88]). On the other hand, the equality (3.4) implies that $T(r, e^{2ih}) = O\{T(r, h) + \log r\} = O\{T(r, h)\}$ outside possibly a set of finite Lebesgue measure by the arithmetic properties of the characteristic and the facts (see [11], [12, p. 99]) that $T(r, F_{z_j}) = O\{T(r, F)\}$ for any meromorphic function F outside a set of finite Lebesgue measure and that $\log r = O\{T(r, g)\}$ for any transcendental meromorphic function g in $\mathbf{C}^n$. We thus obtain a contradiction. This proves the above assertion that $A - iB = 0$ and then by (3.3) that $ipB + pA + iC = 0$. Solving for A and B from these two equalities, we obtain that $A = -\frac{iC}{2p}$ and $B = -\frac{C}{2p}$, which yields by the definitions of A and B that

$$-Dp_3h_{z_1} - Dp_4h_{z_2} = -\frac{iC}{2p} + cD(p_2)_{z_2} - cp_2D_{z_2} + cD(p_1)_{z_1} - cp_1D_{z_1} := a,$$

$$cDp_1h_{z_1} + cDp_2h_{z_2} = -\frac{C}{2p} + D(p_3)_{z_1} - p_3D_{z_1} + D(p_4)_{z_2} - p_4D_{z_2} := b.$$

The above system in h_{z_1} and h_{z_2} has determinant cD^3. Solving for h_{z_1} and h_{z_2} from the system yields that

$$h_{z_1} = \frac{acp_2 + bp_4}{cD^2}, \quad h_{z_2} = -\frac{bp_3 + acp_1}{cD^2}$$

as given in the theorem. Note that each function in the right-hand sides of the above two equalities is a polynomial. Thus, h_{z_1}, h_{z_2} must be polynomials, since they are entire functions. Hence, h is a polynomial. ∎

Proof of Corollary 2.2 Equation (2.5) is equation (2.1) in Theorem 2.1 with $p \equiv 1$. Thus, by Theorem 2.1 entire solutions u are given by

$$u_{z_1} = \frac{1}{D}(p_4 \cos h - cp_2 \sin h), \quad u_{z_2} = \frac{1}{D}(cp_1 \sin h - p_3 \cos h), \tag{3.5}$$

where h is a constant, or a nonconstant polynomial given by (2.3), *i.e.*,

$$h_{z_1} = \frac{acp_2 + bp_4}{cD^2}, \quad h_{z_2} = -\frac{acp_1 + bp_3}{cD^2}, \tag{3.6}$$

$c = \pm 1$, $D = p_1 p_4 - p_3 p_4$, and

$$\begin{aligned} a &= cD(p_2)_{z_2} - cp_2 D_{z_2} + cD(p_1)_{z_1} - cp_1 D_{z_1}, \\ b &= D(p_3)_{z_1} - p_3 D_{z_1} + D(p_4)_{z_2} - p_4 D_{z_2}. \end{aligned} \tag{3.7}$$

Noting that when $p \equiv 1$, we can always take $c = 1$ in the equalities (3.1) and (3.2) in the proof of Theorem 2.1. In fact, if $c = -1$ in (3.2), we then replace h by $h_1 = -h$, then (3.1) and (3.2) become

$$p_1 u_{z_1} + p_2 u_{z_2} = X_1 = \frac{e^{ih_1} + e^{-ih_1}}{2},$$
$$p_3 u_{z_1} + p_4 u_{z_2} = X_2 = \frac{e^{ih_1} - e^{-ih_1}}{2i}.$$

Therefore, in any case we can always have $c = 1$ in (3.1) and (3.2) (for some entire function h or h_1 there) and thus in the entire proof of Theorem 2.1. The above results (3.5), (3.6), and (3.7) with $c = 1$ then yield the equalities (2.6), (2.7), and (2.8), respectively. ∎

Proof of Corollary 2.3 The sufficiency is easy to check. To prove the necessity, assume that u is an entire solution of (2.9), which is equation (2.5) with $p_1 = P$, $p_2 = p_3 = 0$, $p_4 = Q$ and $D = PQ$. Then by Corollary 2.2, u is given by (2.6). If the function h in (2.6) is a polynomial given by (2.7), then by (2.8), we have that

$$a = DP_{z_1} - PD_{z_1}, \quad b = DQ_{z_2} - QD_{z_2}.$$

Thus, we have by (2.7)) that

$$h_{z_1} = \frac{bQ}{D^2} = Q\frac{DQ_{z_2} - QD_{z_2}}{D^2} = Q\Big(\frac{Q}{D}\Big)_{z_2} = Q\Big(\frac{1}{P}\Big)_{z_2} = -Q\frac{P_{z_2}}{P^2}. \tag{3.8}$$

From the given equation (2.9), it is clear that P and Q do not have any common zeros (since the right-hand side of (2.9) is 1). This implies that P_{z_2} in (3.8) must be identically 0, since otherwise $h_{z_1} = -Q(P_{z_2}/P^2)$ has poles at some zeros of P due to the fact that the degree of the polynomial P in the denominator is higher than the one of its derivative P_{z_2} in the numerator. This is absurd, since h is entire. Thus, we have shown by (3.8) that $h_{z_1} \equiv 0$. Also, by (2.7) we have that

$$h_{z_2} = -\frac{aP}{D^2} = -P\frac{DP_{z_1} - PD_{z_1}}{D^2} = -P\Big(\frac{P}{D}\Big)_{z_1} = -P\Big(\frac{1}{Q}\Big)_{z_1}.$$

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Thus, the same argument shows that $h_{z_2} \equiv 0$. Therefore, we must have that h is a constant in (2.6), which gives us

$$u_{z_1} = \frac{1}{D} Q \cos h = \frac{\cos h}{P}, \quad u_{z_2} = \frac{1}{D} P \sin h = \frac{\sin h}{Q}.$$

If $\cos h \equiv 0$, then $u_{z_1} \equiv 0$ is a constant. If $\cos h \not\equiv 0$, then P must be a constant, since otherwise u_{z_1} has poles at zeros of P, which is impossible. This shows that u_{z_1} is also a constant. That is, in any case we always have that u_{z_1} is a constant. The same way shows that u_{z_2} is a constant. Thus, $u = c_1 z_1 + c_2 z_2 + c_3$ is a linear function. Plugging it into the given equation (2.9), we have that

$$(c_1 P)^2 + (c_2 Q)^2 = 1. \tag{3.9}$$

If $c_1 = 0$, then Q must be a constant satisfying $(c_2 Q)^2 = 1$ by (3.9), as given in the conclusion of the corollary. If $c_2 = 0$, then P must be a constant satisfying that $(c_1 P)^2 = 1$ by (3.9), as given in the conclusion of the corollary. If $c_1 c_2 \neq 0$, then we can change (3.9) into $(c_1 P + i c_2 Q)(c_1 P - i c_2 Q) = 1$, which implies that $c_1 P + i c_2 Q = e^{i\alpha}$ and then $(c_1 P - i c_2 Q) = e^{-i\alpha}$ for some entire function α. Solving P and Q from these last two identities yields that

$$P = \frac{1}{2c_1}(e^{i\alpha} + e^{-i\alpha}) = \frac{1}{c_1} \cos \alpha.$$

Similarly, $Q = \frac{1}{c_2} \sin \alpha$. But P and Q are polynomials. Thus, α must be a constant, since otherwise P and Q will become transcendental functions. Now that α is a constant, P and Q are constant and satisfy (3.9), as given in the corollary. ■

Proof of Corollary 2.4 For the sufficiency, suppose that u is given as (2.11). Then

$$u_{z_1} = 2\alpha i(i z_1 - c z_2) + i\beta + \gamma, \quad u_{z_2} = -2\alpha c(i z_1 - c z_2) - c\beta.$$

Note that $c^2 = 1$. It is easy to check that

$$\begin{aligned} u_{z_1}^2 + u_{z_2}^2 &= 4\alpha i(i\beta + \gamma)(i z_1 - c z_2) + (i\beta + \gamma)^2 + +4\alpha c^2 \beta(i z_1 - c z_2) + c^2 \beta^2 \\ &= 4\alpha i\gamma(i z_1 - c z_2) + 2i\beta\gamma + \gamma^2 = c_1(i z_1 - c z_2) + c_2 = p \end{aligned}$$

by the expressions of α, β and p given in the theorem.

To prove the necessity, let u be an entire solution of (2.10). Then by Theorem 2.1 entire solutions u are given by (2.2) with $p_1 = p_4 = 1$ and $p_2 = p_3 = 0$. Thus, $D = 1$ and

$$u_{z_1} = \frac{e^{ih} + p e^{-ih}}{2}, \quad u_{z_2} = c \frac{e^{ih} - p e^{-ih}}{2i}, \tag{3.10}$$

where $c = \pm 1$ and h is a constant or a nonconstant polynomial given in (2.3). In the latter case, we have by (2.4) that

$$a = -\frac{iC}{2p}, \quad b = -\frac{C}{2p}, \quad C = -p_{z_2} - \frac{1}{i} c p_{z_1}.$$

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Thus, the equalities in (2.3) become

$$
(3.11)\qquad
\begin{aligned}
h_{z_1} &= \frac{b}{c} = -\frac{C}{2pc} = -\frac{-\frac{1}{i}cp_{z_1} - p_{z_2}}{2pc} = \frac{cp_{z_1} + ip_{z_2}}{2pci},\\
h_{z_2} &= -a = \frac{iC}{2p} = i\frac{-\frac{1}{i}cp_{z_1} - p_{z_2}}{2p} = -\frac{cp_{z_1} + ip_{z_2}}{2p}.
\end{aligned}
$$

If p is nonconstant, then (3.11) implies that

$$
(3.12)\qquad cp_{z_2} + ip_{z_2} \equiv 0,
$$

since otherwise $cp_{z_1} + ip_{z_2}$ would be a nonzero polynomial with degree less than the degree of p, which implies by (3.11) that h_{z_1}, h_{z_2}, and thus h would have poles, a contradiction to the fact that h is a polynomial. The equality (3.12) is clearly also true when p is a constant. We can treat (3.12) as a linear partial differential equation in p and solve it in a standard way to obtain that $p = f(iz_1 - cz_2)$ for a polynomial f in one complex variable. But, p is an irreducible polynomial. Thus, the polynomial f must be linear, say $f = c_1 z + c_2, z \in \mathbf{C}$, where c_1, c_2 are two complex numbers. Then

$$
(3.13)\qquad p = c_1(iz_1 - cz_2) + c_2,
$$

as given in the corollary. Also, by (3.11) and (3.12), h is a constant. Set $e^{ih} = \gamma$. By (3.10), we have that

$$
u_{z_1} = \frac{\gamma + \gamma^{-1}p}{2}, \quad u_{z_2} = c\frac{\gamma - \gamma^{-1}p}{2i}.
$$

Integrating the above equalities and in view of (3.13), after some algebra manipulation, we obtain that

$$
\begin{aligned}
u &= \frac{1}{2}\gamma z_1 + \frac{\gamma^{-1}}{2}\Big(\frac{c_1}{2i}(iz_1 - cz_2)^2 + c_2 z_1\Big) + \frac{c}{2i}(\gamma - \gamma^{-1}c_2)z_2 + \delta\\
&= \alpha(iz_1 - cz_2)^2 + \beta(iz_1 - cz_2) + \gamma z_1 + \delta,
\end{aligned}
$$

where $\alpha = \frac{c_1\gamma^{-1}}{4i}, \beta = \frac{1}{2i}(\gamma^{-1}c_2 - \gamma)$ and $\gamma \neq 0, \delta$ are constants, as given in Corollary 2.4. ∎

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